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EXISTENCE OF INFINITELY MANY PERIODIC SOLUTIONS FOR SECOND-ORDER NONAUTONOMOUS HAMILTONIAN SYSTEMS

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Abstract. By using minimax methods and critical point theory, we obtain infinitely many periodic solutions for a second-order nonautonomous Hamiltonian systems, when the gradient of potential energy does not exceed linear growth.

1. Introduction and main results

Consider the second-order Hamiltonian system

$$\begin{gathered}\ddot{u}(t)+\nabla F(t,u(t))=0, \quad \text{a.e. } t\in[0,T],\\ u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0\,.\end{gathered}\tag{1.1}$$

Where $T>0$ and $F:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ satisfies the following assumption:

(A1) $F(t,x)$ is measurable in t for every $x\in\mathbb{R}^N$, continuously differentiable in x for a.e. $t\in[0,T]$, and there exist $a\in C(\mathbb{R}^+,\mathbb{R}^+)$, $b\in L^1([0,T],\mathbb{R}^+)$ such that
$$|F(t,x)|\le a(|x|)b(t),\ |\nabla F(t,x)|\le a(|x|)b(t)$$
for all $x\in\mathbb{R}^N$ and a.e. $t\in[0,T]$.

The existence of periodic solutions for problem (1.1) was obtained in [1, 3, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 21, 22, 23] with many solvability conditions by using the least action principle and the minimax methods, such as the coercive type potential condition [3], the convex type potential condition [8], the periodic type potential conditions [18], the even type potential condition [6], the subquadratic potential condition in Rabinowitz's sense [11], the bounded nonlinearity condition (see [9]), the subadditive condition (see [12]), the sublinear nonlinearity condition (see [5, 14]), and the linear nonlinearity condition (see [10, 16, 22, 23]).

In particular, when the nonlinearity $\nabla F(t,x)$ is bounded; that is, there exists $g(t)\in L^1([0,T],\mathbb{R}^+)$ such that $|\nabla F(t,x)|\le g(t)$ for all $x\in\mathbb{R}^N$ and a.e. $t\in[0,T]$, and that
$$\int_0^T F(t,x)dt\to\pm\infty \quad \text{as } |x|\to\infty,$$

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Mawhin and Willem [9] proved that problem (1.1) has at least one periodic solution.

Han and Tang [5,14] generalized these results to the sublinear case:

$$|\nabla F(t,x)| \le f(t)|x|^{\alpha} + g(t) \quad \text{for all } x \in \mathbb{R}^N \text{ and a.e. } t \in [0,T] \tag{1.2}$$

with

$$|x|^{-2\alpha} \int_0^T F(t,x)dt \to \pm\infty \quad \text{as } |x| \to \infty, \tag{1.3}$$

where $f(t), g(t) \in L^1([0,T], \mathbb{R}^+)$ and $\alpha \in [0,1)$.

Subsequently, when $\alpha = 1$ Zhao and Wu [22,23], and Meng and Tang [10,16] proved the existence of periodic solutions for problem (1.1), i.e. $\nabla F(t,x)$ does not exceed linear growth:

$$|\nabla F(t,x)| \le f(t)|x| + g(t) \quad \text{for all } x \in \mathbb{R}^N \text{ and a.e. } t \in [0,T], \tag{1.4}$$

where $f(t), g(t) \in L^1([0,T], \mathbb{R}^+)$.

On the other hand, there are large number of papers that deals with multiplicity results for this problem. In particular, infinitely many solutions for (1.1) are obtained in [2,20,24] when the nonlinearity $F(t,x)$ have symmetry. Since the symmetry assumption on the nonlinearity F has play an important role in [2,21,24], many authors have paid much attention to weak the symmetry condition and some existence results on periodic solutions have been obtained without any symmetry condition [4,7,17,25]. Especially, Zhang and Tang [25] obtained infinitely many periodic solutions for (1.1) when (1.2) holds and F has a suitable oscillating behaviour at infinity:

$$\limsup_{r\to+\infty} \inf_{x\in\mathbb{R}^N, |x|=r} |x|^{-2\alpha} \int_0^T F(t,x)dt = +\infty,$$
$$\liminf_{R\to+\infty} \sup_{x\in\mathbb{R}^N, |x|=R} |x|^{-2\alpha} \int_0^T F(t,x)dt = -\infty,$$

where $\alpha \in [0,1)$.

Motivated by the results mentioned above, especially by ideas in [10,16,22,23,25], in this article, by using the minimax methods in critical point theory, we obtain infinitely many periodic solutions for (1.1).

Let H_T^1 be a Hilbert space $H_T^1 = \{u : [0,T] \to \mathbb{R}^N : u \text{ is absolutely continuous}, u(0) = u(T) \text{ and } \dot{u} \in L^2([0,T], \mathbb{R})\}$, with the norm

$$\|u\| = \Big(\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt\Big)^{1/2}, \tag{1.5}$$

for $u \in H_T^1$. Let

$$J(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t,u(t))dt. \tag{1.6}$$

It is well known that the function J is continuously differentiable and weakly lower semicontinuous on H_T^1 and the solutions of (1.1) correspond to the critical points of J (see [9]). Our main result is the following theorem.

Theorem 1.1. *Suppose that* (A1) *and* (1.4) *with* $\int_0^T f(t)dt < \frac{3}{T}$ *hold and*

$$\limsup_{r\to+\infty} \inf_{x\in\mathbb{R}^N, |x|=r} \int_0^T F(t,x)dt = +\infty, \tag{1.7}$$

$$\liminf_{R\to+\infty}\sup_{x\in\mathbb{R}^N,|x|=R}|x|^{-2}\int_0^T F(t,x)dt < -\frac{3T^2}{2\pi^2(12-T\int_0^T f(t)dt)}\int_0^T f^2(t)dt. \quad (1.8)$$

Then

(i) *There exists a sequence of periodic solutions $\{u_n\}$ which are minimax type critical points of functional J, and $J(u_n)\to+\infty$ as $n\to\infty$;*

(ii) *There exists another sequence of periodic solutions $\{u_m^*\}$ which are local minimum points of functional J, and $J(u_m^*)\to-\infty$ as $m\to\infty$.*

Remark 1.2.

(i) As in [25], in this paper we do not assume any symmetry condition on nonlinearity;

(ii) Our main result in this paper extends main result in [25] corresponding to $\alpha=1$.

2. Proof of main results

For $u\in H_T^1$, let

$$\overline{u}=\frac{1}{T}\int_0^T u(t)dt,\ \widetilde{u}(t)=u(t)-\overline{u}. \quad (2.1)$$

The following inequalities are well known (see [9]):

$$\|\tilde u\|_\infty^2 \le \frac{T}{12}\|\dot u\|_{L^2}^2 \quad \text{(Sobolev's inequality)},$$
$$\|\tilde u\|_{L^2}^2 \le \frac{T^2}{4\pi^2}\|\dot u\|_{L^2}^2 \quad \text{(Wirtinger's inequality)}.$$

For the sake of convenience, we denote

$$M_1=\Big(\int_0^T f^2(t)dt\Big)^{1/2},\quad M_2=\int_0^T f(t)dt,\quad M_3=\int_0^T g(t)dt.$$

Lemma 2.1. *Suppose that $\int_0^T f(t)dt<3/T$ and (1.4) hold, then*

$$J(u)\to+\infty \quad \text{as } \|u\|\to\infty \text{ in } \widetilde{H}_T^1, \quad (2.2)$$

where $\widetilde{H}_T^1=\{u\in H_T^1 \mid \overline{u}=0\}$ be the subspace of H_T^1.

Proof. From (1.4) and Sobolev's inequality, for all u in $\widetilde{H}_T^1$ we have

$$\begin{aligned}
J(u)&=\frac12\int_0^T|\dot u(t)|^2dt-\int_0^T F(t,u(t))dt\\
&\ge\frac12\int_0^T|\dot u(t)|^2dt-\int_0^T f(t)|u(t)|^2dt-\int_0^T g(t)|u(t)|dt\\
&\ge\frac12\int_0^T|\dot u(t)|^2dt-\|\widetilde u\|_\infty^2\int_0^T f(t)dt-\|\widetilde u\|_\infty\int_0^T g(t)dt\\
&\ge\frac12\|\dot u\|_{L^2}^2-\frac{T}{12}\|\dot u\|_{L^2}^2\int_0^T f(t)dt-\big(\frac{T}{12}\big)^{1/2}\|\dot u\|_{L^2}\int_0^T g(t)dt\\
&=\Big(\frac12-\frac{T}{12}\int_0^T f(t)dt\Big)\|\dot u\|_{L^2}^2-C_1\|\dot u\|_{L^2}\,.
\end{aligned}$$

By Wirtinger's inequality, the norm $\|u\|=\big(\int_0^T|\dot u(t)|^2dt\big)^{1/2}$ is an equivalent norm on $\widetilde{H}_T^1$. So, $J(u)\to+\infty$ as $\|u\|\to\infty$ in $\widetilde{H}_T^1$. $\square$

Lemma 2.2. *Suppose that* (1.7) *holds. Then there exists positive real sequence* $\{a_n\}$ *such that*

$$\lim_{n\to\infty} a_n = +\infty, \quad \lim_{n\to\infty} \sup_{u\in\mathbb{R}^N, |u|=a_n} J(u) = -\infty\,.$$

The above lemma follows from (1.7).

Lemma 2.3. *Suppose that* $\int_0^T f(t)dt < \frac{3}{T}$, (1.4) *and* (1.8) *hold. Then there exists positive real sequence* $\{b_m\}$ *such that*

$$\lim_{m\to\infty} b_m = +\infty, \quad \lim_{m\to\infty} \inf_{u\in H_{b_m}} J(u) = +\infty,$$

where $H_{b_m} = \{u \in \mathbb{R}^{\mathbb{N}} : |u| = b_m\} \bigoplus \widetilde{H}_T^1$.

Proof. By (1.8), we can choose an $a > 3T^2/(12\pi^2 - \pi^2 TM_2)$ such that

$$\liminf_{r\to+\infty} \sup_{x\in\mathbb{R}^N, |x|=r} |x|^{-2}\int_0^T F(t,x)dt < -\frac{a}{2}M_1^2.$$

For any $u \in H_{b_m}$, let $u = \overline{u} + \widetilde{u}$, where $|\overline{u}| = b_m$, $\widetilde{u} \in \widetilde{H}_T^1$. So, we have

$$\begin{aligned}
&\Big|\int_0^T F(t,u(t)) - F(t,\overline{u})dt\Big| \\
&= \Big|\int_0^T \int_0^1 (\nabla F(t,\overline{u}+s\widetilde{u}(t)), \widetilde{u}(t))\,ds\,dt\Big| \\
&\le \int_0^T \int_0^1 f(t)|\overline{u}+s\widetilde{u}(t)||\widetilde{u}(t)|\,ds\,dt + \int_0^T \int_0^1 g(t)|\widetilde{u}(t)|\,ds\,dt \\
&\le \int_0^T f(t)\left(|\overline{u}| + \frac{1}{2}|\widetilde{u}(t)|\right)|\widetilde{u}(t)|dt + \int_0^T g(t)|\widetilde{u}(t)|dt \\
&\le |\overline{u}|\Big(\int_0^T f^2(t)dt\Big)^{1/2}\Big(\int_0^T |\widetilde{u}(t)|^2dt\Big)^{1/2} + \frac{1}{2}\|\widetilde{u}\|_\infty^2\int_0^T f(t)dt + \|\widetilde{u}\|_\infty\int_0^T g(t)dt \\
&= M_1|\overline{u}|\|\widetilde{u}\|_{L^2} + \frac{M_2}{2}\|\widetilde{u}\|_\infty^2 + M_3\|\widetilde{u}\|_\infty \\
&\le \frac{1}{2a}\|\widetilde{u}\|_{L^2}^2 + \frac{a}{2}M_1^2|\overline{u}|^2 + \frac{M_2}{2}\|\widetilde{u}^2\|_\infty + M_3\|\widetilde{u}\|_\infty \\
&\le \big(\frac{T^2}{8a\pi^2} + \frac{TM_2}{24}\big)\|\dot{u}\|_{L^2}^2 + \frac{a}{2}M_1^2|\overline{u}|^2 + \big(\frac{T}{12}\big)^{1/2}M_3\|\dot{u}\|_{L^2}
\end{aligned}$$

for all $u \in H_{b_m}$. Hence we have

$$\begin{aligned}
J(u) &= \frac{1}{2}\int_0^T |\dot{u}(t)|^2dt - \int_0^T [F(t,u(t)) - F(t,\overline{u})]dt - \int_0^T F(t,\overline{u})dt \\
&\ge \big(\frac{1}{2} - \frac{T^2}{8a\pi^2} - \frac{TM_2}{24}\big)\|\dot{u}\|_{L^2}^2 - \big(\frac{T}{12}\big)^{1/2}M_3\|\dot{u}\|_{L^2} \\
&- |\overline{u}|^2\Big(|\overline{u}|^{-2}\int_0^T F(t,\overline{u})dt + \frac{a}{2}M_1^2\Big)
\end{aligned}$$

for all $u \in H_{b_m}$. As $\big(|\overline{u}|^2 + \|\dot{u}\|_{L^2}\big)^{\frac{1}{2}} \to \infty$ if and only if $\|u\| \to \infty$, then the Lemma follows from (1.8) and the above inequality. □

Now prove our main result.

Proof of Theorem 1.1. Let B_{a_n} be a ball in $\mathbb{R}^N$ with radius a_n. Then we define a family of maps

$$\Gamma_n = \{\gamma \in C(B_{a_n}, H_T^1) : \gamma\big|_{\partial B_{a_n}} = Id\big|_{\partial B_{a_n}}\}$$

and corresponding minimax values

$$c_n = \inf_{\gamma\in\Gamma_n} \max_{x\in B_{a_n}} J(\gamma(x)).$$

It is easy to see that each γ intersects the hyperplane $\widetilde{H}_T^1$, i.e., for any $\gamma \in \Gamma_n$, $\gamma(Ba_n) \cap \widetilde{H}_T^1 \neq \emptyset$.

By Lemma 2.1, the functional J is coercive on $\widetilde{H}_T^1$. So, there is a constant M such that

$$\max_{x\in B_{a_n}} J(\gamma(x)) \geq \inf_{u\in\widetilde{H}_T^1} J(u) \geq M.$$

Hence

$$c_n \geq \inf_{u\in\widetilde{H}_T^1} J(u) \geq M.$$

By Lemma 2.2, for all large value of n,

$$c_n > \max_{u\in\partial B_{a_n}} J(u).$$

For such n, there exists a sequence $\{\gamma_k\}$ in Γ_n such that

$$\max_{x\in B_{a_n}} J(\gamma_k(x)) \to c_n, k \to \infty.$$

Applying [9, Theorem 4.3 and Corollary 4.3], we know there exists a sequence $\{v_k\}$ in H_T^1 such that

$$J(v_k) \to c_n, \operatorname{dist}(v_k, \gamma_k(B_{a_n})) \to 0, J'(v_k) \to 0, \tag{2.3}$$

as $k \to \infty$. If we can show $\{v_k\}$ is bounded, then there is a subsequence, which is still be denote by $\{v_k\}$ such that

$$v_k \rightharpoonup u_n \quad \text{weakly in } H_T^1,$$
$$v_k \to u_n \quad \text{uniformly in } C([0,T], \mathbb{R}^N).$$

Hence

$$\langle J'(v_k) - J'(u_n), v_k - u_n\rangle \to 0,$$
$$\int_0^T (\nabla F(t, v_k) - \nabla F(t, u_n), v_k - u_n)dt \to 0$$

as $k \to \infty$. Moreover, it is easy to see that

$$\begin{aligned}&\langle J'(v_k) - J'(u_n), v_k - u_n\rangle \\ &= \|\dot{v_k} - \dot{u_n}\|_{L^2}^2 - \int_0^T (\nabla F(t, v_k) - \nabla F(t, u_n), v_k - u_n)dt,\end{aligned}$$

so $\|\dot{v_k} - \dot{u_n}\|_{L^2}^2 \to 0$ as $k \to \infty$. Then, it is not difficult to obtain $\|v_k - v_n\| \to 0$ as $k \to \infty$. So, we have

$$J'(u_n) = \lim_{k\to\infty} J'(v_k) = 0, \quad J(u_n) = \lim_{k\to\infty} J(v_k) = c_n.$$

Thus, u_n is critical point and c_n is critical value of functional J.

Now, let us show the sequence $\{v_k\}$ is bounded in H_T^1. By (2.3), for any large enough k, we have

$$c_n \leq \max_{x\in B_{a_n}} J(\gamma_k(x)) \leq c_n + 1, \tag{2.4}$$

and we can find $w_k \in \gamma_k(B_{a_n})$ such that $\|v_k - w_k\| \leq 1$.

Fix n, by Lemma 2.3, we can choose a large enough m such that

$$b_m > a_n \quad \text{and} \quad \inf_{u\in H_{b_m}} > c_n + 1.$$

This implies $\gamma(B_{a_n})$ cannot intersect the hyperplane H_{b_m} for each k.

Let $w_k = \overline{w}_k + \widetilde{w}_k$, where $\overline{w}_k \in \mathbb{R}^N$ and $\widetilde{w}_k \in \widetilde{H}_T^1$. Then we have $|\overline{w}_k| < b_m$ for each k. Also, by Sobolev's inequality and (1.4), it is obvious that

$$\begin{aligned}
&c_n + 1\\
&\geq J(w_k) = \frac{1}{2}\int_0^T |\dot{w}_k(t)|^2 dt - \int_0^T F(t, w_k(t))dt\\
&\geq \frac{1}{2}\int_0^T |\dot{w}_k(t)|^2 dt - \int_0^T f(t)|w_k(t)|^2 dt - \int_0^T g(t)|w_k(t)|dt\\
&\geq \frac{1}{2}\int_0^T |\dot{w}_k(t)|^2 dt - 2\int_0^T f(t)[|\overline{w}_k|^2 + |\widetilde{w}_k(t)|^2]dt - \int_0^T g(t)[|\overline{w}_k| + |\widetilde{w}_k(t)|]dt\\
&\geq \frac{1}{2}\int_0^T |\dot{w}_k(t)|^2 dt - 2\|\widetilde{w}_k\|_\infty^2 \int_0^T f(t)dt - 2|\overline{w}_k|^2 \int_0^T f(t)dt\\
&\quad - \|\widetilde{w}_k\|_\infty \int_0^T g(t)dt - |\overline{w}_k| \int_0^T g(t)dt\\
&\geq \frac{1}{2}\|\dot{w}_k(t)\|_{L^2}^2 - \frac{T}{6}\|\dot{w}_k(t)\|_{L^2}^2 \int_0^T f(t)dt - 2|\overline{w}_k|^2 \int_0^T f(t)dt\\
&\quad - \big(\frac{T}{12}\big)^{1/2}\|\dot{w}_k(t)\|_{L^2} \int_0^T g(t)dt - |\overline{w}_k| \int_0^T g(t)dt\\
&= \big(\frac{1}{2} - \frac{T}{6}M_2\big)\|\dot{w}_k(t)\|_{L^2}^2 - \big(\frac{T}{12}\big)^{1/2} M_3 \|\dot{w}_k(t)\|_{L^2} - C_2
\end{aligned}$$

As $(|\overline{u}|^2 + \|\dot{u}\|_{L_2})^{1/2}$ is an equivalent norm in H_T^1, it follows that $\widetilde{w}_k(t)$ is bounded. Hence, w_k is bounded. Also, $\{v_k\}$ is bounded in H_T^1.

From the previous discussion we know that accumulation point u_n of $\{v_k\}$ is a critical point and c_n is critical value of J.

If we choose large enough n such that $a_n > b_m$, then $\gamma(B_{a_n})$ intersects the hyperplane H_{b_m} for any $\gamma \in \Gamma_n$.

It follows that

$$\max_{x\in B_{a_n}} J(\gamma(x)) \geq \inf_{u\in H_{b_m}} J(u).$$

From this inequality and Lemma 2.3 we obtain $\lim_{n\to\infty} c_n = +\infty$. Result (i) of Theorem 1.1 is obtained.

Next we prove (ii). For fixed m, define the subset P_m of H_T^1 by

$$P_m = \{u \in H_T^1 : u = \overline{u} + \widetilde{u}, |\overline{u}| \leq b_m, \widetilde{u} \in \widetilde{H}_T^1\}. \tag{2.5}$$

For $u \in P_m$, we have

$$
\begin{aligned}
J(u) &= \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t,u(t))dt \\
&\geq \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T f(t)|u(t)|^2 dt - \int_0^T g(t)|u(t)|dt \\
&\geq \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - 2\int_0^T f(t)[|\overline{u}(t)|^2 + |\widetilde{u}(t)|^2]dt - \int_0^T g(t)[|\overline{u}(t)| + |\widetilde{u}(t)|]dt \\
&\geq \frac{1}{2}\|\dot{u}(t)\|_{L^2}^2 - \frac{T}{6}\|\dot{u}(t)\|_{L^2}^2 \int_0^T f(t)dt - 2|\overline{u}(t)|^2 \int_0^T f(t)dt \\
&\quad - \big(\frac{T}{12}\big)^{1/2}\|\dot{u}(t)\|_{L^2}\int_0^T g(t)dt - |\overline{u}(t)|\int_0^T g(t)dt \\
&= \big(\frac{1}{2} - \frac{T}{6}M_2\big)\|\dot{u}(t)\|_{L^2}^2 - \big(\frac{T}{12}\big)^{1/2} M_3 \|\dot{u}(t)\|_{L^2} - C_3
\end{aligned}
\tag{2.6}
$$

Then J is bounded below on P_m.

Let

$$\mu_m = \inf_{u \in P_m} J(u),$$

and $\{u_k\}$ be a minimizing sequence in P_m; that is,

$$J(u_k) \to \mu_m \quad \text{as } k \to \infty.$$

By (2.6), $\{u_k\}$ is bounded in H_T^1. Then there is a subsequence, which is still be denoted by $\{u_k\}$, such that

$$u_k \rightharpoonup u_m^* \text{ weakly in } H_T^1.$$

Since P_m is a convex closed subset of H_T^1, $u_m^* \in P_m$. As J is weakly lower semi-continuous, we have

$$\mu_m = \lim_{k\to\infty} J(u_k) \geq J(u_m^*).$$

Since $u_m^* \in P_m$, $\mu_m = J(u_m^*)$.

If we can show u_m^* is in the interior of P_m, then u_m^* is a local minimum of functional J. In fact, let $u_m^* = \overline{u}_m^* + \widetilde{u}_m^*$. From Lemmas 2.2 and 2.3, we see $|\overline{u}_m^*| \neq b_m$ for large m, which means that u_m^* is in the interior of P_m.

Since u_m^* is a minimum of J on P_m, we have

$$J(u_m^*) = \inf_{u \in P_m} J(u) \leq \sup_{|u| = b_m} J(u).$$

It follows from Lemma 2.2 that $J(u_m^*) \to -\infty$ as $m \to \infty$. Therefore, the proof is complete. □

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