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HÖLDER CONTINUITY WITH EXPONENT $(1+\alpha)/2$ IN THE TIME VARIABLE FOR SOLUTIONS OF PARABOLIC EQUATIONS

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Abstract. We consider the regularity of solutions for some parabolic equations. We show Hölder continuity with exponent $(1+\alpha)/2$, with respect to the time variable, when the gradient in the space variable of the solution has the Hölder continuity with exponent α.

1. Introduction

In this article we consider the Hölder continuity of solutions for the equation.

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial t} = f \quad \text{in } Q \tag{1.1}$$

where $Q = \Omega \times (0,T]$, $\Omega \subset \mathbb{R}^n$ is a domain and $T > 0$. For the classical solution $u(x,t)$ of (1.1), we shall show the Hölder continuity with exponent $(1+\alpha)/2$ in the time variable t, when the gradient of u with respect to the space variable x has Hölder continuity with exponent α.

We assume that:

(H1) L is parabolic, i.e., for any $(x,t) \in Q$,

$$\sum_{i,j\ 1}^{n} a_{ij}(x,t)\xi_i\xi_j > 0 \quad \text{for all } 0 \neq \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n.$$

Note that L is not necessary uniformly parabolic.

(H2) $a_{ij}, b_i \in C(Q)$ for $i,j = 1,\dots,n$ where $C(Q)$ denotes the space of continuous functions in Q.

(H3) There exist constants $\mu_1, \mu_2 > 0$ such that

$$\sum_{i=1}^{n} a_{ii}(x,t) \le \mu_1, \quad \sum_{i=1}^{n} |b_i(x,t)| \le \mu_2 \quad \text{for all } (x,t) \in Q.$$

(H4) $f = f(x,t)$ is a bounded continuous function in Q satisfying

$$|f(x,t)| \le \mu_3 \quad \text{for all } (x,t) \in Q.$$

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In the following, for non-negative integers k, l and any set $A \subset \mathbb{R}^n$, we denote the space of functions $u \in C(A \times (0,T])$ such that u has continuous partial derivatives $\partial_x^\alpha u$ for $|\alpha| \le k$ and $\partial_t^j u$ for $j \le l$ in $A \times (0,T]$ by $C^{k,l}(A \times (0,T])$. Here

$$\partial_x^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ and $|\alpha| = \sum_{i=1}^n \alpha_i$. We also use the notation $u_t = \partial_t u$, $u_{x_i} = \partial_{x_i} u$, $u_{x_i x_j} = \partial_{x_i} \partial_{x_j} u$ etc. Now we are in a position to state our main result.

Theorem 1.1. *Under the hypotheses* (H1)–(H4)*, let $u \in C^{2,1}(Q)$ be a solution of* (1.1) *in Q. Assume that there exist $\alpha \in (0,1]$ and constants $C_1, C_2 \ge 0$ such that*

$$|\nabla u(x,t) - \nabla u(y,t)| \le C_1 |x-y|^\alpha \tag{1.2}$$

for all $(x,t), (y,t) \in Q$, and

$$|\nabla u(x,t)| \le C_2 \tag{1.3}$$

for all $(x,t) \in Q$. Here and hereafter ∇ denotes the gradient operator with respect to the space variable x.

(i) *Let $\Omega' \subset \Omega$ be a subdomain such that $\operatorname{dist}(\Omega', \partial\Omega) \ge d > 0$, and define $Q' = \Omega' \times (0,T]$. Then there exist $\delta > 0$ depending only on μ_1, μ_2, μ_3 and α, $K > 0$ depending only on $\mu_1, \mu_2, \mu_3, d, \alpha, C_1$ and C_2 such that*

$$|u(x,t) - u(x,t_0)| \le K|t-t_0|^{(1+\alpha)/2} \tag{1.4}$$

for all $(x,t), (x,t_0) \in Q'$ with $|t - t_0| < \delta$.

(ii) *Furthermore, if we assume that $\partial\Omega \ne \emptyset$ and $u \in C^{1,0}(\overline{\Omega} \times (0,T])$ satisfies that there exist $\beta \in (0,1]$ and a constant $D \ge 0$ such that*

$$|\nabla u(x,t) - \nabla u(x,t_0)| \le D|t-t_0|^{(1+\beta)/2}$$

for all $x \in \partial\Omega$ and $t, t_0 \in (0,T]$, then for any $\sigma > 0$ there exists $K > 0$ depending only on $\mu_1, \mu_2, \mu_3, C_1, C_2, D$ and σ such that

$$|u(x,t) - u(x,t_0)| \le K|t-t_0|^{(1+\gamma)/2}, \quad \gamma = \min\{\alpha, \beta\}$$

for any $(x,t), (x,t_0) \in Q$ with $|t-t_0| < \sigma$.

Remark 1.2. Gilding [6] assumed that $|u(x,t) - u(y,t)| \le C_1|x-y|^\alpha$ instead of (1.2) and (1.3), and obtained

$$|u(x,t) - u(x,t_0)| \le K|t-t_0|^\alpha$$

instead of (1.4). Note that the papers of Brandt [4] and Knerr [7] can be viewed as precursors to the present study. See also the discussion of Ladyzhenskaja et al [8] in [7]. Then the author of [6] applied the result to the Cauchy problem for the porous media equation in one dimension. See also Aronson [2] and Bénilan [3]. On the other hand, our result can be applied to the regularity for a quasilinear parabolic type system associated with the Maxwell equation. For such application, see Aramaki [1].

2. Proof of Theorem 1.1

We shall use a modification of the arguments in [6].

(i) Let $\Omega' \subset \Omega$ be a subdomain with $\operatorname{dist}(\Omega', \partial\Omega) \geq d > 0$ and define $Q' = \Omega' \times (0, T]$. Fix arbitrary points $(x_0, t_0), (x_0, t_1) \in Q'$ with $0 < t_0 < t_1 \leq T$ and choose $0 < \rho < d$, and define μ and C so that

$$\mu = \max\{\mu_1, \mu_2, \mu_2 C_2 + \mu_3\} \quad \text{and} \quad C = \frac{C_1}{1+\alpha}.$$

Moreover, we define a set and functions

$$\begin{gathered} N = \{x \in \mathbb{R}^n ; |x - x_0| < \rho\} \times (t_0, t_1] \subset Q, \\ v^{\pm}(x,t) = \mu\{1 + 2s\rho^{-2}(1+\rho)\}(t - t_0) + s\rho^{-2}|x - x_0|^2 + C\rho^{1+\alpha} \\ \pm \{u(x,t) - u(x_0, t_0) - \nabla u(x_0, t_0) \cdot (x - x_0)\} \end{gathered}$$

where "$\cdot$" denotes the inner product in $\mathbb{R}^n$. Let

$$s = \sup_{t_0 \leq t \leq t_1, x \in \Omega'} |u(x,t) - u(x, t_0)|.$$

Since

$$\begin{gathered} v_t^{\pm} = \mu\{1 + 2s\rho^{-2}(1+\rho)\} \pm u_t(x,t), \\ v_{x_i}^{\pm} = 2s\rho^{-2}(x_i - x_{0,i}) \pm \{u_{x_i}(x,t) - u_{x_i}(x_0, t_0)\}, \\ v_{x_i x_j}^{\pm} = 2s\rho^{-2}\delta_{ij} \pm u_{x_i x_j}(x,t) \end{gathered}$$

where δ_{ij} denotes the Kronecker delta, we have

$$\begin{aligned} Lv^{\pm} &= -\mu - 2s\rho^{-2}\mu(1+\rho) + 2s\rho^{-2}\Big\{\sum_{i=1}^n a_{ii}(x,t) + \sum_{i=1}^n b_i(x,t)(x_i - x_{0,i})\Big\} \\ &\quad \pm Lu(x,t) \mp \sum_{i=1}^n b_i(x,t) u_{x_i}(x_0, t_0) \\ &\leq -\mu - 2s\rho^{-2}(\mu + \mu\rho) + 2s\rho^{-2}(\mu_1 + \mu_2\rho) + |f(x,t)| \\ &\quad + \sum_{i=1}^n |b_i(x,t)||u_{x_i}(x_0, t_0)| \\ &\leq -\mu - 2s\rho^{-2}(\mu + \mu\rho) + 2s\rho^{-2}(\mu_1 + \mu_2\rho) + \mu_3 + C_2\mu_2 \leq 0. \end{aligned} \tag{2.1}$$

Here we used the definition of μ.

When $t = t_0$ and $|x - x_0| \le \rho$, from the definition of C, we see that

$$
\begin{aligned}
v^{\pm}(x,t_0) &= s\rho^{-2}|x-x_0|^2 + C\rho^{1+\alpha} \\
&\quad \pm \{u(x,t_0) - u(x_0,t_0) - \nabla u(x_0,t_0)\cdot(x-x_0)\} \\
&= s\rho^{-2}|x-x_0|^2 + C\rho^{1+\alpha} \\
&\quad \pm \int_0^1 (\nabla u(\theta x_0 + (1-\theta)x) - \nabla u(x_0,t_0))\cdot(x-x_0)d\theta \\
&\ge s\rho^{-2}|x-x_0|^2 + C\rho^{1+\alpha} \\
&\quad - C_1\int_0^1 |\theta x_0 + (1-\theta)x - x_0|^{\alpha} d\theta |x-x_0| \\
&\ge s\rho^{-2}|x-x_0|^2 + C\rho^{1+\alpha} - \frac{C_1}{1+\alpha}\rho^{1+\alpha} \ge 0.
\end{aligned} \tag{2.2}
$$

When $|x - x_0| = \rho$ and $t_0 < t \le t_1$, using the definition of s, we can see that

$$
\begin{aligned}
v^{\pm}(x,t) &= \mu\{1 + 2s\rho^{-2}(1+\rho)\}(t-t_0) + s + C\rho^{1+\alpha} \\
&\quad \pm \{u(x,t) - u(x_0,t_0) - \nabla u(x_0,t_0)\cdot(x-x_0)\} \\
&= \mu\{1 + 2s\rho^{-2}(1+\rho)\}(t-t_0) + s + C\rho^{1+\alpha} \\
&\quad \pm \{u(x,t_0) - u(x_0,t_0) - \nabla u(x_0,t_0)\cdot(x-x_0)\} \\
&\quad \pm \{u(x,t) - u(x,t_0)\} \\
&\ge \mu\{1 + 2s\rho^{-2}(1+\rho)\}(t-t_0) + s + C\rho^{1+\alpha} - \frac{C_1}{1+\alpha}\rho^{1+\alpha} - s \\
&\ge 0.
\end{aligned} \tag{2.3}
$$

Thus from (2.1), (2.2) and (2.3), we see that

$$
\begin{gathered}
Lv^{\pm} \le 0 \quad \text{in } N, \\
v^{\pm} \ge 0 \quad \text{on the parabolic boundary of } N.
\end{gathered} \tag{2.4}
$$

By the maximum principle (cf. Friedman [5, p. 34] or Lieberman [9, Chapter 2, Lemma 2.3]), it follows that $v^{\pm} \ge 0$ in N. Hence we have

$$
\begin{aligned}
&\mp \{u(x,t) - u(x_0,t_0) - \nabla u(x_0,t_0)\cdot(x-x_0)\} \\
&\le C\rho^{1+\alpha} + \mu\{1 + 2s\rho^{-2}(1+\rho)\}(t-t_0) + s\rho^{-2}|x-x_0|^2.
\end{aligned}
$$

If we put $x = x_0$, then we see that

$$
|u(x_0,t) - u(x_0,t_0)| \le C\rho^{1+\alpha} + \mu\{1 + 2s\rho^{-2}(1+\rho)\}(t-t_0).
$$

Since $x_0 \in \Omega'$ and $t \in (t_0, t_1]$ are arbitrary, it follows that

$$
\begin{aligned}
s &\le C\rho^{1+\alpha} + \mu\{1 + 2s\rho^{-2}(1+\rho)\}(t_1-t_0) \\
&= C\rho^{1+\alpha} + \mu(t-t_0) + \frac{1}{2}s\{4\mu\rho^{-2}(1+\rho)(t_1-t_0)\}.
\end{aligned} \tag{2.5}
$$

Let ρ^* be the positive root of the quadratic equation $y^2 = 4\mu(1+y)(t_1-t_0)$, i.e.,

$$
\rho^* = 2\mu(t_1-t_0) + 2\{\mu(t_1-t_0) + \mu^2(t_1-t_0)^2\}^{1/2}. \tag{2.6}
$$

If we define $\delta = d^2/(4\mu(1+d))$, for $t_1 < t_0 + \delta$, it is easily seen that $\rho^* < d$. Thus we can replace ρ in (2.5) with ρ^*. Therefore when $t_0 < t_1 < t_0 + \delta$, we see that

$$
s \le C\big(2\mu(t_1-t_0) + 2\{\mu(t_1-t_0) + \mu^2(t_1-t_0)^2\}^{1/2}\big)^{1+\alpha}
$$

$$
\begin{aligned}
&+ \mu(t_1 - t_0) + \frac{1}{2}s \\
&= C\big(2\mu(t_1-t_0)^{1/2} + 2\{\mu + \mu^2(t_1-t_0)\}^{1/2}\big)^{1+\alpha}(t_1-t_0)^{(1+\alpha)/2} \\
&+ \mu(t_1-t_0)^{(1-\alpha)/2}(t_1-t_0)^{(1+\alpha)/2} + \frac{1}{2}s.
\end{aligned}
$$

Since $t_1 - t_0 < \delta$, we have

$$
s \le 2\big[C\big(2\mu\delta^{1/2} + 2\{\mu+\mu^2\delta\}^{1/2}\big)^{1+\alpha} + \mu\delta^{(1-\alpha)/2}\big](t_1-t_0)^{(1+\alpha)/2}.
$$

Thus we have

$$
|u(x_0,t_1) - u(x_0,t_0)| \le K(t_1-t_0)^{(1+\alpha)/2}
$$

where

$$
K = 2\big[C\big(2\mu\delta^{1/2} + 2\{\mu+\mu^2\delta\}^{1/2}\big)^{1+\alpha} + \mu\delta^{(1-\alpha)/2}\big]
$$

for any $t_1 < t_0 + \delta$. Since (x_0,t_0) and (x_0,t_1) with $t_0 < t_1 \le T$ are arbitrary points in Q', we get the conclusion of (i).

(ii) When $(x_0,t_0),(x_0,t_1) \in Q$ with $0 < t_0 < t_1 < t_0 + \sigma$, we choose ρ^* as in (2.6). We define

$$
\begin{gathered}
N^* = \{x \in \mathbb{R}^n : |x - x_0| < \rho^*\} \times (t_0, t_1] \subset \mathbb{R}^n \times (0,T], \\
w^{\pm}(x,t) = v^{\pm}(x,t) + D(t_1-t_0)^{(1+\beta)/2} \text{ in } N^* \cap Q, \\
s = \sup_{t_0 \le t \le t_1, x \in \overline{\Omega}} |u(x,t) - u(x,t_0)|.
\end{gathered}
$$

By a similar argument as in the proof of (i), we have

$$
\begin{gathered}
Lw^{\pm} \le 0 \quad \text{in } N^* \cap Q, \\
w^{\pm} \ge 0 \quad \text{on the parabolic boundary of } N^* \cap Q.
\end{gathered}
$$

If we choose $\mu = \max\{\mu_1, \mu_2, \mu_2 C_2 + \mu_3, D\sigma^{(1+\beta)/2}\}$, from a similar argument as in (i) we can get the conclusion of (ii).

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