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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS FOR A SECOND-ORDER BOUNDARY-VALUE PROBLEM

RAMZI S. ALSAEDI

Abstract. We study the boundary-value problem

$$\begin{gathered} \frac{1}{A(t)}(A(t)u'(t))' = \lambda f(t,u(t)) \quad t \in (0,\infty), \\ \lim_{t\to 0^+} A(t)u'(t) = -a \le 0, \quad \lim_{t\to\infty} u(t) = b > 0, \end{gathered}$$

where $\lambda \ge 0$ and f is nonnegative continuous and nondecreasing with respect to the second variable. Under some assumptions on the nonlinearity f, we prove the existence of a positive solution for λ sufficiently small. Our approach is based on the Schauder fixed point theorem.

1. Introduction

The second-order differential equation

$$\frac{1}{A(t)}(A(t)u'(t))' = g(t,u(t)), \quad t \in (a,b) \tag{1.1}$$

has been extensively studied on both bounded and unbounded intervals with different boundary values (see for example [1, 3, 13, 14, 15, 16, 17, 19] and the reference therein). Many results of existence and uniqueness of positive bounded solutions or unbounded ones have been obtained in the literature. Most of these results treat the case where the nonlinearity g is negative and $A(t) = 1$ or $A(t) = t^{n-1}$ with $n \ge 3$. Boundary-value problems for differential equations of type (1.1) play a very important role in both theory and applications. They are used to describe a large number of physical, biological and chemical phenomena.

Recently in [9], the authors considered the density profile equation

$$\begin{gathered} \psi''(r) + \frac{n-1}{r}\psi'(r) = 4\lambda(\psi(r)+1)\psi(r)(\psi(r)-\xi), \quad r \in (0,\infty), \\ \psi'(0) = 0\,, \quad \psi(\infty) = \xi\,, \end{gathered}$$

where $\psi(r)$ stands for the density of a fluid. This equation has the origins in the Cahn-Hillard theory which is used in hydrodynamics to study the behavior of nonhomogeneous fluids. Analytical aspects concerning this equation, i.e, the

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existence and uniqueness of strictly increasing solutions, their asymptotic behavior at infinity, as well as a various numerical solutions were thoroughly carried out in [10, 11, 12].

In this article, we study the existence of positive solutions for the boundary-value problem

$$\begin{gathered} \frac{1}{A(t)}(A(t)u'(t))' = \lambda f(t,u) \quad t \in (0,\infty), \\ Au'(0) = \lim_{t\to 0} A(t)u'(t) = -a\,, \quad u(\infty) = \lim_{t\to\infty} u(t) = b\,, \end{gathered} \tag{1.2}$$

where $\lambda \geq 0$, $a \geq 0$, $b > 0$ and the function f is nonnegative, continuous and nondecreasing with respect to the second variable and satisfies some integrability condition. Throughout this article, the function A is assumed to satisfy the following condition:

- (A1) A is a continuous function on $[0,\infty)$, differentiable and positive on $(0,\infty)$ such that
$$\int_1^\infty \frac{dt}{A(t)} < \infty.$$

For a function A satisfying (A1), the Green's function of the operator $Lu = \frac{1}{A}(Au')'$ on $(0,\infty)$ with Dirichlet boundary conditions $Au'(0) = 0\,,\ u(\infty) = 0$ is

$$G(x,t) = A(t)\int_{x\vee t}^\infty \frac{1}{A(s)}\,ds\,, \quad \text{for } x,t \in ((0,\infty))^2\,,$$

where $x \vee t = \max(x,t)$.

To state our main result, we adopt the following notation. We denote by $B((0,\infty))$ the set of Borel measurable functions on $(0,\infty)$ and by $B^+((0,\infty))$ the set of nonnegative ones. Also we refer to $C([0,\infty])$ the collection of all continuous functions u in $[0,\infty)$ such that $\lim_{x\to\infty} u(x)$ exists and $C_0([0,\infty))$ the subclass of $C([0,\infty])$ consisting of functions which vanish continuously at ∞.

We refer to the Green potential of a function $h \in B^+((0,\infty))$ by

$$Vh(x) = \int_0^\infty G(x,t)h(t)dt = \int_x^\infty \frac{1}{A(t)}\Big(\int_0^t A(s)h(s)ds\Big)dt.$$

We denote by $\mathcal{K}$ the set of functions defined by

$$\mathcal{K} = \big\{\varphi \in B^+(0,\infty) : V\varphi(0) = \int_0^\infty G(0,t)\varphi(t)dt < \infty\big\}.$$

Finally, we denote by $\omega(x) = a\int_x^\infty \frac{1}{A(t)}\,dt + b$. Taking into account these notations, we assume that the function f satisfies the following assumptions:

- (A2) $f : (0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous and nondecreasing with respect to the second variable.
- (A3) The function $t \mapsto q(t) = \frac{f(t,\omega(t))}{\omega(t)}$ is nontrivial nonnegative and belongs to the class $\mathcal{K}$.

Our main result is the following.

Theorem 1.1. *Assume that* (A1)–(A3) *are satisfied. Then there exists* $\lambda_0 > 0$ *such that for each* $\lambda \in [0,\lambda_0)$*, problem* (1.2) *has a positive solution* $u \in C^2((0,\infty))$ *satisfying*

$$\big(1 - \frac{\lambda_0}{\lambda}\big)\omega(x) \leq u(x) \leq \omega(x), \quad \textit{for } x \in (0,\infty).$$

Remark 1.2. When $a = 0$ the solution given in the previous theorem is bounded, while this solution is unbounded near 0 when $a > 0$ and $\int_0^1 \frac{dt}{A(t)}$ diverges.

Our paper is organized as follows. In section 2, we give some properties related to the Green's function $G(x, y)$. Section 3 is devoted to the proof of Theorem 1.1 and to the study of an example that illustrates our result.

2. Preliminary results

In this section we give some inequalities on the Green function $G(x, y)$ and establish some technical results that will play a crucial role in the proof of our main result. Let A satisfy assumption (A1).

Proposition 2.1. (i) *For each $x, t, s \in (0, \infty)$, we have*
$$\frac{G(x,t)G(t,s)}{G(x,s)} \le G(0,t).$$
(ii) *For each $x, t \in (0, \infty)$, we have*
$$\frac{G(x,t)\omega(t)}{\omega(x)} \le G(0,t).$$

Proof. (i) For $x, t, s \in (0, \infty)$, we have
$$\begin{aligned}\frac{G(x,t)G(t,s)}{G(x,s)} &= \frac{A(t)\Big(\int_{x\vee t}^\infty \frac{1}{A(r)}\,dr\Big)\Big(\int_{t\vee s}^\infty \frac{1}{A(r)}\,dr\Big)}{\int_{x\vee s}^\infty \frac{1}{A(r)}\,dr}\\ &\le \frac{A(t)\Big(\int_{x\vee s}^\infty \frac{1}{A(r)}\,dr\Big)\Big(\int_{t}^\infty \frac{1}{A(r)}\,dr\Big)}{\int_{x\vee s}^\infty \frac{1}{A(r)}\,dr} = G(0,t).\end{aligned}$$

(ii) Let $h(t) = \int_t^\infty \frac{1}{A(r)}\,dr$ for $t > 0$. Since h is non-increasing on $(0, \infty)$, it follows that
$$h(x \vee t)\,(ah(t) + b) = a\,h(x \vee t)h(t) + b\,h(x \vee t) \le h(t)\,(ah(x) + b)\,.$$
This proves that for each $x, t \in (0, \infty)$, we have
$$\frac{G(x,t)\omega(t)}{\omega(x)} = \frac{A(t)h(x\vee t)\,(ah(t)+b)}{ah(x)+b} \le A(t)h(t) = G(0,t).$$
□

Proposition 2.2. *Let q be a nonnegative function in $\mathcal{K}$ and let*
$$\alpha_q = \sup_{x,s\in(0,\infty)} \int_0^\infty \frac{G(x,t)G(t,s)}{G(x,s)}\,q(t)dt.$$
Then we have
(i) $\alpha_q = \|Vq\|_\infty = Vq(0)$.
(ii) $V(q\omega)(x) \le \alpha_q\omega(x)$ *for each $x \in (0, \infty)$.*

Proof. (i) Using Proposition 2.1 (i), we obtain
$$\int_0^\infty \frac{G(x,t)G(t,s)}{G(x,s)}\,q(t)dt \le \int_0^\infty G(0,t)q(t)dt.$$
Hence
$$\alpha_q \le Vq(0) = \|Vq\|_\infty. \tag{2.1}$$

On the other hand, since for each $x,t\in(0,\infty)$ we have $\lim_{s\to\infty}\frac{G(t,s)}{G(x,s)}=1$, Using Fatou's lemma, we have

$$\begin{aligned} Vq(x) &= \int_0^\infty G(x,t)q(t)dt \\ &= \int_0^\infty \lim_{s\to\infty}\frac{G(x,t)G(t,s)}{G(x,s)}\,q(t)dt \\ &\le \liminf_{s\to\infty}\int_0^\infty \frac{G(x,t)G(t,s)}{G(x,s)}\,q(t)dt \le \alpha_q. \end{aligned}$$

This shows that

$$Vq(0)=\|Vq\|_\infty \le \alpha_q. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain that $\alpha_q=\|Vq\|_\infty=Vq(0)$.

(ii) Using assertion (i) and Proposition 2.1 (ii), we obtain

$$\int_0^\infty \frac{G(x,t)\omega(t)}{\omega(x)}\,q(t)dt \le \int_0^\infty G(0,t)q(t)dt = Vq(0)=\alpha_q.$$

Hence $V(q\omega)(x)\le \alpha_q\,\omega(x)$. □

The following continuity result will be used in the proof of Theorem 1.1.

Proposition 2.3. *Let q be a nonnegative function in $\mathcal{K}$. Then the family of functions*

$$S_q=\big\{x\mapsto \frac{1}{\omega(x)}\int_0^\infty G(x,t)\varphi(t)\omega(t)dt : \varphi\in B((0,\infty)) \text{ and } |\varphi|\le q\big\}$$

is relatively compact in $C_0([0,\infty))$.

Proof. Since $b>0$, then for each $x,t\in(0,\infty)$ we have $\frac{G(x,t)}{\omega(t)}\le\frac{G(0,t)}{b}$. Hence

$$\big|\frac{1}{\omega(x)}V(\varphi\,\omega)(x)\big| \le \frac{1}{b}Vq(0).$$

This shows that S_q is uniformly bounded. Next, we consider $x,x'\in[0,\infty)$. Then we have

$$\big|\frac{1}{\omega(x)}V(\varphi\,\omega)(x)-\frac{1}{\omega(x')}V(\varphi\,\omega)(x')\big| \le \int_0^\infty \big|\frac{G(x,t)\omega(t)}{\omega(x)}-\frac{G(x',t)\omega(t)}{\omega(x')}\big|q(t)dt.$$

Using the continuity of the function $x\mapsto\frac{G(x,t)}{\omega(x)}$ on $[0,\infty)$ for each $t\in[0,\infty)$, Proposition 2.1 (ii) and the fact that $q\in\mathcal{K}$, we deduce from the dominated convergence theorem, the equicontinuity of S_q on $[0,\infty)$. Moreover, since $b>0$ and A satisfies (A1), we have

$$\lim_{x\to\infty}\frac{G(x,t)}{\omega(x)}=\lim_{x\to\infty}\frac{A(t)\int_{x\vee t}^\infty \frac{1}{A(s)}\,ds}{b+a\int_x^\infty\frac{1}{A(s)}\,ds}=0.$$

This and Proposition 2.1 (ii) shows that

$$\lim_{x\to\infty}\big|\frac{1}{\omega(x)}V(\varphi\,\omega)(x)\big|=0\,,\quad \text{uniformly in } \varphi.$$

Then by Ascoli's theorem, we deduce that S_q is relatively compact in $C_0([0,\infty))$. □

3. Proof of main result

Lemma 3.1. *If f satisfies (A3), then*

$$\lambda_0 := \inf_{x\in(0,\infty)} \frac{\omega(x)}{V(f(.,\omega))(x)} > 0.$$

Proof. Since f satisfies (A3), the function $q = \frac{f(.,\omega)}{\omega}$ belongs to $\mathcal{K}$. It follows from Proposition 2.2 that

$$V(f(.,\omega))(x) = V\Big(\frac{f(.,\omega)}{\omega}\,\omega\Big)(x) \le \alpha_q\,\omega(x).$$

Or equivalently

$$\frac{\omega(x)}{V(f(.,\omega))(x)} \ge \frac{1}{\alpha_q}.$$

This shows that $\lambda_0 \ge \frac{1}{\alpha_q} > 0$. $\square$

Proof of Theorem 1.1. Let λ_0 be the positive constant given in Lemma 3.1. For $\lambda \in [0,\lambda_0)$, we consider the nonempty closed convex set

$$\Lambda = \big\{v \in C([0,\infty]) : (1-\frac{\lambda}{\lambda_0}) \le v \le 1\big\}$$

and define the operator T on Λ by

$$Tv(x) = 1 - \frac{\lambda}{\omega(x)}\int_0^\infty G(x,t)f(t,\omega(t)v(t))dt = 1 - \frac{\lambda}{\omega(x)}V(f(.,\omega))(x).$$

Since f is non-decreasing with respect to the second variable, then for each $v\in\Lambda$ and $x>0$, we have

$$0 \le \frac{1}{\omega(x)}V(f(.,\omega\,v))(x) \le \frac{1}{\omega(x)}V(f(.,\omega))(x) \le \frac{1}{\lambda_0}.$$

This shows that $(1-\frac{\lambda}{\lambda_0}) \le Tv \le 1$ for each $v\in\Lambda$. On the other hand since $\frac{f(.,\omega)}{\omega}\in\mathcal{K}$, it follows from Proposition 2.3 that the family $\{\frac{1}{\omega}V(f(.,\omega\,v)) : v\in\Lambda\}$ is relatively compact in $C_0([0,\infty))$. Hence $T\Lambda\subset\Lambda$ and $T\Lambda$ is relatively compact in $C([0,\infty])$.

Next, we prove the continuity of the operator T in Λ in the supremum norm. Let $(v_k)_k$ be a sequence in Λ which converges uniformly to a function $v\in\Lambda$. Then we have

$$\begin{aligned}|Tv_k(x) - Tv(x)| &\le \frac{\lambda}{\omega(x)}|V(f(.,\omega\,v_k))(x) - V(f(.,\omega\,v))(x)|\\ &\le \frac{\lambda}{\omega(x)}\int_0^\infty G(x,t)|f(t,\omega(t)v_k(t)) - f(t,\omega(t)v(t))|\,dt.\end{aligned}$$

From the monotonicity of f with respect to the second variable, we have

$$|f(t,\omega(t)v_k(t)) - f(t,\omega(t)v(t))| \le 2f(t,\omega(t)).$$

Since by Proposition 2.3 and (A3), the function $V(f(.,\omega))/\omega \in C_0([0,\infty))$, using the continuity of f with respect to the second variable and the dominated convergence theorem, we conclude that

$$Tv_k(x) \to Tv(x) \quad \text{as } k\to\infty.$$

Consequently, as $T\Lambda$ is relatively compact in $C([0,\infty])$, we deduce that pointwise convergence implies uniform convergence, namely

$$\|Tv_k - Tv\|_\infty \to 0 \quad \text{as } k \to \infty\,.$$

Therefore T is a continuous mapping from Λ to itself. Since $T\Lambda$ is relatively compact in $C([0,\infty])$, it follows that T is a compact mapping on Λ. Finally, the Schauder fixed point theorem implies the existence of $v \in \Lambda$ such that

$$v(x) = 1 - \frac{\lambda}{\omega(x)} \int_0^\infty G(x,t) f(t, \omega(t)v(t)) dt.$$

Put $u(x) = \omega(x)v(x)$ for $x \in (0,\infty)$. Then $u \in C((0,\infty))$ and

$$(1 - \frac{\lambda}{\lambda_0})\omega(x) \le u(x) \le \omega(x) \quad \text{for } x \in (0,\infty).$$

Moreover u satisfies the integral equation

$$\begin{aligned} u(x) &= \omega(x) - \lambda V(f(.,u))(x) \\ &= \omega(x) - \lambda \int_x^\infty \frac{1}{A(t)} \Big(\int_0^t A(s) f(s,u(s)) ds \Big) dt. \end{aligned} \tag{3.1}$$

Now, since $0 \le u \le \omega$,

$$\lim_{x\to\infty} \frac{1}{\omega(x)} \int_0^\infty G(x,t) f(t,u(t)) dt = 0.$$

This limit implies $\lim_{x\to\infty} \int_0^\infty G(x,t) f(t,u(t)) dt = 0$; consequently $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} \omega(x) = b$. On the other hand, using (3.1), we obtain

$$A(x)u'(x) = -a + \int_0^x A(s) f(s,u(s)) ds.$$

This and $\lim_{x\to 0} \int_0^x A(s) f(s,u(s)) ds = 0$ imply that $\lim_{x\to 0} A(x)u'(x) = 0$. $\square$

Example 3.2. Let $\alpha > 1$, $\sigma \ge 0$ and $p : (0,\infty) \to [0,\infty)$ be a nontrivial nonnegative continuous function satisfying

$$\int_0^1 t^{1-(\alpha-1)(\sigma-1)} p(t) dt + \int_1^\infty t p(t) dt < \infty.$$

Then there exists $\lambda_0 > 0$ such that for each $\lambda \in [0,\lambda_0)$, the problem

$$\begin{gathered} u'' + \frac{\alpha}{x} u' = \lambda p(x) u^\sigma(x)\,, \quad x \in (0,\infty) \\ \lim_{x\to 0}, x^\alpha u'(x) = -a < 0\,, \quad \lim_{x\to\infty} u(x) = b > 0\,, \end{gathered}$$

has a positive solution $u \in C^2((0,\infty))$ satisfying

$$(1 - \frac{\lambda}{\lambda_0}) \Big(\frac{a}{(\alpha-1)x^{\alpha-1}} + b \Big) \le u(x) \le \Big(\frac{a}{(\alpha-1)x^{\alpha-1}} + b \Big)\,,$$

for $x \in (0,\infty)$.

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Ramzi S. Alsaedi

Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

E-mail address: `ramzialsaedi@yahoo.co.uk`