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BLOW-UP FOR THE EULER-BERNOULLI VISCOELASTIC EQUATION WITH A NONLINEAR SOURCE

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Abstract. In this article, we consider the Euler-Bernoulli viscoelastic equation

$$u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^t g(t-s)\Delta^2 u(x,s)ds = |u|^{p-1}u$$

together with some suitable initial data and boundary conditions in $\Omega \times (0,+\infty)$. Some sufficient conditions on blow-up of solutions are obtained under different initial energy states. And from these results we can clearly understand the competitive relationship between the viscoelastic damping and source.

1. Introduction

The Euler-Bernoulli equation

$$u_{tt}(x,t) + \Delta^2 u(x,t) + h(u_t) = f(u), \quad (x,t) \in \mathbb{R}^n \times (0,+\infty), \tag{1.1}$$

describes the deflection $u(x,t)$ of a beam (when $n=1$) or a plate (when $n=2$). Where

$$\Delta^2 u := \Delta(\Delta u) = \sum_{j=1}^n \Big(\sum_{i=1}^n u_{x_i x_i}\Big)_{x_j x_j},$$

h and f represent the friction damping and the source respectively. The blow-up properties of this model have been extensively studied. For example, Messaoudi [13] studied the equation

$$u_{tt}(x,t) + \Delta^2 u(x,t) + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \tag{1.2}$$

where $a, b > 0$ and $p, m > 2$ and proved that the solution blows up in finite time with negative initial energy when $m < p$. Later, this result was improved to the case of positive initial energy by Chen and Zhou[5]. All these results reflect a competition between the source and the friction damping.

When we take the viscoelastic materials into consideration, the model (1.1) becomes

$$u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^t g(t-s)\Delta^2 u(x,s)ds + h(u_t) = f(u), \tag{1.3}$$

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where g is so-called viscoelastic kernel. The term $\int_0^t g(t-s)\Delta^2 u(x,s)ds$ describes the hereditary properties of the viscoelastic materials[6]. It expresses the fact that the stress at any instant t depends on the past history of strains which the material has undergone from time 0 up to t. When $h \equiv 0$ and $f \equiv 0$, Tatar [16] obtained the property of the energy decay of the model (1.3). And from this, we know that the term $\int_0^t g(t-s)\Delta^2 u(x,s)ds$, similar to the friction damping, can cause the inhibition of the energy.

When various damping, such as friction damping, strong damping and viscoelastic damping, come together, some blow-up results are obtained. For example, Tahamtani and Peyravi [18] considered the equation

$$u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^t g(t-s)\Delta^2 u(x,s)ds - \Delta u_t - \Delta u_{tt} + |u_t|^{m-1}u_t = |u|^{p-1}u, \tag{1.4}$$

together with some initial-boundary conditions and proved that the L^{p+1} norm of any solution grows as an exponential function if $m < p$ and the initial energy is negative. Very recently, Gang Li et al.[11] studied the asymptotic behavior and blow-up properties of solutions of (1.4) in the case where the positive initial energy has an upper bound. From their result, we can see the competition mechanism between source and all the dampings. For the better comprehension of our motivation, we point out that the system (1.4) has weak damping $|u_t|^{m-1}u_t$, strong damping Δu_t and viscoelastic damping $\int_0^t g(t-s)\Delta^2 u(x,s)ds$ at the same time. But, how much influence on blow-up the each damping has? This question did not seem to be answered in the literature. That is, we can not know the specific effect of each damping among this competition mechanism. And this is the motivation of our present work. More precisely, we will discuss the initial-boundary value problem

$$\begin{gathered} u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^t g(t-s)\Delta^2 u(x,s)ds = |u|^{p-1}u, \\ (x,t) \in \Omega \times (0,T), \\ u(x,t) = \frac{\partial u(x,t)}{\partial \nu} = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{gathered} \tag{1.5}$$

where $\Omega \in \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. ν is the unit outward normal vector on $\partial\Omega$. And u_0, u_1 are given initial data belonging to suitable spaces. We try to discuss the influence of the competition between viscoelastic damping and source on the blow-up of solutions. And we hope that we have a more in-depth understanding of the interaction mechanism between the source and the viscoelastic damping. Compared with the relevant literature, this is a distinctive feature of the present paper.

We will discuss this problem in three cases: $E(0) < 0$, $0 < E(0) < E_0$ and $E(0) > 0$. When $E(0) < 0$ and $0 < E(0) < E_0$, our approach is the same as in [4, 14, 15], in which the authors studied the equation

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds + |u_t|^{m-1}u_t = |u|^{p-1}u \tag{1.6}$$

and obtained some blow-up results. But, in the case of $E(0) > 0$, we here solve (1.5) by introducing the so-called positive type function.

2. Preliminaries and statement of main results

Throughout this article, C denotes a generic positive constant. It may be different from line to line. And we use the standard Lebesgue space $L^p(\Omega)$ with their usual norms $\|\cdot\|_p$. We first state the general assumptions on g and p as follows:

(A1) $g \in C^1([0,\infty))$ is a non-negative and non-increasing function satisfying

$$0 < k := \int_0^\infty g(s)ds < 1. \tag{2.1}$$

(A2) The function $e^{\frac{t}{2}}g(t)$ is of positive type in the following sense:

$$\int_0^t v(s)\int_0^s e^{\frac{s-\tau}{2}} g(s-\tau)v(\tau)d\tau ds \geq 0, \quad \forall v \in C^1([0,\infty)),\ \forall t>0.$$

(A3) If the space dimension $n = 1,2,3,4$, then $1 < p < \infty$; If $n \geq 5$, then $1 < p \leq \frac{n}{n-4}$.

Remark 2.1. Assumption (A1) was used in [4, 14, 15] and it points out that the kernel function g must be "small". For the definition of positive type function in detail, we refer readers to [8]. A typical example of such function is

$$g(t) = \varepsilon e^{-t}, \quad 0 < \varepsilon < 1.$$

Moreover, since $e^{\frac{t}{2}}g(t)$ is of positive type, we have

$$\begin{aligned}
&\int_0^t e^s \int_0^s g(s-\tau)\int_\Omega \Delta u(x,s)\Delta u(x,\tau)\,dx\,d\tau\,ds \\
&= \int_\Omega\int_0^t \Big(e^{s/2}\Delta u(x,s)\Big)\int_0^s \Big(e^{\frac{s-\tau}{2}}g(s-\tau)\Big)\Big(e^{\tau/2}\Delta u(x,\tau)\Big)\,d\tau\,ds\,dx \geq 0.
\end{aligned}$$

Thus, we deduce that

$$\int_\Omega\int_0^t g(t-s)\Delta u(t)\Delta u(s)\,ds\,dx \geq 0.$$

To simplify notation, we set

$$(\phi\circ\psi)(t) := \int_0^t \phi(t-s)\int_\Omega |\psi(t)-\psi(s)|^2\,dx\,ds,$$

where ψ may be a scalar, or a vector valued function. A direct computation shows that, for any $g \in C^1(\mathbf{R})$ and $u \in H^2(0,T,L^2(\Omega))$, the following equality holds:

$$\begin{aligned}
&\int_0^t g(t-s)\,(\Delta u(s),\Delta u_t(t))\,ds \\
&= \frac{1}{2}(g'\circ\Delta u)(t) - \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 \\
&\quad - \frac{1}{2}\frac{d}{dt}\Big\{(g\circ\Delta u)(t) - \Big(\int_0^t g(s)ds\Big)\|\Delta u(t)\|_2^2\Big\}.
\end{aligned} \tag{2.2}$$

Now, we state the existence of a local solution which can be established by adopting the arguments in [13, 1].

Theorem 2.2. *Assume that* (A1) *and* (A3) *hold. Let $u_0 \in H_0^2(\Omega)$ and $u_1 \in H_0^1(\Omega)$ be given. Then, there exists a unique weak solution $u(t)$ of* (1.5) *such that*

$$u \in C([0,T];H_0^2(\Omega)) \cap C^1([0,T];L^2(\Omega)), \quad u_t \in L^2([0,T];H_0^1(\Omega)). \tag{2.3}$$

for $T > 0$ small enough.

Now, we define the following two functionals:

$$I(t) := I(u(t)) = \|\Delta u\|_2^2 - \|u\|_{p+1}^{p+1}, \tag{2.4}$$

$$\begin{aligned} E(t) := E(u(t)) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\Big(1 - \int_0^t g(s)ds\Big)\|\Delta u\|_2^2 \\ + \frac{1}{2}(g \circ \Delta u)(t) - \frac{1}{p+1}\|u\|_{p+1}^{p+1}. \end{aligned} \tag{2.5}$$

By (2.2) and the assumption (A1), some direct computations yield

$$E'(t) = \frac{1}{2}(g' \circ \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u\|_2^2 \le 0. \tag{2.6}$$

The following four lemmas are necessary to prove our main results. The first one is the Sobolev-Poincaré inequality [2, Chapter 4].

Lemma 2.3. *Let q be a number with $2 \le q < \infty (n = 1, 2, 3, 4)$ or $2 \le q \le 2n/(n-4) (n \ge 5)$, then for $u \in H_0^2(\Omega)$ there exists a positive $C_* = C(\Omega, q)$ such that $\|u\|_q \le C_*\|\Delta u\|_2$.*

Lemma 2.4. . *Assume that* (A1) *holds. If $\|u_0\|_{p+1} > \lambda_0 \equiv B_0^{\frac{-2}{p-1}}$ and $E(0) < E_0 = (\frac{1}{2} - \frac{1}{p+1})B_0^{\frac{-2(p+1)}{p-1}}$, then we have $\|u\|_{p+1} > \lambda_0$ and $\|\Delta u\|_2 > B_0^{\frac{-(p+1)}{p-1}}$ for all $t \ge 0$, where $B_0 = \frac{B}{\sqrt{1-k}}$ for $\|u\|_{p+1} \le B\|\Delta u\|_2$.*

Proof. By (2.5), we have

$$\begin{aligned} E(t) &\ge \frac{1}{2}\Big(1 - \int_0^t g(s)ds\Big)\|\Delta u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &\ge \frac{1-k}{2}\|\Delta u\|_{p+1}^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &\ge \frac{1}{2B_0^2}\|u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}. \end{aligned} \tag{2.7}$$

Denote

$$f(x) = \frac{1}{2B_0^2}x^2 - \frac{1}{p+1}x^{p+1}, \quad x \ge 0.$$

Then, we easily deduce that $f(x)$ takes its maximum value E_0 at λ_0. Since $E_0 > E(0) \ge e(t) \ge f(\|u\|_{p+1})$ for all $t \ge 0$, there is no time t^* such that $\|u(t^*)\|_{p+1} = \lambda_0$. By the continuity of the $\|u(t)\|_{p+1}$-norm with respect to the time variable, one has $\|u(t)\|_{p+1} > \lambda_0$ for all $t \ge 0$, and consequently,

$$\|\Delta u\|_2 \ge \frac{1}{\sqrt{1-k}B_0}\|u\|_{p+1} > \frac{1}{\sqrt{1-k}}B_0^{\frac{-(p+1)}{p-1}} > B_0^{\frac{-(p+1)}{p-1}}.$$

This completes the proof. □

Lemma 2.5. *Assume that $g(t)$ satisfies* (A1) *and* (A2)*, and $u(t)$ is the corresponding solution of the problem* (1.5)*. Moreover, the function $\Phi(t)$ is twice continuously differentiable, satisfying*

$$\begin{gathered} \Phi''(t) + \Phi'(t) > \int_0^t g(t-s)\int_\Omega \Delta u(x,s)\Delta u(x,t)\,dx\,ds \\ \Phi(0) > 0, \quad \Phi'(0) > 0, \end{gathered} \tag{2.8}$$

for every $t \in [0, T_0)$. Then $\Phi(t)$ is strictly increasing on $[0, T_0)$.

Proof. We first consider the auxiliary ODE

$$\begin{gathered}\varphi''(t) + \varphi'(t) = \int_0^t g(t-s) \int_\Omega \Delta u(x,s)\Delta u(x,t)\,dx\,ds \\ \varphi(0) = \Phi(0), \quad \varphi'(0) = 0\end{gathered} \tag{2.9}$$

for every $t \in [0, T_0)$. The solution of the problem (2.9) is

$$\varphi(t) = \varphi(0) + \int_0^t \frac{e^{-s} - e^{-t}}{e^{-s}} \int_0^s g(s-\tau) \int_\Omega \Delta u(x,s)\Delta u(x,\tau)\,dx\,d\tau ds, \tag{2.10}$$

$t \in [0, T_0)$. Then, by (A2) we obtain

$$\begin{aligned}&\varphi'(t) \\ &= \int_0^t e^{s-t} \int_0^s g(s-\tau) \int_\Omega \Delta u(x,s)\Delta u(x,\tau)\,dx\,d\tau ds \\ &= e^{-t} \int_\Omega \int_0^t \big(e^{s/2}\Delta u(x,s)\big) \int_0^s \Big(e^{\frac{s-\tau}{2}} g(s-\tau)\Big) \Big(e^{\tau/2}\Delta u(x,\tau)\Big)\,d\tau\,ds\,dx \\ &\geq 0, \quad \forall t \in [0, T_0).\end{aligned} \tag{2.11}$$

Therefore, we have $\varphi(t) \geq \varphi(0) = \Phi(0)$.

Note that $\Phi'(0) > \varphi'(0)$. We next show that

$$\Phi'(t) > \varphi'(t), \quad \forall t \geq 0. \tag{2.12}$$

Assume that (2.12) is not valid. This implies that there exists $t_0 > 0$ satisfying

$$t_0 = \min\{t \geq 0 : \Phi'(t) = \varphi'(t)\}. \tag{2.13}$$

Then we have the problem

$$\begin{gathered}\Phi''(t) - \varphi''(t) + \Phi'(t) - \varphi'(t) > 0 \\ \Phi(0) - \varphi(0) = 0, \quad \Phi'(0) - \varphi'(0) > 0\end{gathered} \tag{2.14}$$

for every $t \in [0, T_0)$. This problem can be solved as

$$\Phi'(t_0) - \varphi'(t_0) > e^{-t_0}(\Phi'(0) - \varphi'(0)) > 0$$

which contradicts (2.13). Thus, we see that $\Phi'(t) > 0$, which implies our desired result. □

Lemma 2.6. *Assume that $(u_0, u_1) \in (H_0^2(\Omega) \times H_0^1(\Omega))$ satisfies $\int_\Omega u_0(x)u_1(x)dx \geq 0$. If the solution $u(t)$ of (1.5) exists on $[0, T)$ and satisfies $I(u) < 0$, then $\|u\|_2^2$ is strictly increasing on $[0, T)$.*

Proof. A direct computation yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d^2}{dt^2}\|u(t)\|_2^2\\
&=\int_\Omega \left(|u_t(t)|^2+uu_{tt}\right)\,dx\\
&=\|u_t\|_2^2+\|u\|_{p+1}^{p+1}-\|\Delta u\|_2^2+\int_\Omega \Delta u(t)\int_0^t g(t-s)\Delta u(s)\,ds\,dx \qquad (2.15)\\
&=\|u_t\|_2^2-I(u(t))+\int_\Omega \Delta u(t)\int_0^t g(t-s)\Delta u(s)\,ds\,dx\\
&>\int_\Omega \Delta u(t)\int_0^t g(t-s)\Delta u(s)\,ds\,dx\ge 0,
\end{aligned}
$$

where the last inequality comes from Remark 2.1. This implies

$$\frac{d}{dt}\|u(t)\|_2^2>2\int_\Omega u_0(x)u_1(x)dx\ge 0. \tag{2.16}$$

Thus, we obtain

$$\frac{d^2}{dt^2}\|u(t)\|_2^2+\frac{d}{dt}\|u(t)\|_2^2>\int_\Omega \Delta u(t)\int_0^t g(t-s)\Delta u(s)\,ds\,dx. \tag{2.17}$$

Therefore, this lemma follows from Lemma 2.5. □

Our main results read as follows.

Theorem 2.7. *Assume that* (A1) *and* (A3) *hold. If* $k<\frac{p^2-1}{p^2}$*, then, for any initial data* $u_0\in H_0^2(\Omega)$ *and* $u_1\in H_0^1(\Omega)$ *satisfying* $E(0)<0$*, the corresponding solution of the problem* (1.5) *blows up in finite time.*

Theorem 2.8. *Assume that* (A1) *and* (A3) *hold. If* $k<\frac{p^2-1}{p^2}$*, then, for any initial data* $u_0\in H_0^2(\Omega)$ *and* $u_1\in H_0^1(\Omega)$ *satisfying* $\|u_0\|_{p+1}>\lambda_0$ *and* $E(0)<E_0$*, the corresponding solution of* (1.5) *blows up in finite time. (Here,* $\lambda_0,E_0>0$*)*

Theorem 2.9. *Assume that* (A1)–(A3) *hold. If* $k<\frac{p-1}{p+1}$*, then, for any initial data* $u_0\in H_0^2(\Omega)$ *and* $u_1\in H_0^1(\Omega)$ *satisfying* $E(0)>0$*,* $\int_\Omega u_0u_1dx>0$*,* $I(u_0)<0$ *and*

$$\|u_0\|_2^2>\frac{4(p+1)C_*^2}{(p-1)-(p+1)k}E(0), \tag{2.18}$$

where C_* *is the constant of Poincare inequality on* Ω*, the corresponding solution of the problem* (1.5) *blows up in finite time.*

Remark 2.10. From condition (2.18), when the initial energy is in the high state, the solution of (1.5) also blows up in finite time if we can ensure that the initial value u_0 satisfy (2.18). In other words, (2.18) is a restrictive condition to u_0, not to $E(0)$.

3. Proof of main results

The method of the proof of Theorem 2.7 and Theorem 2.8 is standard. And the idea comes from H. A. Levine etc.. For the convenience of readers, we here write out the process of the proof in detail. And the improved convexity method will be used to prove Theorem 2.9. Readers can refer to the relevant literatures [3, 7, 9, 10, 12, 17, 19, 20] and the references therein.

3.1. **Proof of Theorem 2.7.** Let

$$L(t) = H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t dx. \tag{3.1}$$

Here, $H(t) = -E(t)$, $0 < \alpha \le \frac{p-1}{2(p+1)}$ and $\varepsilon > 0$ to be choose later. First, by (2.6), we easily obtain $H'(t) \ge 0$ and $0 < H(0) \le H(t) \le \frac{1}{p+1}\|u\|_{p+1}^{p+1}$. Second, by differentiating the equality (3.1) and applying (1.5) and (2.5), we have

$$\begin{aligned}
L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega |u_t|^2 dx + \varepsilon\int_\Omega u u_{tt} dx \\
&\ge \varepsilon\int_\Omega |u_t|^2 dx + \varepsilon\int_\Omega u u_{tt} dx \\
&= \varepsilon\|u_t\|_2^2 + \varepsilon\Big(- \|\Delta u\|_2^2 + \int_\Omega \Delta u(t)\int_0^t g(t-s)\Delta u(s)\,ds\,dx + \|u\|_{p+1}^{p+1}\Big) \\
&= \varepsilon\|u_t\|_2^2 - \varepsilon\|\Delta u\|_2^2 + \varepsilon\int_\Omega \Delta u(t)\int_0^t g(t-s)\Delta u(s)ds - \varepsilon(p+1)E(t) \\
&\quad + \frac{\varepsilon(p+1)}{2}\Big[\|u_t\|_2^2 + \Big(1 - \int_0^t g(s)ds\Big)\|\Delta u\|_2^2 + (g\circ\Delta u)(t)\Big].
\end{aligned} \tag{3.2}$$

In addition, by Young and Schwarz inequalities, we have

$$\begin{aligned}
&\int_\Omega \Delta u(t)\int_0^t g(t-s)\Delta u(s)\,ds\,dx \\
&= -\int_\Omega \Delta u(t)\int_0^t g(t-s)\left(\Delta u(t) - \Delta u(s)\right)\,ds\,dx + \Big(\int_0^t g(s)ds\Big)\|\Delta u\|_2^2 \\
&\ge \Big(\int_0^t g(s)ds\Big)\|\Delta u\|_2^2 - \int_0^t g(t-s)\|\Delta u(t)\|_2\|\Delta u(s) - \Delta u(t)\|_2 ds \\
&\ge \Big(\int_0^t g(s)ds\Big)\|\Delta u\|_2^2 - \frac{\int_0^t g(s)ds}{4\delta}\|\Delta u\|_2^2 - \delta(g\circ\Delta u)(t), \quad \forall\delta > 0
\end{aligned} \tag{3.3}$$

Thus, by (3.2) and (3.3), we obtain

$$\begin{aligned}
L'(t) &\ge \varepsilon\frac{p+3}{2}\|u_t\|_2^2 + \varepsilon(p+1)H(t) + \varepsilon\left(\frac{p+1}{2} - \delta\right)(g\circ\Delta u)(t) \\
&\quad + \varepsilon\Big[\frac{p-1}{2} - \big(\frac{p-1}{2} + \frac{1}{4\delta}\big)\int_0^t g(s)ds\Big]\|\Delta u\|_2^2.
\end{aligned} \tag{3.4}$$

Now, choosing $0 < \delta < \frac{p+1}{2}$ and according to the hypothesis $k < \frac{p^2-1}{p^2}$, we have $\frac{p+1}{2} - \delta > 0$ and $\frac{p-1}{2} - \big(\frac{p-1}{2} + \frac{1}{4\delta}\big)\int_0^t g(s)ds > 0$. So we can deduce that

$$L'(t) \ge C\left(H(t) + \|u_t\|_2^2 + \|\Delta u\|_2^2 + (g\circ\Delta u)(t)\right). \tag{3.5}$$

Next, thanks to Hölder and Young inequalities, we have

$$\begin{aligned} \Big|\int_\Omega uu_t dx\Big|^{\frac{1}{1-\alpha}} &\le \|u\|_2^{\frac{1}{1-\alpha}}\|u_t\|_2^{\frac{1}{1-\alpha}} \\ &\le C\|u\|_{p+1}^{\frac{1}{1-\alpha}}\|u_t\|_2^{\frac{1}{1-\alpha}} \\ &\le C\left(\|u\|_{p+1}^s+\|u_t\|_2^2\right) \\ &\le C\left(\|\Delta u\|_2^2+\|u\|_{p+1}^{p+1}+\|u_t\|_2^2\right) \\ &\le C\left(\|\Delta u\|_2^2+H(t)+(g\circ\Delta u)(t)+\|u_t\|_2^2\right), \end{aligned} \tag{3.6}$$

where $2 \le s = \frac{2}{1-2\alpha} \le p+1$. Thus, we obtain

$$\begin{aligned} L^{\frac{1}{1-\alpha}}(t) &= \Big(H^{1-\alpha}(t)+\varepsilon\int_\Omega uu_t dx\Big)^{\frac{1}{1-\alpha}} \\ &\le 2^{\frac{1}{1-\alpha}}\Big(H(t)+\Big|\int_\Omega uu_t dx\Big|^{\frac{1}{1-\alpha}}\Big) \\ &\le C\big(\|\Delta u\|_2^2+H(t)+(g\circ\Delta u)(t)+\|u_t\|_2^2\big), \end{aligned} \tag{3.7}$$

which implies

$$L'(t) \ge \lambda L^{\frac{1}{1-\alpha}}(t),$$

where λ is a positive constant depending on C and ε. Therefore

$$L(t) = \Big(L^{\frac{\alpha}{\alpha-1}}(0)+\frac{\alpha}{\alpha-1}\lambda t\Big)^{\frac{\alpha-1}{\alpha}}.$$

So $L(t)$ approaches infinite as t tends to $\frac{1-\alpha}{\alpha\lambda L^{\frac{\alpha}{1-\alpha}}(0)}$. And this completes the proof.

3.2. **Proof of Theorem 2.8.** Let $G(t) = E_0 + H(t)$, then we have $G'(t) \ge 0$. By Lemma 2.4 we have

$$\begin{aligned} 0 < G(t) &= \Big(\frac{1}{2}-\frac{1}{p+1}\Big)B_0^{\frac{-2(p+1)}{p-1}}+H(t) \\ &< \Big(\frac{1}{2}-\frac{1}{p+1}\Big)\|\Delta u\|_2^2+H(t) \\ &< C\big(\|\Delta u\|_2^2+H(t)\big). \end{aligned} \tag{3.8}$$

Now, we set

$$F(t) = G^{1-\alpha}(t)+\varepsilon\int_\Omega uu_t dx. \tag{3.9}$$

Then, similar to (3.2), we have

$$\begin{aligned} F'(t) &= (1-\alpha)G^{-\alpha}(t)G'(t)+\varepsilon\int_\Omega |u_t|^2 dx+\varepsilon\int_\Omega uu_{tt}dx \\ &\ge \varepsilon\int_\Omega |u_t|^2 dx+\varepsilon\int_\Omega uu_{tt}dx \\ &\ge \varepsilon\frac{p+3}{2}\|u_t\|_2^2+\varepsilon(p+1)H(t)+\varepsilon\Big(\frac{p+1}{2}-\delta\Big)(g\circ\Delta u)(t) \\ &\quad +\varepsilon\Big[\frac{p-1}{2}-\Big(\frac{p-1}{2}+\frac{1}{4\delta}\Big)\int_0^t g(s)ds\Big]\|\Delta u\|_2^2 \quad \forall\delta>0. \end{aligned} \tag{3.10}$$

Next, similar to the proof of Theorem 2.7, we easily deduce that

$$F'(t) \geq \lambda F^{\frac{1}{1-\alpha}}(t)$$

which shows that $F(t)$ blows up in time $T^* \leq \frac{1-\alpha}{\alpha\lambda F^{\frac{\alpha}{1-\alpha}}(0)}$.

3.3. **Proof of Theorem 2.9.** We first claim that

$$I(u(t)) < 0, \tag{3.11}$$

$$\|u(t,\cdot)\|_2^2 > \frac{4(p+1)C_*^2}{(p-1)-(p+1)k} E(0) \tag{3.12}$$

for every $t \in [0,T)$.

In fact, if (3.11) does not hold, then there exists a time t_1 such that

$$t_1 = \min\{t \in (0,T) : I(u(t)) = 0\} > 0. \tag{3.13}$$

By the continuity of the solution $u(x,t)$ as a function of t, we deduce that $I(u(t)) < 0$, for all $t \in [0,t_1)$ and $I(u(t_1)) = 0$. Thus, by Lemma 2.6 we obtain

$$\|u(t,\cdot)\|_2^2 > \|u_0\|_2^2 > \frac{4(p+1)C_*^2}{(p-1)-(p+1)k} E(0), \quad \forall t \in [0,t_1).$$

In addition, it is obvious that $\|u(t,\cdot)\|_2^2$ is continuous on $[0,t_1]$, which implies that

$$\|u(t_1,\cdot)\|_2^2 > \frac{4(p+1)C_*^2}{(p-1)-(p+1)k} E(0). \tag{3.14}$$

On the other hand, it follows from (2.5), (2.6) and (3.13) that

$$\begin{aligned} &\Big(\frac{1-k}{2} - \frac{1}{p+1}\Big)\|\Delta u(t_1,\cdot)\|_2^2 \\ &< \frac{1}{2}\Big(1 - \int_0^{t_1} g(s)ds\Big)\|\Delta u(t_1,\cdot)\|_2^2 - \frac{1}{p+1}\|u(t_1,\cdot)\|_{p+1}^{p+1} \leq E(0). \end{aligned} \tag{3.15}$$

Thus, by Lemma 2.3 and the hypothesis $k < \frac{p-1}{p+1}$, we deduce that

$$\|u(t_1,\cdot)\|_2^2 \leq \frac{2(p+1)C_*^2}{(p-1)-(p+1)k} E(0). \tag{3.16}$$

Obviously, there is a contradiction between (3.14) and (3.16). Thus, we have proved that (3.11) is true for every $t \in [0,T)$. Furthermore, by Lemma 2.6 we see that (3.12) is also valid on $[0,T)$.

Now, we prove that the solution of (1.5) blows up in a finite time. To this end, we define the following auxiliary function

$$M(t) = \|u(t,\cdot)\|_2^2 + \int_0^t \|u(s,\cdot)\|_2^2 ds + (b-t)\|u_0\|_2^2 + a(c+t)^2, \tag{3.17}$$

where a, b and c are positive constants which will be determined in the sequel.

Direct computation yields

$$M'(t) = 2(u(t),u_t(t)) + 2\int_0^t (u(s),u_s(s))ds + 2a(c+t), \tag{3.18}$$

$$M''(t) = 2\|u_t\|_2^2 + 2(u,u_{tt}) + 2(u,u_t) + 2a. \tag{3.19}$$

By (2.16) and (1.5), we see that

$$
\begin{aligned}
&M''(t)\\
&\geq 2\|u_t\|_2^2 + 2(u, u_{tt}) + 2a\\
&= 2\|u_t\|_2^2 + 2(u, -\Delta^2 u + \int_0^t g(t-s)\Delta^2 u(s)ds + |u|^{p-1}u) + 2a\\
&= 2\|u_t\|_2^2 + 2\|u\|_{p+1}^{p+1} - 2\|\Delta u\|_2^2 + 2a + 2\int_0^t g(t-s)\int_\Omega |\Delta u(t)|^2\,dx\,ds\\
&\quad + 2\int_0^t g(t-s)\int_\Omega \Delta u(t)(\Delta u(s) - \Delta u(t))ds.
\end{aligned}
\tag{3.20}
$$

Next, by using Young's inequality to estimate the last term in (3.20), we have

$$
\int_0^t g(t-s)\int_\Omega \Delta u(t)(\Delta u(s) - \Delta u(t))ds \leq \frac{\int_0^t g(s)ds}{2\delta}\|\Delta u\|_2^2 + \frac{\delta}{2}(g\circ\Delta u)(t), \tag{3.21}
$$

for all $\delta > 0$. Now, we pick $\delta = \frac{2k}{(p-1)(1-k)}$. Since $k < \frac{p-1}{p+1} < \frac{p^2-1}{p^2+1}$, we have $p+1-\delta > 0$. And we easily obtain

$$
\begin{aligned}
&M''(t)\\
&\geq (p+3)\|u_t\|_2^2 + \Big[(p-1)\Big(1 - \int_0^t g(s)ds\Big) - \frac{1}{\delta}\int_0^t g(s)ds\Big]\|\Delta u\|_2^2\\
&\quad + (p+1-\delta)(g\circ\Delta u)(t) - 2(p+1)E(t) + 2a\\
&\geq (p+3)\|u_t\|_2^2 + \Big[(p-1)\,(1-k) - \frac{k}{\delta}\Big]\|\Delta u\|_2^2 + (p+1-\delta)(g\circ\Delta u)(t)\\
&\quad - 2(p+1)E(0) + 2(p+1)\int_0^t \|u_s(s,\cdot)\|_2^2 ds + 2a\\
&\geq (p+3)\|u_t\|_2^2 + 2(p+1)\int_0^t \|u_s(s,\cdot)\|_2^2 ds + 2a\\
&\quad + \frac{(p-1)(1-k)}{2C_*^2}\|u_0\|_2^2 - 2(p+1)E(0),
\end{aligned}
\tag{3.22}
$$

where the last inequality follows from Lemma 2.3 and Lemma 2.6. Noting the condition (2.18) and $k < \frac{p-1}{p+1}$, we see that

$$
\frac{(p-1)(1-k)}{2C_*^2}\|u_0\|_2^2 - 2(p+1)E(0) > 0.
$$

Thus, we have $M''(t) > 0$ for every $t \in (0,T)$. Then, by $M'(0) > 0$, we see that $M(t)$ and $M'(t)$ are strictly increasing on $[0,T)$.

Next, we select the positive constants a, b and c such that

$$
\begin{gathered}
(p+1)a < \frac{(p-1)(1-k)}{2C_*^2}\|u_0\|_2^2 - 2(p+1)E(0),\\
b \geq \frac{p-1}{4}\frac{M(0)}{M'(0)},\\
\frac{p-1}{2}\Big(\int_\Omega u_0 u_1 dx + ac\Big) \geq \|u_0\|_2^2.
\end{gathered}
$$

In addition, we denote

$$P := \|u(t,\cdot)\|_2^2 + \int_0^t \|u(s,\cdot)\|_2^2 ds + a(c+t)^2,$$

$$Q := \frac{M'(t)}{2}, \quad R := \|u_t(t,\cdot)\|_2^2 + \int_0^t \|u_s(s,\cdot)\|_2^2 ds + a.$$

Thus, we have $M(t) \geq P$ and $M''(t) \geq (p+3)R$ for every $t \in [0,b)$. It follows that

$$M''(t)M'(t) - \frac{p+3}{4}(M'(t))^2 \geq (p+3)(PR - Q^2) \geq 0, \tag{3.23}$$

where the last inequality comes from $P\theta^2 - 2Q\theta + R \geq 0$ for every $\theta \in \mathbf{R}$.

Now, we pick $\beta = \frac{p-1}{4} > 0$. Direct computation yields

$$(M^{-\beta})' = -\beta M^{-\beta-1}M'(t) < 0,$$

$$(M^{-\beta})'' = -\beta M^{-\beta-2}\Big(M''(t)M'(t) - \frac{p+3}{4}(M'(t))^2\Big) < 0.$$

This means that $M^{-\beta}$ is concave. Noting that $M(0) > 0$, we see that the function $M^{-\beta} \to 0$ as $t \to T^{*-}$ and $T^* < \frac{(p-1)M(0)}{4M'(0)}$. Therefore, there exists a finite time $T^* > 0$ such that

$$\lim_{t\to T^{*-}} \|u(t,\cdot)\|_2^2 \to \infty.$$

Thus, the proof of Theorem 2.9 is complete.

Remark 3.1. A comparison of Theorems 2.7, 2.8 and 2.9 indicates that we must reduce the effect of viscoelastic damping, or increase the source, or both of them to ensure that the solution of the concerned system blows up in finite time when the initial energy state transitions from low to high.

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References

[1] N. E. Amroun, A. Benaissa; Global existence and energy decay of solutions to a Petrovsky equation with general nonlinear dissipation and source term, *Georgian Math. J.*, 2006, **13 (3)**: 397-410.

[2] R. A. Adams, J. F. John; *Sobolev Spaces*, 2nd edition. Elsevier, Singapore, 2009.

[3] J. M. Ball; Finite time blow-up in nonlinear problem, in: *M.G. Grandall(Ed.), Nonlinear Evolution equations, Academic Press*, New York, 1978:189-205.

[4] M. M. Cavalcanti, V. N. Domingos, J. A. Soriano; Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping, *Electron. J. Diff. Eq.*, **2002 (2002)**: 1-14.

[5] W. Chen, Y. Zhou; Global nonexistence for a semilinear Petrovsky equation, *Nonlinear Analysis:TMA*, 2009, **70(9)**: 3203-3208.

[6] M. Fabrizio, A. Morro; *Mathematical problems in linear viscoelasticity*, SIAM Stud. Appl. Math., 1992, **12(1)**.

[7] V. Georgiev, G. Todorova; Existence of a solution of the wave equation with nonlinear damping and source terms, *J. Diff. Eq.*, 1994, **109**: 295-308.

[8] K. B. Hannsgen; Indirect abelian theorems and a linear Volterra equation, *Trans. Amer. Math. Soc.*, 1969, **142**: 539-555.

[9] H. A. Levine; Instability and nonexistence of global solution of nonlinear wave equations of the form $Pu_t = -Au + F(u)$, *Trans. Amer. Math. Soc.*, 1974, **192:**1-21.

[10] H. A. Levine; Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, *SIAM. J. Math. Anal.*, 1974, **5**: 138-146.
[11] G. Li, Y. Sun, W. Liu; On asymptotic behavior and blow-up of solutions for a nonlinear viscoelastic Petrovsky equation with positive initial energy, *J. Function Spaces and Applications*, 2013, Article ID 905867, 7 pages,http://dx.doi.org/10.1155/2013/905867.
[12] H. A. Levine, G. Todorova; Blow-up of solutions of the Cauchy problem for a wave equation with nonlinear damping and source terms and positive initial energy, *Proc. Amer. Math. Soc.*, 2000, **129**: 793-805.
[13] S. A. Messaoudi; Global existence and nonexistence in s system of Petrovsky, *J. Math. Anal. Appl.*, 2002, **265(2)**: 296-308.
[14] S. A. Messaoudi; Blow-up and global existence in a nonlinear viscoelastic wave equation, *Math. Nachr.*, 2003, **260**: 58-66.
[15] S. A. Messaoudi; Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation, *J. Math. Anal. Appl.*, 2006, **320(2)**: 902-915.
[16] N.-e. Tatar; Arbitrary decay in linear viacoelasticity, *J. Math. Phy.*, 2011, **52(1)**: 01350201-01350212.
[17] G. Todorova; Cauchy problm for a nonlinear wave equation with nonlinear damping and source terms, *Nonlinear Analysis*, 2000, **41**: 891-905.
[18] F. Tahamtani and A. Pyravi, Global existence, uniform decay and exponential growth of solutions for a system of viscoelastic Petrovsky equations, *Turkish J. Math.*, 2014, **38(1)**: 87-109.
[19] Y. Wang; A sufficient condition for finite time blow-up of the nonlinear Klein-Gordon equations with arbitrarily positive initial energy, *Proc. Amer. Math. Soc.*, 2008, **136**: 3477-3482.
[20] Y. Xiong; Blow-up and polynomial decay of solutions for a viscoelastic equation with a nonlinear source, *J. Anal. Appl.*, 2012, **31:**251-266.

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