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EXISTENCE OF INFINITELY MANY SOLUTIONS FOR QUASILINEAR PROBLEMS WITH A $p(x)$-BIHARMONIC OPERATOR

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Abstract. By using critical point theory, we establish the existence of infinitely many weak solutions for a class of Navier boundary-value problem depending on two parameters and involving the $p(x)$-biharmonic operator. Under an appropriate oscillatory behaviour of the nonlinearity and suitable assumptions on the variable exponent, we obtain a sequence of pairwise distinct solutions.

1. Introduction

In this work we study the existence of infinitely many weak solutions for Navier boundary-value problem

$$\begin{gathered} \Delta^2_{p(x)} u = \lambda f(x,u) + \mu g(x,u), \quad x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with boundary of class C^1, λ is a positive parameter, μ is a non-negative parameter, $f, g \in C^0(\overline{\Omega} \times \mathbb{R})$, $p(\cdot) \in C^0(\overline{\Omega})$ with

$$\max\{2, \frac{N}{2}\} < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x)$$

and $\Delta^2_{p(x)} u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the operator of fourth order called the $p(x)$-biharmonic operator, which is a natural generalization of the p-biharmonic operator (where $p > 1$ is a constant).

The study of differential equations and variational problems with variable exponents has attracted intense research interests in recent years. Such problems arise from the study of electrorheological fluids, image processing, and the theory of nonlinear elasticity. So the investigation of existence and multiplicity of solutions for problems involving biharmonic, p-biharmonic and $p(x)$-biharmonic operators has drawn the attention of many authors see [9, 10, 14, 15, 19, 24, 25, 26, 27]. In particular, in [25], the authors studied the following $p(x)$-biharmonic elliptic problem

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with Navier boundary conditions:

$$\begin{gathered}\Delta^2_{p(x)} u = \lambda a(x) f(x,u) + \mu g(x,u), \quad x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega,\end{gathered} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with boundary of class C^1, λ, μ are non-negative parameters, $p(\cdot) \in C^0(\overline{\Omega})$ with

$$\max\{2, \frac{N}{2}\} < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x).$$

By the three critical points theorem obtained by Ricceri [21], they established the existence of three weak solutions to problem (1.2).

In the case when $p(x) \equiv p$ is a constant, we know that the problem (1.1) has infinitely many solutions from [9].

Here we point out that the $p(x)$-biharmonic operator possesses more complicated nonlinearities than p-biharmonic, for example, it is inhomogeneous and usually it does not have the so called first eigenvalue, since the infimum of its principle eigenvalue is zero.

Recently in [2], presenting a version of the infinitely many critical points theorem of Ricceri (see [20, Theorem 2.5]), the existence of an unbounded sequence of weak solutions for a Strum-Liouville problem, having discontinuous nonlinearities, has been established. In a such approach, an appropriate oscillating behavior of the nonlinear term either at infinity or at zero is required. This type of methodology has been used then in several works in order to obtain existence results for different kinds of problems (see, for instance, [1, 3, 4, 5, 6, 7, 8, 9, 12] and references therein).

Our goal in this article is to obtain some sufficient conditions to guarantee that problem (1.1) has infinitely many weak solutions. To this end, we require that the primitive F of f satisfies a suitable oscillatory behavior either at infinity (for obtaining unbounded solutions) or at zero (for finding arbitrarily small solutions), while G, the primitive of g, has an appropriate growth (see Theorems 3.1 and 4.6). Our approach is fully variational and the main tool is a general critical point theorem (see Lemma 2.1 below) contained in [2]; see also [20].

The plan of this article is as follows. In Section 2, some known definitions and results on variable exponent Lebesgue and Sobolev spaces, which will be used in sequel, are collected. Moreover, the abstract critical points theorem (Lemma 2.1) is recalled. Section 3 is devoted to main theorem and finally, in Section 4, some applications are presented.

2. Preliminaries

The goal of this work is to establish some new criteria for problem (1.1) to have infinitely many weak solutions. Our analysis is mainly based on a recent critical point theorem of Bonanno and Molica Bisci [2] (see Lemma (2.1) below) which is a more precise version of Ricceri's variational principle [20, Theorem 2.5].

Lemma 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semi-continuous, strongly continuous and coercive, and Ψ is sequentially weakly upper* *semicontinuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\big(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)\big) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \text{and} \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional*
$$I_\lambda := \Phi - \lambda\Psi$$
to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) *If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*
  (1) *I_λ possesses a global minimum, or*
  (2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*
$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$, then for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either*
  (1) *there is a global minimum of Φ which is a local minimum of I_λ, or*
  (2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ that converges weakly to a global minimum of Φ.*

For the reader's convenience, we recall some background facts concerning the Lebesgue-Sobolev spaces variable exponent and introduce some notation. For more details, we refer the reader to [11, 13, 16, 17, 18, 22, 23]. Set

$$C_+(\Omega) := \big\{h \in C(\overline{\Omega}) : h(x) > 1, \forall x \in \overline{\Omega}\big\}.$$

For $p(\cdot) \in C_+(\Omega)$, define

$$L^{p(\cdot)}(\Omega) := \big\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty\big\}.$$

We introduce a norm on $L^{p(\cdot)}(\Omega)$ by

$$|u|_{p(\cdot)} = \inf\big\{\beta > 0 : \int_\Omega |\frac{u(x)}{\beta}|^{p(x)} dx \le 1\big\}.$$

The space $(L^{p(\cdot)}(\Omega), |u|_{p(\cdot)})$ is a Banach space called a variable exponent Lebesgue space. Define the Sobolev space with variable exponent

$$W^{m,p(\cdot)}(\Omega) = \big\{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \le m\big\},$$

where

$$D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}} u$$

with $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{m,p(\cdot)}(\Omega)$, equipped with the norm

$$\|u\|_{m,p(\cdot)} := \sum_{|\alpha|\le m} |D^\alpha u|_{p(\cdot)},$$

becomes a separable, reflexive and uniformly convex Banach space. We denote by $W_0^{m,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p(\cdot)}(\Omega)$.

Now we denote

$$X := W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega).$$

For $u \in X$, we define

$$\|u\| = \inf\Big\{\beta > 0 : \int_\Omega |\frac{\Delta u(x)}{\beta}|^{p(x)} dx \le 1\Big\}.$$

It is easy to see that X endowed with the above norm is a separable and reflexive Banach space. We denote by X^* its dual.

Remark 2.2. According to [28], the norm $\|u\|_{2,p(\cdot)}$ is equivalent to the norm $|\Delta u|_{p(\cdot)}$ in the space X. Consequently, the norms $\|u\|_{2,p(\cdot)}, \|u\|$ and $|\Delta u|_{p(\cdot)}$ are equivalent.

For the rest of this article, we use $\|u\|$ instead of $\|u\|_{2,p(\cdot)}$ on X.

Proposition 2.3 ([11, 22]). *The conjugate space of $L^{p(\cdot)}(\Omega)$ is $L^{q(\cdot)}(\Omega)$ where $q(\cdot)$ is the conjugate function of $p(\cdot)$; i.e.,*

$$\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1.$$

For $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, we have

$$\Big|\int_\Omega u(x)v(x)dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(\cdot)}|v|_{q(\cdot)} \le 2|u|_{p(\cdot)}|v|_{q(\cdot)}.$$

Proposition 2.4 ([11, 22]). *Set $\rho(u) = \int_\Omega |u|^{p(x)}dx$. For $u, u_n \in L^{p(\cdot)}(\Omega)$, we have*

(1) $|u|_{p(\cdot)} < (=;>)1 \Leftrightarrow \rho(u) < (=;>)1$,
(2) $|u|_{p(\cdot)} > 1 \Rightarrow |u|_{p(\cdot)}^{p^-} \le \rho(u) \le |u|_{p(\cdot)}^{p^+}$,
(3) $|u|_{p(\cdot)} < 1 \Rightarrow |u|_{p(\cdot)}^{p^+} \le \rho(u) \le |u|_{p(\cdot)}^{p^-}$,
(4) $|u_n|_{p(\cdot)} \to 0 \Leftrightarrow \rho(u_n) \to 0$,
(5) $|u_n|_{p(\cdot)} \to \infty \Leftrightarrow \rho(u_n) \to \infty$.

From Proposition 2.4, for $u \in L^{p(\cdot)}(\Omega)$ the following inequalities hold:

$$\|u\|^{p^-} \le \int_\Omega |\Delta u(x)|^{p(x)}\, dx \le \|u\|^{p^+} \quad \text{if } \|u\| > 1; \tag{2.1}$$

$$\|u\|^{p^+} \le \int_\Omega |\Delta u(x)|^{p(x)}\, dx \le \|u\|^{p^-} \quad \text{if } \|u\| < 1. \tag{2.2}$$

Proposition 2.5 ([26]). *If $\Omega \subset \mathbb{R}^N$ is a bounded domain, then the embedding $X \hookrightarrow C^0(\overline{\Omega})$ is compact whenever $N/2 < p^-$.*

From Proposition 2.5, there exists a positive constant c depending on $p(\cdot), N$ and Ω such that

$$\|u\|_\infty = \sup_{x\in\overline{\Omega}} |u(x)| \le c\|u\|, \quad \forall u \in X. \tag{2.3}$$

Corresponding to f and g we introduce the functions $F, G : \Omega \times \mathbb{R} \to \mathbb{R}$, as follows

$$F(x,t) := \int_0^t f(x,\xi)\, d\xi, \quad G(x,t) := \int_0^t g(x,\xi)\, d\xi$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.

We say that a function $u \in X$ is a *weak solution* of problem (1.1) if

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v dx - \lambda \int_\Omega f(x,u(x))v(x)\,dx - \mu \int_\Omega g(x,u(x))v(x)\,dx = 0$$

holds for all $v \in X$.

3. Main results

Fix $x^0 \in \Omega$ and choose r_1, r_2 with $0 < r_1 < r_2$, such that $B(x^0, r_2) \subseteq \Omega$ where $B(x,r)$ stands for the open ball in $\mathbb{R}^N$ of radius r centered x. Let

$$\begin{aligned} \sigma := &\frac{2c^{p^-}\pi^{\frac{N}{2}}(r_2^N - r_1^N)}{N\Gamma(\frac{N}{2})} \\ &\times \max\Big\{\big[\frac{12(N+2)^2(r_1+r_2)}{(r_2-r_1)^3}\big]^{p^-}, \big[\frac{12(N+2)^2(r_1+r_2)}{(r_2-r_1)^3}\big]^{p^+}\Big\}, \end{aligned} \tag{3.1}$$

$$\eta := \liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{|t|\le\xi} F(x,t)\,dx}{\xi^{p^-}},$$

$$\theta := \limsup_{\xi\to+\infty} \frac{\int_{B(x^0,r_1)} F(x,\xi)\,dx}{\xi^{p^+}},$$

$$\lambda_1 := \frac{\sigma}{p^- c^{p^-}\theta}, \quad \lambda_2 := \frac{1}{p^+ c^{p^-}\eta},$$

where Γ denotes the Gamma function and c is defined by (2.3).

Theorem 3.1. *Assume that*

- (A1) $F(x,t) \ge 0$ *for every* $(x,t) \in \Omega \times [0,+\infty[$;
- (A2) *there exist* $x^0 \in \Omega$, *and* $0 < r_1 < r_2$ *as considered in* (3.1) *such that* $\eta < \frac{p^-}{p^+\sigma}\theta$.

Then, for each $\lambda \in \Lambda := (\lambda_1, \lambda_2)$ *and for every* $g \in C^0(\overline{\Omega}\times\mathbb{R})$ *whose potential* $G(x,t) := \int_0^t g(x,\xi)\,d\xi$ *for all* $(x,t) \in \overline{\Omega}\times[0,+\infty[$, *is a non-negative function satisfying the condition*

$$g_\infty := \limsup_{\xi\to+\infty} \frac{\int_\Omega \sup_{|t|\le\xi} G(x,t)\,dx}{\xi^{p^-}} < +\infty, \tag{3.2}$$

if we put

$$\mu_{g,\lambda} := \frac{1}{p^+c^{p^-}g_\infty}\Big(1 - \lambda p^+ c^{p^-}\eta\Big),$$

where $\mu_{g,\lambda} = +\infty$ *when* $g_\infty = 0$, *problem* (1.1) *has an unbounded sequence of weak solutions for every* $\mu \in [0, \mu_{g,\lambda})$ *in* X.

Proof. Our aim is to apply Lemma 2.1(b) to problem (1.1). To this end, fix $\overline{\lambda} \in (\lambda_1,\lambda_2)$ and g satisfying our assumptions. Since $\overline{\lambda} < \lambda_2$, we have

$$\mu_{g,\overline{\lambda}} = \frac{1}{p^+c^{p^-}g_\infty}\Big(1 - \overline{\lambda}p^+c^{p^-}\eta\Big) > 0.$$

Now fix $\overline{\mu} \in (0, \mu_{g,\overline{\lambda}})$ and set

$$J(x,\xi) := F(x,\xi) + \frac{\overline{\mu}}{\overline{\lambda}}G(x,\xi)$$

for all $(x,\xi) \in \Omega \times \mathbb{R}$. For each $u \in X$, we let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \int_\Omega \frac{1}{p(x)} |\Delta u(x)|^{p(x)} dx, \quad \Psi(u) := \int_\Omega J(x,u(x))\, dx,$$

and put

$$I_{\overline{\lambda}}(u) := \Phi(u) - \overline{\lambda}\Psi(u), \quad u \in X.$$

Note that the weak solutions of (1.1) are exactly the critical points of $I_{\overline{\lambda}}$. The functionals Φ, Ψ satisfy the regularity assumptions of Lemma 2.1. Indeed, by standard arguments, we have that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v dx$$

for any $v \in X$. Furthermore, the differential $\Phi' : X \to X^*$ admits a continuous inverse (see [25, Lemma 3.1]). On the other hand, the fact that X is compactly embedded into $C^0([0,1])$ implies that the functional Ψ is well defined, continuously Gâteaux differentiable and with compact derivative, whose Gâteaux derivative is given by

$$\Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)\, dx + \frac{\overline{\mu}}{\overline{\lambda}} \int_\Omega g(x,u(x))v(x)\, dx.$$

Furthermore, we have from (2.1) that

$$\Phi(u) \geq \frac{1}{p^+} \|u\|^{p^-} \tag{3.3}$$

for all $u \in X$ such that $\|u\| > 1$, and so Φ is coercive. First of all, we will show that $\overline{\lambda} < 1/\gamma$. Hence, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \xi_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_\Omega \sup_{|t|\leq \xi_n} F(x,t)\, dx}{\xi_n^{p^-}} = \eta.$$

Put

$$r_n := \frac{1}{p^+} \Big(\frac{\xi_n}{c}\Big)^{p^-}$$

for all $n \in \mathbb{N}$. Then, for all $v \in X$ with $\Phi(v) < r_n$, taking (2.1) and (2.2) into account, one has

$$\|v\| \leq \max\big\{(p^+ r_n)^{\frac{1}{p^+}}, (p^+ r_n)^{\frac{1}{p^-}}\big\}.$$

So, thanks to the embedding $X \hookrightarrow C^0(\overline{\Omega})$ (see (2.3)), one has $\|v\|_\infty < \xi_n$. Note that $\Phi(0) = \Psi(0) = 0$. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{\big(\sup_{v\in\Phi^{-1}(-\infty,r_n)} \Psi(v)\big) - \Psi(u)}{r_n - \Phi(u)} \\
&\leq \frac{\sup_{v\in\Phi^{-1}(-\infty,r_n)} \Psi(v)}{r_n} \\
&\leq \frac{\int_\Omega \sup_{|t|\leq\xi_n} J(x,t)\, dx}{\frac{1}{p^+}\big(\frac{\xi_n}{c}\big)^{p^-}}
\end{aligned}$$

$$\le p^+ c^{p^-}\Big[\frac{\int_\Omega \sup_{|t|\le\xi_n} F(x,t)\,dx}{\xi_n^{p^-}}+\frac{\overline{\mu}}{\overline{\lambda}}\frac{\int_\Omega \sup_{|t|\le\xi_n} G(x,t)\,dx}{\xi_n^{p^-}}\Big].$$

Moreover, from the assumption (A2) and the condition (3.2), we have $\eta<+\infty$ and

$$\lim_{n\to+\infty}\frac{\int_\Omega \sup_{|t|\le\xi_n} G(x,t)\,dx}{\xi_n^{p^-}}=g_\infty.$$

Therefore,

$$\gamma\le\liminf_{n\to+\infty}\varphi(r_n)\le p^+c^{p^-}\Big(\eta+\frac{\overline{\mu}}{\overline{\lambda}}g_\infty\Big)<+\infty. \tag{3.4}$$

The assumption $\overline{\mu}\in(0,\mu_{G,\overline{\lambda}})$ immediately yields

$$\gamma\le p^+c^{p^-}\Big(\eta+\frac{\overline{\mu}}{\overline{\lambda}}g_\infty\Big)<p^+c^{p^-}\eta+\frac{1-\overline{\lambda}p^+c^{p^-}\eta}{\overline{\lambda}}.$$

Hence,

$$\overline{\lambda}=\frac{1}{p^+c^{p^-}\eta+(1-\overline{\lambda}p^+c^{p^-}\eta)/\overline{\lambda}}<\frac{1}{\gamma}.$$

Let $\overline{\lambda}$ be fixed. We claim that the functional $I_{\overline{\lambda}}$ is unbounded from below. Since

$$\frac{1}{\overline{\lambda}}<\frac{p^-c^{p^-}}{\sigma}\theta,$$

there exist a sequence $\{\tau_n\}$ of positive numbers and $\tau>0$ such that $\lim_{n\to+\infty}\tau_n=+\infty$ and

$$\frac{1}{\overline{\lambda}}<\tau<\frac{p^-c^{p^-}}{\sigma}\frac{\int_{B(x^0,r_1)}F(x,\tau_n)\,dx}{\tau_n^{p^+}} \tag{3.5}$$

for each $n\in\mathbb{N}$ large enough. For all $n\in\mathbb{N}$ define $w_n\in X$ by

$$w_n(x):=\begin{cases}0, & x\in\overline{\Omega}\setminus B(x^0,r_2),\\ \frac{\tau_n[3(l^4-r_2^4)-4(r_1+r_2)(l^3-r_2^3)+6r_1r_2(l^2-r_2^2)]}{(r_2-r_1)^3(r_1+r_2)}, & x\in B(x^0,r_2)\setminus B(x^0,r_1),\\ \tau_n, & x\in B(x^0,r_1),\end{cases} \tag{3.6}$$

where $l=\operatorname{dist}(x,x^0)=\sqrt{\sum_{i=1}^N(x_i-x_i^0)^2}$. Then

$$\frac{\partial w_n(x)}{\partial x_i}=\begin{cases}0,\quad \text{if } x\in\overline{\Omega}\setminus B(x^0,r_2)\cup B(x^0,r_1),\\ \frac{12\tau_n[l^2(x_i-x_i^0)-l(r_1+r_2)(x_i-x_i^0)+r_1r_2(x_i-x_i^0)]}{(r_2-r_1)^3(r_1+r_2)},\\ \quad\text{if } x\in B(x^0,r_2)\setminus B(x^0,r_1),\end{cases}$$

$$\frac{\partial^2 w_n(x)}{\partial x_i^2}=\begin{cases}0,\quad \text{if } x\in\overline{\Omega}\setminus B(x^0,r_2)\cup B(x^0,r_1),\\ \frac{12\tau_n[r_1r_2+(2l-r_1-r_2)(x_i-x_i^0)^2/l-(r_1+r_2-l)l]}{(r_2-r_1)^3(r_1+r_2)},\\ \quad\text{if } x\in B(x^0,r_2)\setminus B(x^0,r_1),\end{cases}$$

$$\sum_{i=1}^N\frac{\partial^2 w_n(x)}{\partial x_i^2}=\begin{cases}0,\quad \text{if } x\in\overline{\Omega}\setminus B(x^0,r_2)\cup B(x^0,r_1),\\ \frac{12\tau_n[(N+2)l^2-(N+1)(r_1+r_2)l+Nr_1r_2]}{(r_2-r_1)^3(r_1+r_2)},\\ \quad\text{if } x\in B(x^0,r_2)\setminus B(x^0,r_1).\end{cases}$$

For any fixed $n \in \mathbb{N}$, one has

$$\Phi(w_n) = \int_{B(x^0,r_2)\setminus B(x^0,r_1)} \frac{1}{p(x)} |\Delta w_n(x)|^{p(x)} dx \leq \frac{\sigma \tau_n^{p^+}}{p^- c^{p^-}}. \tag{3.7}$$

On the other hand, bearing (A1) in mind and since G is non-negative, from the definition of Ψ, we infer

$$\Psi(w_n) = \int_\Omega \Big[F(x, w_n(x)) + \frac{\overline{\mu}}{\overline{\lambda}} G(x, w_n(x))\Big]\, dx \geq \int_{B(x^0,r_1)} F(x, \tau_n)\, dx. \tag{3.8}$$

By (3.5), (3.7) and (3.8), we observe that

$$I_{\overline{\lambda}}(w_n) \leq \frac{\sigma \tau_n^{p^+}}{p^- c^{p^-}} - \overline{\lambda} \int_{B(x^0,r_1)} F(x, \tau_n)\, dx < \frac{\sigma \tau_n^{p^+}}{p^- c^{p^-}} (1 - \overline{\lambda}\tau) \tag{3.9}$$

for every $n \in \mathbb{N}$ large enough. Since $\overline{\lambda}\tau > 1$ and $\lim_{n\to+\infty} \tau_n = +\infty$, we have

$$\lim_{n\to+\infty} I_{\overline{\lambda}}(w_n) = -\infty.$$

Then, the functional $I_{\overline{\lambda}}$ is unbounded from below, and it follows that $I_{\overline{\lambda}}$ has no global minimum. Therefore, by Lemma 2.1(b), there exists a sequence $\{u_n\}$ of critical points of $I_{\overline{\lambda}}$ such that $\lim_{n\to+\infty} \|u_n\| = +\infty$, and the conclusion is achieved. □

Remark 3.2. Under the conditions $\eta = 0$ and $\theta = +\infty$, from Theorem 3.1 we see that for every $\lambda > 0$ and for each $\mu \in \big[0, \frac{1}{p^+ c^{p^-} g_\infty}\big)$, problem (1.1) admits a sequence of weak solutions which is unbounded in X. Moreover, if $g_\infty = 0$, the result holds for every $\lambda > 0$ and $\mu \geq 0$.

4. Applications

In this section, we point out some consequences and applications of the results previously obtained. First, we present the following consequence of Theorem 3.1 with $\mu = 0$.

Theorem 4.1. *Assume that all the assumptions in the Theorem 3.1 hold. Then, for each*

$$\lambda \in \Big(\frac{\sigma}{p^- c^{p^-} \theta}, \frac{1}{p^+ c^{p^-} \eta}\Big),$$

the problem

$$\begin{gathered} \Delta^2_{p(x)} u = \lambda f(x, u), \quad x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega \end{gathered} \tag{4.1}$$

has an unbounded sequence of weak solutions in X.

Here we point out the following consequence of Theorem 3.1.

Corollary 4.2. *Assume that the assumption* (A1) *in Theorem 3.1 holds. Suppose that*

$$\eta < \frac{1}{p^+ c^{p^-}}, \quad \theta > \frac{\sigma}{p^- c^{p^-}}.$$

Then, the problem

$$\begin{gathered} \Delta^2_{p(x)} u = f(x, u) + \mu g(x, u), \quad x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega \end{gathered} \tag{4.2}$$

has an unbounded sequence of weak solutions in X.

Corollary 4.3. *Let $g_1 : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function. Put $G_1(\xi) := \int_0^\xi g_1(t)\,dt$ for all $\xi \in \mathbb{R}$ and assume that*

(A3) $\liminf_{\xi\to+\infty} \frac{G_1(\xi)}{\xi^{p^-}} < +\infty$;

(A4) $\limsup_{\xi\to+\infty} \frac{G_1(\xi)}{\xi^{p^+}} = +\infty$.

Then, for every $\alpha_i \in L^1(\Omega)$ for $1 \le i \le n$, with $\min_{x\in\Omega}\{\alpha_i(x) : 1 \le i \le n\} \ge 0$ and with $\alpha_1 \neq 0$, and for every non-negative continuous $g_i : \mathbb{R} \to \mathbb{R}$ for $2 \le i \le n$, satisfying

$$\max\Big\{\sup_{\xi\in\mathbb{R}} G_i(\xi) : 2 \le i \le n\Big\} \le 0,$$
$$\min\Big\{\liminf_{\xi\to+\infty} \frac{G_i(\xi)}{\xi^{p^-}} : 2 \le i \le n\Big\} > -\infty,$$

where $G_i(\xi) := \int_0^\xi g_i(t)\,dt$ for all $\xi \in \mathbb{R}$ for $2 \le i \le n$, for each

$$\lambda \in \Big]0, \frac{1}{p^+c^{p^-}\liminf_{\xi\to+\infty}\frac{G_1(\xi)}{\xi^{p^-}}\int_\Omega \alpha_1(x)\,dx}\Big[,$$

the problem

$$\Delta^2_{p(x)}u = \lambda\sum_{i=1}^n \alpha_i(x)g_i(u), \quad x \in \Omega,$$
$$u = \Delta u = 0, \quad x \in \partial\Omega$$

has an unbounded sequence of weak solutions in X.

Proof. Set $f(x,t) = \sum_{i=1}^n \alpha_i(x)g_i(t)$ for all $(x,t) \in \Omega\times\mathbb{R}$. From the assumption (A4) and the condition $\min\{\liminf_{\xi\to+\infty}\frac{G_i(\xi)}{\xi^{p^-}} : 2 \le i \le n\} > -\infty$, we have

$$\limsup_{\xi\to+\infty}\frac{\int_\Omega F(x,\xi)\,dx}{\xi^{p^+}} = \limsup_{\xi\to+\infty}\frac{\sum_{i=1}^n\big(G_i(\xi)\int_\Omega \alpha_i(x)\,dx\big)}{\xi^{p^+}} = +\infty.$$

Moreover, from the assumption (A_3) and the condition $\max\{\sup_{\xi\in\mathbb{R}} G_i(\xi) : 2 \le i \le n\} \le 0$, we have

$$\liminf_{\xi\to+\infty}\frac{\int_\Omega \sup_{|t|\le\xi} F(x,t)\,dx}{\xi^{p^-}} \le \Big(\int_\Omega \alpha_1(x)\,dx\Big)\liminf_{\xi\to+\infty}\frac{G_1(\xi)}{\xi^2} < +\infty.$$

Hence, applying Theorem 3.1 the desired conclusion follows. □

Let us observe that the function $s : \overline{\Omega} \to \mathbb{R}_0^+$ defined by

$$s(x) = d(x,\partial\Omega) \quad \forall x \in \overline{\Omega}$$

is Lipschitz continuous. Hence, there exists $y^0 \in \Omega$ such that

$$\overline{s} = s(y^0) = \max_{x\in\Omega} s(x).$$

Moreover, put

$$\begin{aligned}\sigma' := \frac{|\Omega|c^{p^-}(1-\overline{\mu}^N)}{\overline{\mu}^N} \\ \times \max\Big\{\big[\frac{12(N+2)^2(1+\overline{\mu})}{\overline{s}^2(1-\overline{\mu})^3}\big]^{p^-}, \big[\frac{12(N+2)^2(1+\overline{\mu})}{\overline{s}^2(1-\overline{\mu})^3}\big]^{p^+}\Big\},\end{aligned} \tag{4.3}$$

where $\overline{\mu} \in]0,1[$.

The following is an autonomous version of Theorem 3.1.

Theorem 4.4. *Let $h:\mathbb{R}\to\mathbb{R}$ be a continuous function such that:*

(A1') $H(t)=\int_0^t h(\xi)d\xi \geq 0$ *for every* $t\in[0,+\infty[$.

(A2') *Putting*

$$\eta' := \liminf_{t\to+\infty} \frac{\max_{|\xi|\leq t} H(\xi)}{t^{p^-}}, \quad \theta' := \limsup_{t\to+\infty} \frac{H(t)}{t^{p^+}},$$

one has $\eta' < \frac{p^-}{p^+\sigma'}\theta'$, *where* σ' *is defined in* (4.3).

Then, for each $\lambda \in \frac{1}{c^{p^-}|\Omega|}(\frac{\sigma'}{p^-\theta'}, \frac{1}{p^+\eta'})$ *and for every* $q\in C^0(\mathbb{R})$ *such that*

$$Q(t)=\int_0^t q(\xi)d\xi \geq 0 \quad \text{for every } t\in[0,+\infty[, \tag{4.4}$$

$$Q_\infty := \limsup_{\xi\to+\infty} \frac{\max_{|\xi|\leq t} Q(\xi)}{t^{p^-}} < +\infty, \tag{4.5}$$

if we put $\mu^* := \frac{1}{p^+c^{p^-}|\Omega|Q_\infty}(1-\lambda\eta' p^+ c^{p^-})$, *for every* $\mu\in[0,\mu^*[$ *the problem*

$$\begin{gathered} \Delta^2_{p(x)}u = \lambda h(u)+\mu q(u), \quad x\in\Omega, \\ u=\Delta u = 0, \quad x\in\partial\Omega \end{gathered} \tag{4.6}$$

admits an unbounded sequence of weak solutions.

Proof. Put $x^0=y^0$, $s_2=\overline{s}$, $s_1=\bar{\mu}\bar{s}$, $f(x,t)=h(t)$ and $g(x,t)=q(t)$ for every $(x,t)\in\overline{\Omega}\times\mathbb{R}$. obviously (A1') implies (A1). Moreover,

$$\eta=|\Omega|\eta', \quad \theta=\frac{\pi^{N/2}}{\Gamma(1+N/2)}(\bar{s}\bar{\mu})^N\theta', \quad \sigma=\frac{(\bar{s}\bar{\mu})^N\pi^{N/2}}{|\Omega|\Gamma(1+N/2)}\sigma'.$$

Hence, in view of (A2'), one has

$$\eta < \frac{p^-}{p^+\sigma'}|\Omega|\theta' = \frac{p^-}{p^+\sigma}\theta;$$

that is (A2) holds and the conclusion follows directly from Theorem 3.1 upon observing that $G(x,t)=Q(t)$ for every $(x,t)\in\Omega\times\mathbb{R}$ and $g_\infty=|\Omega|Q_\infty$. □

Corollary 4.5. *Let $h:\mathbb{R}\to\mathbb{R}$ be a continuous and non-negative function such that*

$$\liminf_{t\to+\infty}\frac{H(t)}{t^{p^-}} < \frac{p^-}{p^+\sigma'}\limsup_{t\to+\infty}\frac{H(t)}{t^{p^+}},$$

where σ' *is defined by* (4.3). *Then, for every*

$$\lambda\in\frac{1}{c^{p^-}|\Omega|}\Big(\frac{\sigma'}{p^-\limsup_{t\to+\infty}\frac{H(t)}{t^{p^+}}}, \frac{1}{p^+\liminf_{t\to+\infty}\frac{H(t)}{t^{p^-}}}\Big),$$

for every $q\in C^0(\mathbb{R})$ *such that:*

$$tq(t)\geq 0 \quad \text{for every } t\in\mathbb{R}, \tag{4.7}$$

$$\lim_{|t|\to+\infty}\frac{q(t)}{|t|^{p^--1}}=0 \tag{4.8}$$

and for every $\mu\geq 0$, *problem* (4.6) *admits an unbounded sequence of weak solutions.*

Proof. It follows from Theorem 4.4 on observing that, in view of the non-negativity of h, (A1') holds and $\eta' = \liminf_{t\to+\infty} \frac{H(t)}{t^{p^-}}$, and also (4.7) implies (4.4). Moreover, by (4.7) one has

$$0 \le \limsup_{t\to+\infty} \frac{\max_{|\xi|\le t} Q(\xi)}{t^{p^-}} = \limsup_{t\to+\infty} \frac{\{Q(t), Q(-t)\}}{t^{p^-}}.$$

Exploiting, (2) and owing to the Hôpital rule we have

$$\lim_{t\to+\infty} \frac{Q(t)}{t^{p^-}} = \lim_{t\to+\infty} \frac{Q(-t)}{t^{p^-}} = \pm \lim_{t\to+\infty} \frac{q(\pm t)}{t^{p^- -1}} = 0.$$

Hence $Q_\infty = 0$ and our conclusion follows. $\square$

Now, put

$$\begin{gathered}
\sigma^0 := \frac{2c^{p^+} \pi^{\frac{N}{2}} (r_2^N - r_1^N)}{N\Gamma(\frac{N}{2})} \times \\
\max\Big\{ \big[\frac{12(N+2)^2(r_1+r_2)}{(r_2-r_1)^3}\big]^{p^-}, \big[\frac{12(N+2)^2(r_1+r_2)}{(r_2-r_1)^3}\big]^{p^+} \Big\},
\end{gathered} \tag{4.9}$$

$$\eta^0 := \liminf_{\xi\to 0^+} \frac{\int_\Omega \sup_{|t|\le\xi} F(x,t)\,dx}{\xi^{p^+}},$$

$$\theta^0 := \limsup_{\xi\to 0^+} \frac{\int_{B(x^0,r_1)} F(x,\xi)\,dx}{\xi^{p^-}},$$

$$\lambda_3 := \frac{\sigma^0}{p^- c^{p^+} \theta^0}, \quad \lambda_4 := \frac{1}{p^+ c^{p^+} \eta^0}.$$

Using Lemma 2.1(c) and arguing as in the proof of Theorem 3.1, we can obtain the following result.

Theorem 4.6. *Assume that (A1) holds and*

(A5) $\eta^0 < \frac{p^-}{p^+\sigma}\theta^0$.

Then, for every $\lambda \in (\lambda_3, \lambda_4)$ and for every $g \in C^0(\bar\Omega \times \mathbb{R})$, such that

$$\text{There exists } \tau > 0 \text{ such that } G(x,t) \ge 0 \text{ for every } (x,t) \in \bar\Omega \times [0,\tau], \tag{4.10}$$

$$g_0 := \limsup_{t\to 0^+} \frac{\int_\Omega \max_{|\xi|\le t} G(x,\xi)dx}{t^{p^+}} < +\infty, \tag{4.11}$$

if we put

$$\mu'_{g,\lambda} := \frac{1}{p^+ c^{p^+} g_0} \Big(1 - \lambda p^+ c^{p^+} \eta^0\Big),$$

where $\mu'_{g,\lambda} = +\infty$ when $g_0 = 0$, then for every $\mu \in [0, \mu'_{g,\lambda})$ problem (1.1) has a sequence of weak solutions, which converges strongly to zero in X.

Proof. Fix $\overline{\lambda} \in (\lambda_3, \lambda_4)$ and let g be a function that satisfies the condition (4.11). Since $\overline{\lambda} < \lambda_4$, we obtain

$$\mu'_{g,\overline{\lambda}} := \frac{1}{p^+ c^{p^-} g_0} \big(1 - \overline{\lambda} p^+ c^{p^-} \eta^0\big) > 0.$$

Now fix $\overline{\mu} \in (0, \mu'_{g,\overline{\lambda}})$ and set

$$J(x,t) := F(x,\xi) + \frac{\overline{\mu}}{\overline{\lambda}} G(x,\xi),$$

for all $(x,t)\in\Omega\times\mathbb{R}$. We take Φ,Ψ and $I_{\overline{\lambda}}$ as in the proof of Theorem 3.1. Now, as it has been pointed out before, the functionals Φ and Ψ satisfy the regularity assumptions required in Lemma 2.1. As first step, we will prove that $\overline{\lambda}<1/\delta$. Then, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty}\xi_n=0$ and

$$\lim_{n\to+\infty}\frac{\int_\Omega \sup_{|t|\le\xi_n}F(x,t)\,dx}{\xi_n^{p^+}}=\eta^0.$$

By the fact that $\inf_X\Phi=0$ and the definition of δ, we have $\delta=\liminf_{r\to 0^+}\varphi(r)$. Putting $r_n=\frac{1}{p^+}\big(\frac{\xi_n}{c}\big)^{p^+}$. Then, as in showing (3.4) in the proof of Theorem 3.1, we can prove that $\delta<+\infty$. From $\overline{\mu}\in(0,\mu'_{g,\overline{\lambda}})$, the following inequalities hold

$$\delta\le p^+c^{p^+}\Big(\eta^0+\frac{\overline{\mu}}{\overline{\lambda}}g_0\Big)<p^+c^{p^+}\eta^0+\frac{1-\overline{\lambda}p^+c^{p^+}\eta^0}{\overline{\lambda}}.$$

Therefore,

$$\overline{\lambda}=\frac{1}{p^+c^{p^+}\eta^0+\big(1-\overline{\lambda}p^+c^{p^+}\eta^0\big)/\overline{\lambda}}<\frac{1}{\delta}\,.$$

Let $\overline{\lambda}$ be fixed. We claim that the functional $I_{\overline{\lambda}}$ has not a local minimum at zero. Since

$$\frac{1}{\overline{\overline{\lambda}}}<\frac{p^-c^{p^+}\theta^0}{\sigma^0},$$

there exist a sequence $\{\tau_n\}$ of positive numbers in $]0,\tau[$ and $\zeta>0$ such that $\lim_{n\to+\infty}\tau_n=0^+$ and

$$\frac{1}{\overline{\overline{\lambda}}}<\zeta<\frac{p^-c^{p^+}}{\sigma^0}\frac{\int_{B(x^0,r_1)}F(x,\tau_n)\,dx}{\tau_n^{p^-}}$$

for each $n\in\mathbb{N}$ large enough. Let $\{w_n\}$ be the sequence in X defined in (3.6). From (k_1) and (A_1) one has (3.8) holds. Note that $\overline{\lambda}\zeta>1$. Then, as in showing (3.9), we can obtain

$$I_{\overline{\lambda}}(w_n)<\frac{\tau_n^{p^-}\sigma^0}{p^-c^{p^+}}(1-\overline{\lambda}\zeta)<0=\Phi(0)-\overline{\lambda}\Psi(0)$$

for every $n\in\mathbb{N}$ large enough. Then, we see that zero is not a local minimum of $I_{\overline{\lambda}}$. This, together with the fact that zero is the only global minimum of Φ, we deduce that the energy functional $I_{\overline{\lambda}}$ has not a local minimum at the unique global minimum of Φ. Therefore, by Lemma 2.1(c), there exists a sequence $\{u_n\}$ of critical points of $I_{\overline{\lambda}}$ which converges weakly to zero. In view of the fact that the embedding $X\hookrightarrow C^0(\overline{\Omega})$ is compact, we know that the critical points converge strongly to zero, and the proof is complete. □

Remark 4.7. Under the conditions $\eta^0=0$ and $\theta^0=+\infty$, Theorem 4.6 ensures that for every $\lambda>0$ and for each $\mu\in\big[0,\frac{1}{p^+c^{p^+}g_0}\big)$, problem (1.1) admits a sequence of weak solutions which strongly converges to 0 in X. Moreover, if $g_0=0$, the result holds for every $\lambda>0$ and $\mu\ge 0$.

Remark 4.8. Applying Theorem 4.6, results similar to Theorem 4.1 Corollaries 4.2 and 4.3, can be obtained. We omit the discussions here.

We conclude this article with the following example that illustrates our results.

Example 4.9. Let $\Omega = \{(x,y) \in \mathbb{R}^2; x^2 + y^2 < 3\}$. Then consider the problem

$$\begin{gathered} \Delta^2_{p(x,y)} u = \lambda f(x,y,u) + \mu g(x,y,u), \quad x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{4.12}$$

where $p(x,y) = x^2 + y^2 + 3$ for all $(x,y) \in \Omega$,

$$f(x,y,t) = \begin{cases} f^*(x,y)t^6\big(7 + \sin(\ln(|t|)) - 7\cos(\ln(|t|))\big), & (x,y,t) \in \Omega \times (\mathbb{R} - \{0\}), \\ 0, & (x,y,t) \in \Omega \times \{0\}, \end{cases}$$

where $f^* : \Omega \to \mathbb{R}$ is a non-negative continuous function, and

$$g(x,y,t) = e^{x+y-t^+}(t^+)^{\varsigma-1}(\varsigma - t^+)$$

for all $(x,y) \in \Omega$ and $t \in \mathbb{R}$, where $t^+ = \max\{t,0\}$ and ς is a positive real number. It is obvious that $p^- = 3$ and $p^+ = 6$. A direct calculation shows that

$$F(x,y,t) = \begin{cases} f^*(x,y)t^7\big(1 - \cos(\ln(|t|))\big), & (x,y,t) \in \Omega \times (\mathbb{R} - \{0\}), \\ 0, & (x,y,t) \in \Omega \times \{0\}. \end{cases}$$

So,

$$\begin{gathered} \liminf_{\xi \to +\infty} \frac{\int_\Omega \max_{|t| \le \xi} F(x,y,t)\, d\sigma}{\xi^3} = 0, \\ \limsup_{\xi \to +\infty} \frac{\int_{B((0,0),1)} F(x,y,\xi)\, dx}{\xi^6} = +\infty. \end{gathered}$$

Hence, using Theorem 3.1, since $g_\infty = 0$, the problem (4.12) for every $(\lambda,\mu) \in]0,+\infty[\times[0,+\infty[$ admits infinitely many weak solutions in X.

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