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SINGULAR LIMITING SOLUTIONS TO 4-DIMENSIONAL ELLIPTIC PROBLEMS INVOLVING EXPONENTIALLY DOMINATED NONLINEARITY AND NONLINEAR TERMS

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Abstract. Let $\Omega \in \mathbb{R}^4$ be a bounded open regular set, $x_1, x_2, \dots, x_m \in \Omega$, $\lambda, \rho > 0$ and $\mathscr{Q}_\lambda$ be a non linear operator (which will be defined later). We prove that the problem

$$\Delta^2 u + \mathscr{Q}_\lambda(u) = \rho^4 e^u$$

has a positive weak solution in Ω with $u = \Delta u = 0$ on $\partial\Omega$, which is singular at each x_i as the parameters λ and ρ tends to 0.

1. Introduction and statement of results

Semilinear equations involving fourth order elliptic operator and exponential nonlinearity appear naturally in conformal geometry and in particular in the prescription of the so called Q-curvature in four-dimensional Riemannian manifolds [7, 8]

$$Q_g = \frac{1}{12}(-\Delta_g S_g + S_g^2 - 3|\operatorname{Ric}_g|^2),$$

where Ric_g denotes the Ricci tensor and S_g is the scalar curvature of the metric g. Recall that the Q-curvature changes under a conformal change of metric

$$g_w = e^{2w} g,$$

according to

$$P_g w + 2Q_g = 2\tilde{Q}_{g_w} e^{4w}, \tag{1.1}$$

where

$$P_g := \Delta_g^2 + \delta(\frac{2}{3} S_g I - 2\operatorname{Ric}_g)\, d,$$

is the Panietz operator, which is an elliptic 4-th order partial differential operator [8] and which transforms according to

$$e^{4w} P_{e^{2w} g} = P_g,$$

under a conformal change of metric $g_w := e^{2w} g$. In the special case where the manifold is the Euclidean space, the Panietz operator is simply given by

$$P_{g_{\mathrm{eucl}}} = \Delta^2,$$

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in which case (1.1) can be written as

$$\Delta^2 w = \tilde{Q} e^{4w},$$

the solutions of which give rise to conformal metric $g_w = e^{2w} g_{\rm eucl}$ whose Q-curvature is given by $\tilde{Q}$. There is by now an extensive literature about this problem and we refer to [8] and [13] for references and recent developments.

Wei in [18], have studied the behavior of solutions to the following nonlinear problem in $\mathbb{R}^4$. More precisely, consider the problem

$$\begin{gathered} \Delta^2 u = \lambda\, f(u) \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

Before showing his result, we introduce some notation. Let $G(x,x')$ defined, over $\Omega\times\Omega$, the Green function associated to the bi-laplacian operator with a Navier boundary conditions, which is the solution of

$$\begin{gathered} \Delta_x^2 G(x,x') = 64\pi^2 \delta_{x=x'} \quad \text{in } \Omega \\ G(x,x') = \Delta_x G(x,x') = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.3}$$

and denote by $H(x,x') = G(x,x') + 8\log|x-x'|$ its smooth part. Consider now the functional

$$E : (x^1,\dots,x^m) \in (\mathbb{R}^4)^m \mapsto \sum_{j=1}^m H(x^j,x^j) + \sum_{j\neq l} G(x^j,x^l) \tag{1.4}$$

and u^* the solution of

$$\begin{gathered} \Delta^2 u^* = 64\pi^2 \sum_{i=1}^m \delta_{x^i} \quad \text{in } \Omega \\ u^* = \Delta u^* = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.5}$$

The author proved the following result.

Theorem 1.1 ([18]). *Let Ω be a smooth bounded domain in $\mathbb{R}^4$ and f a smooth nonnegative increasing function such that*

$$e^{-u} f(u) \text{ and } \varepsilon^{-u}\int_0^u f(s)ds \text{ tends to } 1 \text{ as } u \to +\infty.$$

For u_λ solution of (1.2), denote by $\Sigma_\lambda = \lambda\int_\Omega f(u_\lambda)dx$. Then, three cases occur:

1. *$\Sigma_\lambda \to 0$ therefore, $\|u_\lambda\|_{L^\infty(\Omega)} \to 0$ as $\lambda \to 0$.*
2. *$\Sigma_\lambda \to +\infty$ then $u_\lambda \to +\infty$ as $\lambda \to 0$.*
3. *$\Sigma_\lambda \to 64\pi^2 m$, for some positive integer m. Then the limiting Function $u^* = \lim_{\lambda\to 0} u_\lambda$ has m blow-up points, $\{x^1,\dots,x^m\}$, where $u_\lambda(x^i) \to +\infty$ as $\lambda \to 0$. Moreover, $(x^1,\dots,x^m)$ is a critical point of E.*

Now, we are interested in positive solutions of the problem

$$\begin{gathered} \Delta^2 u = \rho^4 e^u \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.6}$$

when the parameter ρ tends to 0. Obviously, the application of the implicit function theorem yields the existence of a smooth one parameter family of solutions $(u_\rho)_\rho$ which converges uniformly to 0 as ρ tends to 0. This branch of solutions is usually referred to as the branch of *minimal solutions* which gives the converse of the case (1) given in the last Theorem.

First, let us mention that in [4], Ben Ayed, El Mehdi and Grossi considered a bi-harmonic equation with large exponent in the non linear term; that is $\Delta^2 u = u^p$, under Navier boundary conditions. The authors have studied the asymptotic behavior of positive solutions obtained by minimizing suitable functionals.

In [9], the authors studied existence and qualitative properties of positives solutions to the boundary-value problem

$$\begin{gathered} \Delta^2 u = \rho^4\, k(x) e^u \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.7}$$

where $k \in C^2(\Omega)$, is a non-negative, not identically zero function, Ω a bounded open regular domain in $\mathbb{R}^4$ and $\rho > 0$ is a small, positive parameter which tends to 0.

Recently, the existence of other branches of solutions as ρ tends to 0 is studied in [3]. The authors construct a non-minimal solutions with singular limit as the parameter ρ tends to 0. Their results which give the converse of the case (3) given in the last Theorem, can be stated as follows.

Theorem 1.2 ([3]). *Let Ω be a smooth open subset of $\mathbb{R}^4$. Assume $(x^1, \dots, x^m)$ is a nondegenerate critical point of E. Then there exist $\rho_0 > 0$ and a one parameter family $(u_\rho)_{\rho \in (0,\rho_0)}$ of solutions of (1.6), such that*

$$\lim_{\rho \to 0} u_\rho = u^*, \quad \text{in } \mathcal{C}^{4,\alpha}_{\rm loc}(\Omega - \{x^1, \dots, x^m\}).$$

To prove Theorem 1.2, the authors present, for the first time, a rather efficient method to solve such singularly perturbed problems in the context of partial differential equations. This method based on some nonlinear domain decomposition has already been used successfully in geometric context (constant mean curvature surfaces, constant scalar curvature metrics, extremal Kähler metrics, ...). In this article, we adopt this method in the study of the following problem.

Let $\Omega \subset \mathbb{R}^4$ be a regular bounded open domain in $\mathbb{R}^4$. We are interested in positive solutions of

$$\Delta^2 u + \mathscr{Q}_\lambda(u) = \rho^4 e^u \text{ in } \Omega \tag{1.8}$$

satisfying $u = \Delta u = 0$ on $\partial\Omega$ and $\mathscr{Q}_\lambda$ is the nonlinear operator given by

$$\begin{aligned} \mathscr{Q}_\lambda(u) := \lambda\big[(\Delta u)^2 + \Delta(|\nabla u|^2) + 2\nabla u \cdot \nabla(\Delta u)\big] \\ + 2\lambda^2\big[\Delta u |\nabla u|^2 + \nabla u \cdot \nabla(|\nabla u|^2)\big] + \lambda^3 |\nabla u|^4. \end{aligned} \tag{1.9}$$

Using the transformation

$$w := (\lambda \rho^4 e^u)^\lambda, \tag{1.10}$$

if u is a solution of (1.8) then w solves the equation

$$\Delta^2 w = w^{\frac{\lambda+1}{\lambda}} \quad \text{in } \Omega. \tag{1.11}$$

Remark that the exponent $q = \frac{\lambda+1}{\lambda}$ tends to ∞ as λ tends to 0.

We denote by ε the smallest positive parameter satisfying

$$\rho^4 = \frac{384\varepsilon^4}{(1+\varepsilon^2)^4}. \tag{1.12}$$

We remark that $\rho \sim \varepsilon$ as $\varepsilon \to 0$. We will suppose in the following that

(A1) $\lambda^{1+\delta/2} \varepsilon^{-\delta} = \mathcal{O}(1)$ as $\varepsilon \to 0$ for any $\delta \in (0,1)$.

In particular, if we take $\lambda = \mathcal{O}(\varepsilon^{2/3})$, then condition (A1) is satisfied. Under the assumption (A1), we can treat equation (1.8) as a perturbation of the equation

$$\Delta^2 u = \rho^4 e^u \quad \text{in } \Omega \subset \mathbb{R}^4.$$

Our question is: Does there exist u_ε a sequence of solutions which converges to some singular function as the parameters ε tend to 0?

Our main result reads as follows.

Theorem 1.3. *Given $\alpha \in (0,1)$. Let Ω be an open smooth bounded set of $\mathbb{R}^4$, $\lambda > 0$ satisfy condition* (A1)*, and $S = \{x_1, \dots, x_m\} \subset \Omega$ be a non empty set. Assume that $(x_1, \dots, x_m)$ is a nondegenerate critical point of the function*

$$\mathscr{F}(x_1, \dots, x_m) = \sum_{j=1}^m H(x_j, x_j) + \sum_{i \neq j} G(x_i, x_j) \quad \text{in } (\Omega)^m,$$

then there exist $\rho_0 > 0$, $\lambda_0 > 0$ and a family $\{u_{\rho,\lambda}\}$ with $0 < \rho < \rho_0, 0 < \lambda < \lambda_0$ of solutions of (1.8)*, such that*

$$\lim_{\rho \to 0,\, \lambda \to 0} u_{\rho,\lambda} = \sum_{j=1}^m G(x_j, \cdot) \quad \text{in } \mathcal{C}^{4,\alpha}_{\text{loc}}(\Omega - \{x_1, \dots, x_m\}).$$

2. Construction of the approximate solution

We first describe the rotationally symmetric approximate solutions of

$$\Delta^2 u - \rho^4 e^u = 0, \tag{2.1}$$

in $\mathbb{R}^4$, which will be crucial in the construction of the approximate solution. Given $\varepsilon > 0$, we define

$$u_{\varepsilon,\tau}(x) := 4 \log(1 + \varepsilon^2) + 4 \log \tau - 4 \log(\varepsilon^2 + (\tau |x|)^2).$$

which is clearly a solution of (2.1) when

$$\rho^4 = \frac{384 \varepsilon^4}{(1 + \varepsilon^2)^4}.$$

For $\tau > 0$, we remark that equation (2.1) is invariant under some dilation in the following sense: If u is solution of (2.1), then

$$\tau \mapsto u(\tau \cdot) + 4 \log \tau.$$

is also a solution of (2.1). So, for $\varepsilon > 0$ and $\tau > 0$ we denote by $u_{\varepsilon,\tau}$ the element of this new family of radial solutions of (2.1).

For $\varepsilon = \tau = 1$ and we denote by $u_1 = u_{1,1}$ this particular solution. We also define the following linear fourth order elliptic operator

$$\mathscr{L} := \Delta^2 - \frac{384}{(1 + |x|^2)^4},$$

which corresponds to the linearization of (2.1) about the solution u_1.

2.1. **Radial solution on $\mathbb{R}^4$.** For all $\varepsilon, \tau, \lambda > 0$, we set

$$R_{\varepsilon,\lambda} := \tau r_{\varepsilon,\lambda}/\varepsilon, \quad \text{where } r_{\varepsilon,\lambda} := \max(\sqrt{\varepsilon}, \sqrt{\lambda}). \tag{2.2}$$

The classification of bounded solutions of $\mathbb{L}w = 0$ in $\mathbb{R}^4$ is well known. Some solutions are easy to find. For example, we can define

$$\phi_0(x) := r\partial_r u_1(x) + 4 = 4\frac{1-r^2}{1+r^2},$$

where $r = |x|$. Clearly $\mathbb{L}\phi_0 = 0$ and this reflects the fact that (2.1) is invariant under the group of dilations $\tau \mapsto u(\tau\cdot) + 4\log\tau$. We also define, for $i = 1, \dots, 4$

$$\phi_i(x) := -\partial_{x_i} u_1(x) = \frac{8x_i}{1+|x|^2},$$

which are also solutions of $\mathbb{L}\phi_i = 0$ since these solutions correspond to the invariance of the equation under the group of translations $a \mapsto u(\cdot + a)$. Then, we have the following classification.

Lemma 2.1 ([3])**.** *Any bounded solution of $\mathscr{L}w = 0$ defined in $\mathbb{R}^4$ is a linear combination of ϕ_i for $i = 0, 1, \dots, 4$.*

Let B_r denote the ball of radius r centered at the origin in $\mathbb{R}^4$.

Definition 2.2. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, we introduce the Hölder weighted spaces $\mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4)$ as the space of functions $w \in \mathcal{C}^{k,\alpha}_{\rm loc}(\mathbb{R}^4)$ for which the norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4)} := \|w\|_{\mathcal{C}^{k,\alpha}(\bar{B}_1)} + \sup_{r\geqslant 1}\big((1+r^2)^{-\delta/2}\|w(r\cdot)\|_{\mathcal{C}^{k,\alpha}_\mu(\bar{B}_1 - B_{1/2})}\big),$$

is finite.

Also, we define

$$\mathcal{C}^{k,\alpha}_{{\rm rad},\mu}(\mathbb{R}^4) = \{f \in \mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4) textsuchthat f(x) = f(|x|), \forall x \in \mathbb{R}^4\}.$$

As a consequence of Lemma 2.1, we recall the surjectivity result of $\mathscr{L}$.

Proposition 2.3 ([3])**.**
(1) *Assume that $\mu > 1$ and $\mu \notin \mathbb{Z}$, then the operator $L_\mu : \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \to \mathcal{C}^{0,\alpha}_{\mu-4}(\mathbb{R}^4)$ defined by $L_\mu(w) = \mathscr{L}w$ is surjective.*
(2) *Assume that $\delta > 0$ and $\delta \notin \mathbb{Z}$, then the operator $L_\delta : \mathcal{C}^{4,\alpha}_{{\rm rad},\delta}(\mathbb{R}^4)) \to \mathcal{C}^{0,\alpha}_{{\rm rad},\delta-4}(\mathbb{R}^4)$ defined by $L_\delta(w) = \mathscr{L}w$ is surjective.*

We set $\bar{B}_1^* = \bar{B}_1 - \{0\}$.

Definition 2.4. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, we introduce the Hölder weighted spaces $\mathcal{C}^{k,\alpha}_\mu(\bar{B}_1^*)$ as the space of functions in $\mathcal{C}^{k,\alpha}_{\rm loc}(\bar{B}_1^*)$ for which the norm

$$\|u\|_{\mathcal{C}^{k,\alpha}_\mu(\bar{B}_1^*)} = \sup_{r\leqslant 1/2}(r^{-\mu}\|u(r\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_2 - B_1)}),$$

is finite.

Then, we define the subspace of radial functions in $\mathcal{C}^{k,\alpha}_\delta(\bar{B}_1^*)$ by

$$\mathcal{C}^{k,\alpha}_{{\rm rad},\delta}(\bar{B}_1^*) = \{f \in \mathcal{C}^{k,\alpha}_\delta(\mathbb{R}^4);\ \textit{such that } f(x) = f(|x|), \forall x \in \bar{B}_1^*\}.$$

Our aim now is the construction of a radial solution u of

$$\Delta^2 u + \mathscr{Q}_\lambda(u) - \rho^4 e^u = 0 \quad \text{in } \bar{B}_{r_{\varepsilon,\lambda}}. \tag{2.3}$$

Thanks to the transformation
$$v(x) = u(\frac{\varepsilon}{\tau}x) + 8\log\varepsilon - 4\log(\tau(1+\varepsilon^2)/2),$$
Equation (2.3) can be written as
$$\Delta^2 v + \mathscr{Q}_\lambda(v) - 24e^v = 0 \quad \text{in } \bar{B}_{R_{\varepsilon,\lambda}}. \tag{2.4}$$
Now, we look for a solution of (2.4) of the form $v(x) = u_1(x)+h(x)$; this amounts to solving
$$\mathscr{L}h = \frac{384}{(1+|x|^2)^4}(e^h - h - 1) - \mathscr{Q}_\lambda(u_1+h) \quad \text{in } \bar{B}_{R_{\varepsilon,\lambda}}. \tag{2.5}$$
We will need the following definition.

Definition 2.5. Given $\bar{r} \geqslant 1$, $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$, the weighted space $\mathcal{C}^{k,\alpha}_\delta(B_{\bar{r}})$ is defined to be the space of functions $w \in \mathcal{C}^{k,\alpha}(B_{\bar{r}})$ endowed with the norm
$$\|w\|_{\mathcal{C}^{k,\alpha}_\delta(\bar{B}_{\bar{r}})} := \|w\|_{\mathcal{C}^{k,\alpha}(B_1)} + \sup_{1\leqslant r\leqslant \bar{r}} (r^{-\delta}\|w(r\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_1 - B_{1/2})}).$$

For $\sigma \geqslant 1$, we denote by
$$\mathscr{E}_\sigma : \mathcal{C}^{0,\alpha}_\delta(\bar{B}_\sigma) \to \mathcal{C}^{0,\alpha}_\delta(\mathbb{R}^4),$$
the extension operator defined by
$$\mathscr{E}_\sigma(f)(x) = \chi(\frac{|x|}{\sigma}) f(\sigma\frac{x}{|x|}),$$
where $t \mapsto \chi(t)$ is a smooth nonnegative cutoff function identically equal to 1 for $t \geqslant 2$ and identically equal to 0 for $t \leqslant 1$. It is easy to check that there exists a constant $c = c(\delta) > 0$, independent of $\sigma \geqslant 1$, such that
$$\|\mathscr{E}_\sigma(w)\|_{\mathcal{C}^{0,\alpha}_\delta(\mathbb{R}^4)} \leqslant c\|w\|_{\mathcal{C}^{0,\alpha}_\delta(\bar{B}_\sigma)}. \tag{2.6}$$

We fix $\delta \in (0,1)$, and denote by $\mathscr{G}_\delta$ to be a right inverse of $\mathscr{L}_\delta$ assured by Proposition 2.3. Now, we use the result of Proposition 2.3 to rephrase the nonlinear equation (2.5) as a fixed point problem. Hence, to obtain a solution of (2.5), it is enough to find a fixed point h in a small ball of $\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)$ for the mapping
$$h \mapsto \mathscr{N}(h) := \mathscr{G}_\delta \circ \mathscr{E}_\delta \circ \mathscr{R}(h), \tag{2.7}$$
where
$$\mathscr{R}(h) := \frac{384}{(1+|x|^2)^4}(e^h - h - 1) - \mathscr{Q}_\lambda(u_1+h).$$
We have
$$\begin{aligned}\mathscr{R}(0) = &-\lambda\big[(\Delta u)^2 + \Delta(|\nabla u|^2) + 2\nabla u\cdot\nabla(\Delta u)\big]\\ &- 2\lambda^2\big[\Delta u|\nabla u|^2 + \nabla u\cdot\nabla(|\nabla u|^2)\big] - \lambda^3|\nabla u|^4.\end{aligned}$$
Recall that
$$u_1 = 4\log(2) - 4\log(1+r^2).$$
Then
$$|\nabla u_1|^2 = 64\frac{r^2}{(1+r^2)^2}, \quad \Delta u_1 = -16\frac{2+r^2}{(1+r^2)^2}, \quad \Delta(|\nabla u_1|^2) = 512\frac{1-2\,r^2}{(1+r^2)^4}.$$
Hence,
$$(1+r^2)^{2-\frac{\delta}{2}}|(\Delta u_1)^2 + \Delta(|\nabla u_1|^2) + 2\nabla u_1\cdot\nabla(\Delta u_1)| \leqslant c(1+r^2)^{-\frac{\delta}{2}},$$

$$(1+r^2)^{2-\frac{\delta}{2}}|\Delta u_1|\nabla u_1|^2+\nabla u_1\cdot\nabla(|\nabla u_1|^2)|\leqslant c(1+r^2)^{-1-\frac{\delta}{2}},$$
$$(1+r^2)^{2-\frac{\delta}{2}}|\nabla u_1|^4\leqslant c(1+r^2)^{-\frac{\delta}{2}}.$$

This implies that given $\kappa>0$, there exists $c_\kappa>0$ (which can depend only on κ), such that for $\delta\in(0,1)$ and $|x|=r$, we have

$$\sup_{r\leqslant R_{\varepsilon,\lambda}}(1+r^2)^{2-\frac{\delta}{2}}|\mathscr{R}(0)|\leqslant c_\kappa\lambda\,.$$

So

$$\|\mathscr{N}(0)\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}\leqslant c_\kappa r^2_{\varepsilon,\lambda}. \tag{2.8}$$

Using Proposition 2.3 and (2.6), we deduce that

$$\|h\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}\leqslant 2c_\kappa r^2_{\varepsilon,\lambda}. \tag{2.9}$$

Now let h_1,h_2 in $B(0,2c_\kappa\, r^2_{\varepsilon,\lambda})$ of $\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)$ and for $\delta\in(0,1)$, then

$$|\mathscr{R}(h_2)-\mathscr{R}(h_1)|\leqslant|e^{h_2}-e^{h_1}+h_1-h_2|+|\mathscr{Q}_\lambda(u_1+h_2)-\mathscr{Q}_\lambda(u_1+h_1)|.$$

Furthermore,

$$\begin{aligned}r^{4-\delta}|e^{h_2}-e^{h_1}+h_1-h_2|&\leqslant cr^{4-\delta}|h_2-h_1||h_2+h_1|\\&\leqslant c_\kappa r^\delta r^2_{\varepsilon,\lambda}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}.\end{aligned}$$

$$\begin{aligned}r^{4-\delta}|(\Delta(u_1+h_1))^2-(\Delta(u_1+h_2))^2|&=r^{4-\delta}|(\Delta(h_1-h_2))(\Delta(2u_1+h_1+h_2))|\\&\leqslant c_\kappa\big(1+r^\delta r^2_{\varepsilon,\lambda}\big)\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}.\end{aligned}$$

$$\begin{aligned}r^{4-\delta}|\Delta|\nabla(u_1+h_2)|^2-\Delta|\nabla(u_1+h_1)|^2|&=r^{4-\delta}|\Delta(\nabla(h_1-h_2)\cdot\nabla(2u_1+h_1+h_2))|\\&\leqslant c_\kappa\big(1+r^\delta r^2_{\varepsilon,\lambda}\big)\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}.\end{aligned}$$

$$\begin{aligned}&r^{4-\delta}\big|\nabla(\Delta(u_1+h_2))\cdot\nabla(u_1+h_2)-\nabla(\Delta(u_1+h_1))\cdot\nabla(u_1+h_1)\big|\\&=r^{4-\delta}\big|\nabla(\Delta(h_1-h_2))\cdot\nabla(2u_1+h_1+h_2)+\nabla(h_2-h_1)\cdot\nabla(\Delta(2u_1+h_1+h_2))\big|\\&\quad\times r^{4-\delta}\big|\nabla(\Delta(u_1+h_2))\cdot\nabla(u_1+h_2)-\nabla(\Delta(u_1+h_1))\cdot\nabla(u_1+h_1)\big|\\&\leqslant c_\kappa\big(1+r^\delta r^2_{\varepsilon,\lambda}\big)\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}.\end{aligned}$$

Since

$$\begin{aligned}&|\nabla(u_1+h_1)|^2\Delta(u_1+h_1)-|\nabla(u_1+h_2)|^2\Delta(u_1+h_2)\\&=\Delta(h_1-h_2)[|\nabla(u_1+h_1)|^2+|\nabla(u_1+h_2)|^2]\\&\quad+\Delta(2u_1+h_1+h_2)[|\nabla(u_1+h_1)|^2-|\nabla(u_1+h_2)|^2],\end{aligned}$$

it follows that

$$\begin{aligned}&r^{4-\delta}\big||\nabla(u_1+h_1)|^2\Delta(u_1+h_1)-|\nabla(u_1+h_2)|^2\Delta(u_1+h_2)\big|\\&\leqslant c_\kappa\big(1+r^\delta r^2_{\varepsilon,\lambda}+r^{2\delta}r^4_{\varepsilon,\lambda}\big)\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}.\end{aligned}$$

Its easy to see that

$$\begin{aligned}&\nabla(|\nabla(u_1+h_2)|^2)\nabla(u_1+h_2)-\nabla(|\nabla(u_1+h_1)|^2)\nabla(u_1+h_1)\\&=\nabla(h_2-h_1)\nabla(|\nabla(u_1+h_2)|^2+|\nabla(u_1+h_1)|^2)\\&\quad+\nabla(2u_1+h_1+h_2)\nabla(|\nabla(u_1+h_2)|^2-|\nabla(u_1+h_1)|^2);\end{aligned}$$

hence

$$\begin{aligned}&r^{4-\delta}\Big|\nabla(|\nabla(u_1+h_2)|^2)\nabla(u_1+h_2)-\nabla(|\nabla(u_1+h_1)|^2)\nabla(u_1+h_1)\Big|\\&\leqslant c_\kappa\big(1+r^\delta r_{\varepsilon,\lambda}^2+r^{2\delta}r_{\varepsilon,\lambda}^4\big)\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}.\end{aligned}$$

Finally, since

$$\begin{aligned}&|\nabla(u_1+h_2)|^4-|\nabla(u_1+h_1)|^4\\&=\nabla(h_2-h_1)\nabla(2u_1+h_2+h_1)(|\nabla(u_1+h_2)|^2+|\nabla(u_1+h_1)|^2),\end{aligned}$$

it follows that

$$\begin{aligned}&r^{4-\delta}\Big||\nabla(u_1+h_2)|^4-|\nabla(u_1+h_1)|^4\Big|\\&\leqslant c_\kappa\big(1+r^\delta r_{\varepsilon,\lambda}^2+r^{2\delta}r_{\varepsilon,\lambda}^4+r^{3\delta}r_{\varepsilon,\lambda}^6\big)\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}.\end{aligned}$$

Thanks to condition (A1),

$$\sup_{r\leqslant R_{\varepsilon,\lambda}} r^{4-\delta}\,|\mathscr{R}(h_2)-\mathscr{R}(h_1)|\leqslant c_\kappa r_{\varepsilon,\lambda}^2\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}.$$

Similarly, by Proposition 2.3 and (2.6), we conclude that given $\kappa>0$, then there exist $\bar c_\kappa>0$ (independent of ε and λ), λ_κ and ε_κ such that

$$\|\mathscr{N}(h_2)-\mathscr{N}(h_1)\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}\leqslant \bar c_\kappa r_{\varepsilon,\lambda}^2\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}. \tag{2.10}$$

Reducing $\lambda_\kappa>0$ and $\varepsilon_\kappa>0$ if necessary, we can assume that $\bar c_\kappa r_{\varepsilon,\lambda}^2\leqslant 1/2$ for all $\lambda\in(0,\lambda_\kappa)$ and $\varepsilon\in(0,\varepsilon_\kappa)$. Then, (2.10) and (2.9) are enough to show that $h\mapsto\mathscr{N}(h)$ is a contraction from $\{h\in\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4):\|h\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}\leqslant 2c_\kappa r_{\varepsilon,\lambda}^2\}$ into itself and hence has a unique fixed point h in this set. This fixed point is solution of (2.7) in $\bar B_{R_{\varepsilon,\lambda}}$. We summarize this in the following proposition.

Proposition 2.6. *Given $\delta\in(0,1)$ and $\kappa>0$, then there exist $\varepsilon_\kappa>0$, $\lambda_\kappa>0$ and $\bar c_\kappa>0$ (depending on κ) such that for all $\lambda\in(0,\lambda_\kappa)$, and for $\varepsilon\in(0,\varepsilon_\kappa)$, there exists a unique solution $h\in\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)$ solution of (2.7) such that*

$$v(x)=u_1(x)+h(x)$$

solves (2.4) in $\bar B_{R_{\varepsilon,\lambda}}$. In addition

$$\|h\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\delta}(\mathbb{R}^4)}\leqslant 2c_\kappa r_{\varepsilon,\lambda}^2.$$

2.2. **Analysis of the Bi-Laplace operator in weighted spaces.** In this section, we prove a surjectivity result of the bi-laplace operator in some weighted spaces and recall some estimations concerning the bi-harmonic extensions. First, given $x^1,\dots,x^m\in\Omega$ we define

$$\bar\Omega^*:=\bar\Omega-\{x^1,\dots x^m\},$$

and we choose $r_0>0$ so that the balls $B_{r_0}(x^i)$ of center x^i and radius r_0 are mutually disjoint and included in Ω. For $k\in\mathbb{N}$, $\alpha\in(0,1)$ and $\nu\in\mathbb{R}$, we introduce the Hölder weighted space $\mathcal{C}^{k,\alpha}_\nu(\bar\Omega^*)$ as the space of functions $w\in\mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\bar\Omega^*)$ endowed with the norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\nu(\bar\Omega^*)}:=\|w\|_{\mathcal{C}^{k,\alpha}(\bar\Omega-\cup_{j=1}^m B_{r_0/2}(x^j))}+\sum_{j=1}^m\sup_{0<r\leqslant r_0/2} r^{-\nu}\|w(x^j+r\cdot)\|_{\mathcal{C}^{k,\alpha}(B_2-B_1)}.$$

When $k\geqslant 2$, we let $[\mathcal{C}^{k,\alpha}_\nu(\bar\Omega^*)]_0$ be the subspace of functions $w\in\mathcal{C}^{k,\alpha}_\nu(\bar\Omega^*)$ satisfying $w=\Delta w=0$.

In this article, we need the following mapping properties of Δ^2.

Proposition 2.7 ([3]). *Assume that $\nu < 0$ and $\nu \notin \mathbb{Z}$, then*

$$\Delta^2 : [\mathcal{C}^{4,\alpha}_{\nu}(\bar{\Omega}^*)]_0 \to \mathcal{C}^{0,\alpha}_{\nu-4}(\bar{\Omega}^*)$$

is surjective.

Remark 2.8 ([3]). It is interesting to observe that, when $\nu < 0$, $\nu \notin \mathbb{Z}$, the right inverse even though it is not unique can be chosen to depend smoothly on the points $x^1, \dots, x^m$, at least locally. Once a right inverse is fixed for some choice of the points $x^1, \dots, x^m$, a right inverse which depends smoothly on the points $\tilde{x}^1, \dots, \tilde{x}^m$ close to $x^1, \dots, x^m$ can be obtained using a simple perturbation argument.

Proof of Proposition 2.7. Given $(\tilde{x}^i)$ close enough to (x^i), we define a family of diffeomorphism $D : \Omega \to \Omega$ depending smoothly on $(\tilde{x}^i)$ by

$$D(x) = x + \sum_{j=1}^{m} \chi_{r_0}(x - x^j)(x^j - \tilde{x}^j),$$

where χ_{r_0} is a cut-off function identically equal to 1 in $B_{r_0/2}$ and identically equal to 0 outside B_{r_0}. Hence $D(\tilde{x}^j) = x^j$ for each j. Then the equation $\Delta^2 \tilde{w} = \tilde{f}$ where $\tilde{f} \in \mathcal{C}^{0,\alpha}_{\nu-4}(\bar{\Omega} - \{\tilde{x}^i, 1 \leqslant i \leqslant m\})$ can be solved by considering $\tilde{w} = w \circ D$ where w is a solution of the problem

$$\Delta^2 w + \big[\Delta^2(w \circ D) - \Delta^2 w \circ D\big] \circ D^{-1} = \tilde{f} \circ D^{-1} \tag{2.11}$$

and this time $\tilde{f} \circ D^{-1} \in \mathcal{C}^{0,\alpha}_{\nu}(\bar{\Omega} - \{x^1, \dots, x^m\})$. It should be clear that

$$\big\| \big[\Delta^2(w \circ D) - \Delta^2 w \circ D\big] \circ D^{-1} \big\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(\bar{\Omega}^*)} \leqslant C \|w\|_{\mathcal{C}^{4,\alpha}_{\nu}(\bar{\Omega}^*)} \sup_{j=1,\dots,m} |\tilde{x}^j - x^j|.$$

Since we have a fixed right inverse for $\Delta^2 : \mathcal{C}^{4,\alpha}_{\nu}(\bar{\Omega}^*) \to \mathcal{C}^{0,\alpha}_{\nu-4}(\bar{\Omega}^*)$, a perturbation argument shows that (2.11) is solvable provided the $\tilde{x}^j$ are close enough to the x^j. This provides a right inverse which depends smoothly on the choice of the points $\tilde{x}^i$. □

2.3. **Bi-harmonic extensions.** Now, we give some estimates. More precisely, given $\varphi \in \mathcal{C}^{4,\alpha}(S^3)$ and $\psi \in \mathcal{C}^{2,\alpha}(S^3)$, we define $H^i (= H^i_{\varphi,\psi})$ to be the solution of

$$\begin{gathered} \Delta^2 H^i = 0 \quad \text{in } B_1 \\ H^i = \varphi \quad \text{on } \partial B_1 \\ \Delta H^i = \psi \quad \text{on } \partial B_1, \end{gathered}$$

where, as already mentioned, B_1 denotes the unit ball in $\mathbb{R}^4$. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we introduce the Hölder weighted spaces $\mathcal{C}^{k,\alpha}_{\nu}(\bar{B}_1^*)$ as the space of function in $\mathcal{C}^{k,\alpha}_{\rm loc}(\bar{B}_1^*)$ for which the following norm

$$\|u\|_{\mathcal{C}^{k,\alpha}_{\nu}(\bar{B}_1^*)} = \sup_{r \leqslant 1/2} r^{-\nu} \|u(r\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_2 - B_1)}$$

is finite. Here $\bar{B}_1^* = \bar{B}_1 - \{0\}$, therefore, this norm corresponds to the norm already defined in the previous section when $\Omega = B_1$, $m = 1$ and $x^1 = 0$. We denote by $e_1, \dots, e_4$ the coordinate functions on S^3.

Lemma 2.9 ([3]). *Assume that*

$$\int_{S^3}(8\varphi-\psi)\,d\sigma=0 \quad \textit{and} \quad \int_{S^3}(12\varphi-\psi)e_\ell\,d\sigma=0, \tag{2.12}$$

for $\ell=1,\dots,4$. *Then there exists* $c>0$ *such that*

$$\|H^i_{\varphi,\psi}\|_{\mathcal{C}^{4,\alpha}_2(\bar B_1^*)}\leqslant c(\|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)}+\|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)}).$$

Given $\varphi\in\mathcal{C}^{4,\alpha}(S^3)$ and $\psi\in\mathcal{C}^{2,\alpha}(S^3)$ we define (when it exists!) $H^e(=H^e_{\varphi,\psi})$ to be a solution of

$$\begin{gathered}\Delta^2H^e=0 \quad \text{in } \mathbb{R}^4-B_1\\ H^e=\varphi \quad \text{on } \partial B_1\\ \Delta H^e=\psi \quad \text{on } \partial B_1.\end{gathered}$$

which decays at infinity. Given $k\in\mathbb{N}$, $\alpha\in(0,1)$ and $\mu\in\mathbb{R}$, we introduce the Hölder weighted spaces $\mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4-B_1)$ as the space of function $w\in\mathcal{C}^{k,\alpha}_{\rm loc}(\mathbb{R}^4-B_1)$ for which the norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4-B_1)}=\sup_{r\geq1} r^{-\mu}\|w(r\cdot)\|_{\mathcal{C}^{k,\alpha}_\mu(\bar B_2-B_1)},$$

is finite.

Lemma 2.10 ([3]). *Assume that*

$$\int_{S^3}\psi\,d\sigma=0. \tag{2.13}$$

Then there exists $c>0$ *such that*

$$\|H^e_{\varphi,\psi}\|_{\mathcal{C}^{4,\alpha}_{-1}(\mathbb{R}^4-B_1)}\leqslant c(\|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)}+\|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)}).$$

Observe that, under the hypothesis of the Lemma, there is uniqueness of the bi-harmonic extension of the boundary data which decays at infinity.

If $E\subset L^2(S^3)$ is a space of functions defined on S^3, we define the space $E^\perp$ to be the subspace of functions which are L^2-orthogonal to the functions $1,e_1,\dots,e_4$.

Lemma 2.11 ([3]). *The mapping*

$$\begin{aligned}\mathcal{P}:\ \mathcal{C}^{4,\alpha}(S^3)^\perp\times\mathcal{C}^{2,\alpha}(S^3)^\perp&\to\mathcal{C}^{3,\alpha}(S^3)^\perp\times\mathcal{C}^{1,\alpha}(S^3)^\perp\\ (\varphi,\psi)&\mapsto(\partial_r(H^i_{\varphi,\psi}-H^e_{\varphi,\psi}),\partial_r(\Delta H^i_{\varphi,\psi}-\Delta H^e_{\varphi,\psi}))\end{aligned}$$

is an isomorphism.

3. Nonlinear interior problem

We are interested in studying equations of type

$$\Delta^2w+\mathscr{Q}_\lambda(w)-24e^w=0 \tag{3.1}$$

in $\bar B_{R_{\varepsilon,\lambda}}$. Given $\varphi\in\mathcal{C}^{4,\alpha}(S^3)$ and $\psi\in\mathcal{C}^{2,\alpha}(S^3)$. Let $\kappa>0$ (whose value will be fixed later on), we further assume that the functions φ, ψ satisfy

$$\|\varphi\|_{\mathcal{C}^{4,\alpha}}\leqslant\kappa r^2_{\varepsilon,\lambda} \quad \text{and} \quad \|\psi\|_{\mathcal{C}^{2,\alpha}}\leqslant\kappa r^2_{\varepsilon,\lambda}. \tag{3.2}$$

Define

$$\mathbf{v}:=u_1+H^i(\varphi,\psi,\cdot/R_{\varepsilon,\lambda})+h,$$

then we look for a solution of (3.1) of the form $w = \mathbf{v} + v$ and using the fact that H^i is biharmonic, this amounts to solving

$$\begin{aligned}\mathscr{L} v = \frac{384}{(1+|x|^2)^4}\ e^h(e^{H^i(\varphi,\psi,\cdot/R_{\varepsilon,\lambda})+v} - v - 1) + \frac{384}{(1+|x|^2)^4}\ (e^h-1)v \\ + \mathscr{Q}_\lambda(u_1+h) - \mathscr{Q}_\lambda\Big(u_1 + H^i(\varphi,\psi,\cdot/R_{\varepsilon,\lambda}) + h + v\Big).\end{aligned} \tag{3.3}$$

We fix $\mu \in (1,2)$ and denote by $\mathscr{G}_\mu$ the right inverse of $\mathscr{L}_\mu$ provided by Proposition 2.3. To obtain a solution of (3.3) it is sufficient to find $v \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ solution of

$$v = \mathscr{N}(v) := \mathscr{G}_\mu \circ \mathcal{E}_\mu \circ \mathscr{S}(v), \tag{3.4}$$

where

$$\begin{aligned}\mathscr{S}(v) := \frac{384}{(1+|x|^2)^4}\ e^h(e^{H^i(\varphi,\psi,\cdot/R_{\varepsilon,\lambda})+v} - v - 1) + \frac{384}{(1+|x|^2)^4}\ (e^h-1)v \\ + \mathscr{Q}_\lambda(u_1+h) - \mathscr{Q}_\lambda\Big(u_1 + H^i(\varphi,\psi,\cdot/R_{\varepsilon,\lambda}) + h + v\Big).\end{aligned} \tag{3.5}$$

We denote by $\mathscr{N}(=\mathscr{N}_{\varepsilon,\lambda,\varphi,\psi})$ the nonlinear operator appearing on the right-hand side of(3.4); then we have the following result.

Lemma 3.1. *For $\mu \in (1,2)$ and $\kappa > 0$, then there exist $\lambda_\kappa > 0$, $\varepsilon_\kappa > 0$, $c_\kappa > 0$ and $\bar{c}_\kappa > 0$ (depending on κ) such that for all $\lambda \in (0,\lambda_\kappa)$ and $\varepsilon \in (0,\varepsilon_\kappa)$,*

$$\|\mathscr{N}(0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant c_\kappa r^2_{\varepsilon,\lambda}. \tag{3.6}$$

Moreover,

$$\|\mathscr{N}(v_2) - \mathscr{N}(v_1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant \bar{c}_\kappa r^2_{\varepsilon,\lambda} \|v_2 - v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}, \tag{3.7}$$

provided that $v_1, v_2 \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$, satisfy

$$\|v_i\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant 2c_\kappa r^2_{\varepsilon,\lambda}.$$

Proof. The proof of the first estimate follows from the asymptotic behavior of H^i together with the assumption on the norm of boundary data φ and ψ given by (3.2). Indeed, let c_κ be a constant depending only on κ (provided ε and λ are chosen small enough) it follows from the estimate of H^i, given by Lemma 2.9, that

$$\|H^i(\varphi,\psi,\cdot/R_{\varepsilon,\lambda})\|_{\mathcal{C}^{4,\alpha}_2(\bar{B}_{R_{\varepsilon,\lambda}})} \leqslant cR^{-2}_{\varepsilon,\lambda}(\|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)}) \leqslant c_\kappa \varepsilon^2.$$

Since for each $x \in \bar{B}_{R_{\varepsilon,\lambda}}$, we have

$$|h(x)| \leqslant c_\kappa r^{2+\delta}_{\varepsilon,\lambda}\varepsilon^{-\delta} \leqslant \begin{cases}\varepsilon^{1-\delta/2} & \text{for } \varepsilon \geqslant \lambda \\ \lambda^{1+\delta/2}\varepsilon^{-\delta} & \text{for } \lambda > \varepsilon.\end{cases}$$

Then, using condition (A1) , we prove that $|h(x)| \to 0$ as ε and λ tend to 0. Given $\kappa > 0$, there exist $c_\kappa > 0$ such that

$$\|(1+|\cdot|^2)^{-2}e^h\big(e^{H^i(\varphi,\psi,\cdot/R_{\varepsilon,\lambda})} - 1\big)\|_{\mathcal{C}^{0,\alpha}_{\mu-2}(\bar{B}_{R_{\varepsilon,\lambda}})} \leqslant c_\kappa \varepsilon^2.$$

On the other hand, using condition (A1), we obtain

$$\sup_{r \leqslant R_{\varepsilon,\lambda}} (1+r^2)^{2-\frac{\mu}{2}} |\mathscr{Q}_\lambda(u_1+h) - \mathscr{Q}_\lambda\big(u_1 + H^i(\varphi,\psi,\cdot/R_{\varepsilon,\lambda}) + h\big)| \leqslant c_\kappa r^2_{\varepsilon,\lambda}.$$

By Proposition 2.3 and (2.6), for $\mu \in (1,\ 2)$, we obtain

$$\|\mathscr{N}(0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant c_\kappa r^2_{\varepsilon,\lambda}.$$

To derive the second estimate, let $v_i \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ satisfy $\|v_i\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant 2c_\kappa r^2_{\varepsilon,\lambda}$, $i = 1, 2$, $\mu \in (1, 2)$ and condition (A1). Hence there exist $c_\kappa > 0$ such that

$$
\begin{aligned}
&\|(1 + |\cdot|^2)^{-4} e^{H^i(\varphi,\psi,\cdot/R_{\varepsilon,\lambda})}(e^{v_2} - e^{v_1} - (v_2 - v_1))\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar B_{R_{\varepsilon,\lambda}})}\\
&\leqslant c_\kappa \varepsilon^2 \|v_2 - v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)},\\
&\quad \|(e^h - 1)(v_2 - v_1)\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar B_{R_{\varepsilon,\lambda}})} \leqslant c_\kappa r^2_{\varepsilon,\lambda} \|v_2 - v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)},\\
&\qquad \big\|\mathscr{Q}_\lambda\big(u_1 + H^i(\varphi, \psi, \cdot/R_{\varepsilon,\lambda}) + h + v_2\big)\\
&\qquad - \mathscr{Q}_\lambda\big(u_1 + H^i(\varphi, \psi, \cdot/R_{\varepsilon,\lambda}) + h + v_1\big)\big\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(\bar B_{R_{\varepsilon,\lambda}})}\\
&\qquad \leqslant c_\kappa r^2_{\varepsilon,\lambda} \big\|v_2 - v_1\big\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.
\end{aligned}
$$

So

$$
\sup_{r \leqslant R_{\varepsilon,\lambda}} (1 + r^2)^{2-\frac{\mu}{2}} |\mathscr{S}(v_2) - \mathscr{S}(v_1)| \leqslant c_\kappa r^2_{\varepsilon,\lambda} \|v_2 - v_1\|_{\mathcal{C}^{4,\alpha}_{\mathrm{rad},\mu}(\mathbb{R}^4)}.
$$

Similarly, using Proposition 2.3 and (2.6), we conclude that there exist $\bar c_\kappa > 0$ such that

$$
\|\mathscr{N}(v_2) - \mathscr{N}(v_1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant \bar c_\kappa r^2_{\varepsilon,\lambda} \|v_2 - v_1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.
$$

□

Reducing $\lambda_\kappa > 0$ and $\varepsilon_\kappa > 0$ if necessary, we can assume that

$$
\bar c_\kappa r^2_{\varepsilon,\lambda} \leqslant \frac{1}{2}, \tag{3.8}
$$

for all $\lambda \in (0, \lambda_\kappa)$ and $\varepsilon \in (0, \varepsilon_\kappa)$. Then, (3.6) and (3.7) in Lemma 3.1 are sufficient to show that $v \mapsto \mathscr{N}(v)$ is a contraction from

$$
\big\{v \in \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \ :\ \|v\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant 2c_\kappa \varepsilon^2\big\}
$$

into itself and hence has a unique fixed point $v = v(\varepsilon, \tau, \varphi, \psi; \cdot)$ in this set. This fixed point is a solution of (3.4) in $\mathbb{R}^4$. We summarize this in the following proposition.

Proposition 3.2. *For $\mu \in (1, 2)$ and $\kappa > 0$ there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$ and $c_\kappa > 0$ (depending on κ) such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\lambda \in (0, \lambda_\kappa)$ satisfying* (A1)*, for all τ in some fixed compact subset of $[\tau_-, \tau^+] \subset (0, \infty)$ and for a given φ and ψ satisfying* (2.12)-(3.2)*, then there exists a unique $v(:= \bar v_{\varepsilon,\tau,\varphi,\psi})$ solution of* (3.4) *such that*

$$
w := u_1 + H^i(\varphi, \psi, \cdot/R_{\varepsilon,\lambda}) + h + \bar v_{\varepsilon,\tau,\varphi,\psi}
$$

solve (3.1) *in $\bar B_{R_{\varepsilon,\lambda}}$. In addition*

$$
\|v\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant 2c_\kappa r^2_{\varepsilon,\lambda}.
$$

4. Nonlinear exterior problem

Denote $G_{\tilde x} = G(x, \tilde x)$ where G is the Green function given by (1.3) and $H(x, \tilde x)$ its regular part. Clearly $x \mapsto H(x, \tilde x)$ is a smooth function.

Let $\tilde{\mathbf{x}} = (\tilde x^j) \in \Omega^m$ close to $\mathbf{x} = (x^j)$, $\tilde{\boldsymbol\eta} = (\tilde\eta^j) \in \mathbb{R}^m$ close to 0. Let $\tilde{\boldsymbol\varphi} = (\tilde\varphi^j) \in (\mathcal{C}^{4,\alpha}(S^3))^m$ and $\tilde{\boldsymbol\psi} = (\tilde\psi^j) \in (\mathcal{C}^{2,\alpha}(S^3))^m$ satisfy (2.13). We define

$$
\tilde{\mathbf{u}} = \tilde u_{\varepsilon,\tilde{\boldsymbol\eta},\tilde{\mathbf{x}},\tilde{\boldsymbol\varphi},\tilde{\boldsymbol\psi}} := \sum_{j=1}^m (1 + \tilde\eta^j) G_{\tilde x^j} + \sum_{j=1}^m \chi_{r_0}(x - \tilde x^j) H^e_{\tilde\varphi^j,\tilde\psi^j}\big(\frac{x - \tilde x^j}{r_{\varepsilon,\lambda}}\big),
$$

where χ_{r_0} is a cut-off function identically equal to 1 in $B_{r_0/2}$ and identically equal to 0 outside B_{r_0}. We would like to solve the equation

$$\Delta^2 u + \mathscr{Q}_\lambda(u) - \rho^4 e^u = 0, \quad \text{in } \Omega - \cup_{1\leqslant j\leqslant m} B_{r_{\varepsilon,\lambda}(\tilde{x}^j)}, \tag{4.1}$$

with $u = \tilde{\mathbf{u}} + \tilde{v}$ is a perturbation of $\tilde{\mathbf{u}}$. This amounts to solve

$$\Delta^2 \tilde{v} = \rho^4 e^{\tilde{\mathbf{u}}}\, e^{\tilde{v}} - \mathscr{Q}_\lambda(\tilde{\mathbf{u}} + \tilde{v}) - \Delta^2 \tilde{\mathbf{u}} = \tilde{\mathscr{S}}(\tilde{v}).$$

Denote $\Omega_{R,\tilde{x}} = \Omega - \cup_{1\leqslant j\leqslant m} B_R(\tilde{x}^j)$ for any $R > 0$. We denote by $\tilde{\xi}_R : \mathcal{C}^{0,\alpha}_\nu(\bar{\Omega}_{R,\tilde{x}}) \to \mathcal{C}^{0,\alpha}_\nu(\bar{\Omega}^*)$ the extension operator defined by

$$\tilde{\xi}_R(f) \equiv f \quad \text{in } \Omega_{R,\tilde{x}},$$
$$\tilde{\xi}_R(f)(x_i + x) = \frac{2|x| - R}{R} f(x_i + \frac{Rx}{|x|}) \quad \text{in } B_R(\tilde{x}^j)\backslash B_{R/2}(\tilde{x}^j), \quad \forall 1 \leqslant j \leqslant m,$$
$$\tilde{\xi}_R(f) \equiv 0 \quad \text{in } \cup_j B_{R/2}(\tilde{x}^j).$$

It easy to check that there exist a constant $c = c(\nu) > 0$, only depending on ν such that

$$\|\tilde{\xi_R}(w)\|_{\mathcal{C}^{0,\alpha}_\nu(\bar{\Omega}^*)} \leqslant c\|w\|_{\mathcal{C}^{0,\alpha}_\nu(\bar{\Omega}_{R,\tilde{x}})}. \tag{4.2}$$

We fix $\nu \in (-1, 0)$, and denote by $\tilde{\mathscr{G}_\nu}$ the right inverse provided by Proposition 2.7. Clearly, it is enough to find $\tilde{v} \in \mathcal{C}^{4,\alpha}_\nu(\Omega^*)$ solution of

$$\tilde{v} = \tilde{\mathscr{G}_\nu} \circ \tilde{\xi}_{r_{\varepsilon,\lambda}} \circ \tilde{\mathscr{S}}(\tilde{v}). \tag{4.3}$$

We denote by $\tilde{\mathscr{N}}(\tilde{v})$ $(= \tilde{\mathscr{N}}_{\varepsilon,\boldsymbol{\eta},\tilde{\mathbf{x}},\tilde{\boldsymbol{\varphi}},\tilde{\boldsymbol{\psi}}}(\tilde{v})) = \tilde{\mathscr{G}_\nu} \circ \tilde{\xi}_{r_{\varepsilon,\lambda}} \circ \tilde{\mathscr{S}}(\tilde{v})$, the nonlinear operator on the right-hand side. Even though this is not notified in the notation, $\tilde{\mathscr{G}_\nu} : \mathcal{C}^{0,\alpha}_{\nu-4}(\bar{\Omega}^*) \to \mathcal{C}^{4,\alpha}_\nu(\bar{\Omega}^*)$ is the right inverse defined in Remark 2.8 with $\bar{\Omega}^* = \bar{\Omega} - \{\tilde{x}^1, \dots, \tilde{x}^m\}$.

Given $\kappa > 0$ (whose value will be fixed later on), we further assume that, the functions $\tilde{\varphi}^j$ and $\tilde{\psi}^j$ satisfy

$$\|\tilde{\varphi}^j\|_{\mathcal{C}^{4,\alpha}} \leqslant \kappa r^2_{\varepsilon,\lambda} \quad \text{and} \quad \|\tilde{\psi}^j\|_{\mathcal{C}^{2,\alpha}} \leqslant \kappa r^2_{\varepsilon,\lambda}, \quad \forall j = 1, \dots, m. \tag{4.4}$$

Moreover, we assume that the parameters $\tilde{\eta}^j$ and the points $\tilde{x}^j$ are chosen to verify

$$|\tilde{\eta}^j| \leqslant \kappa r^2_{\varepsilon,\lambda} \quad \text{and} \quad r_{\varepsilon,\lambda}|\tilde{x}^j - x^j| \leqslant \kappa r^2_{\varepsilon,\lambda}. \tag{4.5}$$

Then the following result holds.

Lemma 4.1. *Given $\nu \in (-1, 0)$ and $\kappa > 0$, there exist $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ (depending on κ) such that for all $\varepsilon \in (0, \varepsilon_\kappa)$ and under the assumptions* (4.4) *and* (4.5)*, we have*

$$\|\tilde{\mathscr{N}}(0)\|_{\mathcal{C}^{4,\alpha}_\nu(\bar{\Omega}^*)} \leqslant c_\kappa r^2_{\varepsilon,\lambda},$$
$$\|\tilde{\mathscr{N}}(\tilde{v}_2) - \tilde{\mathscr{N}}(\tilde{v}_1)\|_{\mathcal{C}^{4,\alpha}_\nu(\bar{\Omega}^*)} \leqslant \bar{c}_\kappa r^2_{\varepsilon,\lambda}\|\tilde{v}_2 - \tilde{v}_1\|_{\mathcal{C}^{4,\alpha}_\nu(\bar{\Omega}^*)},$$

provided that $\tilde{v}_1, \tilde{v}_2 \in \mathcal{C}^{4,\alpha}_\nu(\bar{\Omega}^*)$ and $\|\tilde{v}_i\|_{\mathcal{C}^{4,\alpha}_\nu(\bar{\Omega}^*)} \leqslant 2c_\kappa r^2_{\varepsilon,\lambda}$.

Proof. The proof of the first estimate follows from the asymptotic behavior of H_e together with the assumption on the norm of boundary data $\tilde{\varphi}_j$ and $\tilde{\psi}_j$ given by (4.4). Indeed, let c_κ be a constant depending only on κ (provided ε and λ are chosen small enough), it follows from the estimate of H_e, given by Lemma 2.10, that

$$\big|H^e_{\tilde{\varphi}_j,\tilde{\psi}_j}(\frac{x - \tilde{x}^j}{r_{\varepsilon,\lambda}})\big| \leqslant c_\kappa\, r^3_{\varepsilon,\lambda}\, r^{-1}. \tag{4.6}$$

Recall that $\tilde{\mathscr{N}}(\tilde{v}) = \tilde{\mathscr{G}}_\nu \circ \tilde{\xi}_{r_\varepsilon} \circ \tilde{\mathscr{S}}(\tilde{v})$, we will estimate $\tilde{\mathscr{N}}(0)$ in different subregions of $\bar{\Omega}^*$.

- In $B_{r_0}(\tilde{x}^j)$ for $1 \leqslant j \leqslant m$, we have $\chi_{r_0}(x - \tilde{x}^j) = 1$ and $\Delta^2 \tilde{\mathbf{u}} = 0$, so that

$$
\begin{aligned}
|\tilde{\mathscr{S}}(0)| &\leqslant c\varepsilon^4 \prod_{j=1}^m \big[e^{(1+\tilde{\eta}^j) G_{\tilde{x}^j}(x) + H^e_{\tilde{\varphi}^j, \tilde{\psi}^j}((x-\tilde{x}^j)/r_\varepsilon)} - \mathscr{Q}_\lambda(\tilde{\mathbf{u}}) \big] \\
&\leqslant c\varepsilon^4 \prod_{j=1}^m |x - \tilde{x}^j|^{-8(1+\tilde{\eta}^j)} + |\mathscr{Q}_\lambda(\tilde{\mathbf{u}})|.
\end{aligned}
$$

So, by an easy computation, for $\nu \in (-1, 0)$ and $\tilde{\eta}^j$ small enough, we obtain

$$
\|\tilde{\mathscr{S}}(0)\|_{\mathcal{C}^{4,\alpha}_\nu \left(\cup_{j=1}^m B(\tilde{x}^j, r_0) \right)} \leqslant \sup_{r_{\varepsilon,\lambda} \leqslant r \leqslant r_0/2} r^{4-\nu} |\tilde{\mathscr{S}}(0)| \leqslant c_\kappa \big(\varepsilon^4 r_{\varepsilon,\lambda}^{-4} + \lambda \big).
$$

- In $\Omega - B_{r_0}(\tilde{x}^j)$, we have $\chi_{r_0}(x - \tilde{x}^j) = 0$ and $\Delta^2 \tilde{\mathbf{u}} = 0$, then

$$
|\tilde{\mathscr{S}}(0)| \leqslant c \big(\varepsilon^4 \prod_{j=1}^m e^{(1+\tilde{\eta}^j) G_{\tilde{x}^j}} + |\mathscr{Q}_\lambda(\tilde{\mathbf{u}})| \big).
$$

Thus

$$
\|\tilde{\mathscr{S}}(0)\|_{\mathcal{C}^{4,\alpha}_\nu (\Omega_{r_0, \tilde{x}})} \leqslant c_\kappa \sup_{r \geqslant r_0} r^{4-\nu} |\tilde{\mathscr{S}}(0)| \leqslant c_\kappa \big(\varepsilon^4 + \lambda \big).
$$

- In $B_{r_0}(\tilde{x}^j) - B_{r_0/2}(\tilde{x}^j)$, using estimate (4.6), we have

$$
\begin{aligned}
|\tilde{\mathscr{S}}(0)| &\leqslant c_\kappa \varepsilon^4 \prod_{j=1}^m |x - \tilde{x}^j|^{-8-\tilde{\eta}^j} + |\mathscr{Q}_\lambda(\tilde{\mathbf{u}})| \\
&\quad + c\varepsilon^4 \sum_{j=1}^m |[\Delta^2, \chi_{r_0}](x - \tilde{x}^j)| |H^{\rm ext}_{\tilde{\varphi}_j, \tilde{\psi}_j}((x - \tilde{x}^j)/r_{\varepsilon,\lambda})|.
\end{aligned}
$$

Here

$$
[\Delta^2, \chi_{r_0}] w = 2\Delta\chi_{r_0} \Delta w + w \Delta^2 \chi_{r_0} + 4 \nabla \chi_{r_0} \cdot \nabla(\Delta w) + 4 \nabla w \cdot \nabla(\Delta \chi_{r_0}) + 4 \nabla^2 \chi_{r_0} \cdot \nabla^2 w.
$$

So,

$$
\|\tilde{\mathscr{S}}(0)\|_{\mathcal{C}^{4,\alpha}_\nu (B(\tilde{x}^j, r_0) - B(\tilde{x}^j, r_0/2))} \leqslant c_\kappa \sup_{r_0/2 \leqslant r \leqslant r_0} r^{4-\nu} |\tilde{\mathscr{S}}(0)| \leqslant c_\kappa \big(r_{\varepsilon,\lambda}^2 + \lambda \big).
$$

Finally, using Proposition 2.7 with (4.2), we conclude that

$$
\|\tilde{\mathscr{N}}(0)\|_{\mathcal{C}^{4,\alpha}_\nu (\bar{\Omega}^*)} \leqslant c_\kappa r_{\varepsilon,\lambda}^2. \tag{4.7}
$$

For the proof of the second estimate, let $\tilde{v}_1$ and $\tilde{v}_2 \in C^{4,\alpha}_\nu(\bar{\Omega}^*)$ satisfying $\|\tilde{v}_i\|_{\mathcal{C}^{4,\alpha}_\nu} \leqslant c_\kappa r_{\varepsilon,\lambda}^2$, so

$$
|\big(\tilde{\mathscr{S}}(\tilde{v}_2) - \tilde{\mathscr{S}}(\tilde{v}_1) \big)| \leqslant c_\kappa |\rho^4 e^{\tilde{\mathbf{u}}} (e^{\tilde{v}_2} - e^{\tilde{v}_1}) - \big(\mathscr{Q}_\lambda(\tilde{\mathbf{u}} + \tilde{v}_2) - \mathscr{Q}_\lambda(\tilde{\mathbf{u}} + \tilde{v}_1) \big)|.
$$

Then, for $\tilde{\eta}^j$ small enough and using estimate (4.2), there exist $\bar{c}_\kappa > 0$ (depending on κ) such that

$$
\|\tilde{\mathscr{N}}(\tilde{v}_2) - \tilde{\mathscr{N}}(\tilde{v}_1)\|_{\mathcal{C}^{4,\alpha}_\nu (\bar{\Omega}^*)} \leqslant \bar{c}_\kappa r_{\varepsilon,\lambda}^2 \|\tilde{v}_2 - \tilde{v}_1\|_{\mathcal{C}^{4,\alpha}_\nu (\bar{\Omega}^*)}. \tag{4.8}
$$

Then we get the second estimate. $\square$

Reducing $\lambda_\kappa > 0$ and $\varepsilon_\kappa > 0$ if necessary, we can assume that

$$\bar{c}_\kappa r_{\varepsilon,\lambda}^2 \leqslant \frac{1}{2}\,, \tag{4.9}$$

for all $\lambda \in (0, \lambda_\kappa)$ and $\varepsilon \in (0, \varepsilon_\kappa)$. Then, (4.8) and (4.7) are sufficient to show that $\tilde{v} \mapsto \tilde{\mathscr{N}}(\tilde{v})$ is a contraction from

$$\Big\{\tilde{v} \in \mathcal{C}_\nu^{4,\alpha}(\mathbb{R}^4) \ :\ \|\tilde{v}\|_{\mathcal{C}_\nu^{4,\alpha}(\mathbb{R}^4)} \leqslant 2c_\kappa r_{\varepsilon,\lambda}^2\Big\}$$

into itself and hence has a unique fixed point $\tilde{v} = \tilde{v}(\varepsilon, \tau, \varphi, \psi; \cdot)$ in this set. This fixed point is a solution of (4.3) in $\mathbb{R}^4$. We summarize this in the following proposition.

Proposition 4.2. *Given $\nu \in (-1, 0)$ and $\kappa > 0$, there exist $\varepsilon_\kappa > 0$, $\lambda_\kappa > 0$ and $c_\kappa > 0$ (depending on κ) such that for all $\varepsilon \in (0, \varepsilon_\kappa)$ and $\lambda \in (0, \lambda_\kappa)$, for all set of parameters $\tilde{\eta}^j$ and points $\tilde{x}^j$ satisfying* (4.5)*, all functions $\tilde{\varphi}^j$, $\tilde{\psi}^j$ satisfying* (2.13) *and* (4.4)*, there exists a unique $\tilde{v}$ $(= \tilde{v}_{\varepsilon,\boldsymbol{\eta},\tilde{\mathbf{x}},\tilde{\boldsymbol{\varphi}},\tilde{\boldsymbol{\psi}}})$ solution of* (4.3)*, such that*

$$\tilde{u}_{\varepsilon,\boldsymbol{\eta},\tilde{\mathbf{x}},\tilde{\boldsymbol{\varphi}},\tilde{\boldsymbol{\psi}}} := \sum_{j=1}^m (1+\tilde{\eta}^j) G_{\tilde{x}^j} + \sum_{j=1}^m \chi_{r_0}(\cdot - \tilde{x}^j) H^e_{\tilde{\varphi}^j,\tilde{\psi}^j}\Big(\frac{x-\tilde{x}^j}{r_\varepsilon}\Big) + \tilde{v}_{\varepsilon,\tilde{\eta},\tilde{x},\tilde{\varphi},\tilde{\psi}}$$

solves (4.1) *in $\bar{\Omega}^*$. In addition*

$$\|\tilde{v}\|_{\mathcal{C}_\nu^{4,\alpha}(\bar{\Omega}^*)} \leqslant 2c_\kappa r_{\varepsilon,\lambda}^2.$$

As in the previous section, observe that the function $\tilde{v}_{\varepsilon,\tilde{\eta},\tilde{x},\tilde{\varphi},\tilde{\psi}}$ being obtained as a fixed point for contraction mapping, it depends smoothly on the parameters $\tilde{\eta}^j$, the points $\tilde{x}^j$ and the boundary data $\tilde{\varphi}^j$ and $\tilde{\psi}^j$, for $j = 1, \dots, m$. Moreover, as in the previous section, the mapping

$$(\boldsymbol{\eta}, \tilde{\mathbf{x}}, \tilde{\boldsymbol{\varphi}}, \tilde{\boldsymbol{\psi}}) \mapsto \tilde{v}_{\varepsilon,\boldsymbol{\eta},\tilde{\mathbf{x}},\tilde{\boldsymbol{\varphi}},\tilde{\boldsymbol{\psi}}} \circ D^{-1}|_{\Omega_{r_{\varepsilon,\lambda},\tilde{x}}} \in \mathcal{C}^{4,\alpha}(\Omega_{r_{\varepsilon,\lambda},\tilde{x}})$$

is compact (here D is the diffeomorphism defined in §2.2). Again this follows from the fact that the equation we solve is semilinear and in (4.3) the right hand side belongs to $\mathcal{C}^{8,\alpha}(\bar{\Omega}^*)$.

5. Nonlinear Cauchy-data matching

We will gather the results of the previous sections, keeping the notations, applying the result of § 2, § 3, as well as the results of § 4. Assume that $\tilde{\mathbf{x}} = (\tilde{x}^i) \in \Omega^m$ are given close enough to $\mathbf{x} = (x^i)$ such that it satisfies (4.5), assume also $\boldsymbol{\tau} = (\tau^i) \in [\tau^-, \tau^+]^m \subset (0, \infty)^m$ (the values of τ^- and τ^+ will be fixed shortly). First, we consider some set of boundary data $\varphi = (\varphi^i) \in (\mathcal{C}^{4,\alpha}(S^3))^m$ and $\psi = (\psi^i) \in (\mathcal{C}^{2,\alpha}(S^3))^m$ satisfying (2.12) and (3.2). According to Proposition 3.2, and provided $\varepsilon \in (0, \varepsilon_\kappa)$, we can find a solution of

$$\Delta^2 u + \mathscr{Q}_\lambda(u) - \rho^4 e^u = 0 \quad \text{in } B_{r_{\varepsilon,\lambda}}(\tilde{x}^j)\ \forall 1 \leqslant j \leqslant m.$$

These solutions can be decomposed (in each $B_{r_{\varepsilon,\lambda}}(\tilde{x}^j)$) as

$$\begin{aligned} u_{int,j}(x) &= u_{\varepsilon,\tau^j}(x-\tilde{x}^j) + h\Big(\frac{R^j_{\varepsilon,\lambda}(x-\tilde{x}^j)}{r_{\varepsilon,\lambda}}\Big) + H^i_{\varphi^j,\psi^j}\Big(\frac{x-\tilde{x}^j}{r_{\varepsilon,\lambda}}\Big) \\ &\quad + v_{\varepsilon,\tau^j,\varphi^j,\psi^j}\Big(\frac{R^j_{\varepsilon,\lambda}(x-\tilde{x}^j)}{r_{\varepsilon,\lambda}}\Big) \end{aligned}$$

where $R^j_{\varepsilon,\lambda} = \tau^j r_{\varepsilon,\lambda}/\varepsilon$ and the function $v^j = v_{\varepsilon,\tau^j,\varphi^j,\psi^j}$ satisfies

$$\|v^j\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \leqslant 2\, c_\kappa r^2_{\varepsilon,\lambda}. \tag{5.1}$$

Similarly, given some boundary data $\tilde{\boldsymbol{\varphi}} := (\tilde{\varphi}^i) \in (\mathcal{C}^{4,\alpha}(S^3))^m$ and $\tilde{\boldsymbol{\psi}} = (\tilde{\psi}^i) \in (\mathcal{C}^{2,\alpha}(S^3))^m$ satisfying (2.13) and (4.4), some parameters $\tilde{\boldsymbol{\eta}} := (\tilde{\eta}^i) \in \mathbb{R}^m$ satisfying (4.5), we can use Proposition 4.2 to find a solution u_{ext} (provided $\varepsilon \in (0, \varepsilon_\kappa)$) of

$$\Delta^2 u + \mathscr{Q}_\lambda(u) - \rho^4 e^u = 0, \quad \text{in } B_{r_{\varepsilon,\lambda}}(\tilde{x}^j),\ \forall 1 \leqslant j \leqslant m.$$

Here the solution can be decomposed as

$$u_{ext}(x) = \sum_{j=1}^m (1+\tilde{\eta}^j) G_{\tilde{x}^j}(x) + \sum_{j=1}^m \chi_{r_0}(x-\tilde{x}^j) H^e_{\tilde{\varphi}^j,\tilde{\psi}^j}\Big(\frac{x-\tilde{x}^j}{r_{\varepsilon,\lambda}}\Big) + \tilde{v}_{\varepsilon,\tilde{\boldsymbol{\eta}},\tilde{\mathbf{x}},\tilde{\boldsymbol{\varphi}},\tilde{\boldsymbol{\psi}}}(x),$$

where the function $\tilde{v}^j := \tilde{v}_{\varepsilon,\tilde{\boldsymbol{\eta}},\tilde{\mathbf{x}},\tilde{\boldsymbol{\varphi}},\tilde{\boldsymbol{\psi}}} \in \mathcal{C}^{4,\alpha}_\nu(\bar{\Omega}^*)$ satisfies

$$\|\tilde{v}^j\|_{\mathcal{C}^{4,\alpha}_\nu(\bar{\Omega}^*)} \leqslant c_\kappa r^2_{\varepsilon,\lambda}. \tag{5.2}$$

It remains to determine the parameters and the functions is such a way that the function which is equal to $u_{int,j}$ in $B_{r_{\varepsilon,\lambda}}(\tilde{x}^j)$ and which is equal to u_{ext} in $\Omega_{r_{\varepsilon,\lambda},\tilde{x}}$ will become a smooth function. This amounts to find the boundary data and the parameters so that, for each $j = 1,\dots,m$

$$u_{int,j} = u_{ext}, \quad \partial_r u_{int,j} = \partial_r u_{ext}, \quad \Delta u_{int,j} = \Delta u_{ext}, \quad \partial_r \Delta u_{int,j} = \partial_r \Delta u_{ext} \tag{5.3}$$

on $\partial B_{r_{\varepsilon,\lambda}}(\tilde{x}^j)$. Assuming we have already (5.3) (for all ε small enough), the function $u_\varepsilon \in \mathcal{C}^{4,\alpha}$ obtained by patching together the functions $u_{int,j}$ and the function u_{ext}, is a solution of our equation. Then the elliptic regularity theory implies that this solution is in fact smooth. This will complete the proof of our result. Because when as ε tends to 0, the sequence of solutions constructed will satisfy the required properties, namely, away from the points x^j the sequence u_ε converges to $\sum_j G_{x^j}$. Before we proceed, the following remarks are important. It will be convenient to observe that the functions u_{ε,τ^j} can be expanded as

$$u_{\varepsilon,\tau^j}(x) = -8\log|x| - 4\log\tau^j + \mathcal{O}(r^2_{\varepsilon,\lambda}) \tag{5.4}$$

near $\partial B_{r_{\varepsilon,\lambda}}$. Moreover, the function

$$\sum_{1\leqslant j\leqslant m} (1+\tilde{\eta}^j) G_{\tilde{x}^j}(x)$$

which appears in the expression of u_{ext} can be expanded as

$$\sum_{\ell=1}^m (1+\tilde{\eta}^\ell) G_{\tilde{x}^\ell}(\tilde{x}^j + x) = -8(1+\tilde{\eta}^j)\log|x| + E_j(\tilde{x}^j,\tilde{\mathbf{x}}) + \nabla_x E_j(\tilde{x}^j,\tilde{\mathbf{x}})\cdot x + \mathcal{O}(r^2_{\varepsilon,\lambda}) \tag{5.5}$$

near $\partial B_{r_{\varepsilon,\lambda}}$, where we define

$$E_j(x,\tilde{\mathbf{x}}) := H(x,\tilde{x}^j) + \sum_{\ell\neq j} G(x,\tilde{x}^\ell).$$

Next, in (5.3), all functions are defined on $\partial B_{r_{\varepsilon,\lambda}}(\tilde{x}^j)$, nevertheless, it will be convenient to solve, instead of (5.3) the following set of equations

$$\begin{gathered}(u_{int,j} - u_{ext})(\tilde{x}^j + r_{\varepsilon,\lambda} y) = 0, \quad \partial_r(u_{int,j} - u_{ext})(\tilde{x}^j + r_{\varepsilon,\lambda} y) = 0,\\ \Delta(u_{int,j} - u_{ext})(\tilde{x}^j + r_{\varepsilon,\lambda} y) = 0, \quad \partial_r\Delta(u_{int,j} - u_{ext})(\tilde{x}^j + r_{\varepsilon,\lambda} y) = 0,\end{gathered} \tag{5.6}$$

on S^3.

Also we decompose

$$\varphi^j = \varphi_0^j + \varphi_1^j + \varphi_\perp^j, \quad \psi^j = 8\varphi_0^j + 12\varphi_1^j + \psi_\perp^j, \quad \tilde\varphi^j = \tilde\varphi_0^j + \tilde\varphi_1^j + \tilde\varphi_\perp^j \quad \tilde\psi^j = \tilde\psi_1^j + \tilde\psi_\perp^j$$

where $\varphi_0^j, \tilde\varphi_0 \in \mathbb{E}_0 = \mathbb{R}$, $\varphi_1^j, \tilde\varphi_1^j, \tilde\psi_1^j \in \mathbb{E}_1 = \operatorname{span}\{e_1, \dots, e_4\}$ and $\varphi_\perp^j, \psi_\perp^j, \tilde\varphi_\perp^j, \tilde\psi_\perp^j \in L^2(S^3)^\perp$, the subspace of functions which are orthogonal to $\mathbb{E}_0$ and $\mathbb{E}_1$.

Projecting the set of equations (5.6) over $\mathbb{E}_0$ will yield the system

$$\begin{gathered} -4\log\tau^j - 8\log r_{\varepsilon,\lambda} + \varphi_0^j + 8(1+\tilde\eta^j)\log r_{\varepsilon,\lambda} - \tilde\varphi_0^j - E_j(\tilde x^j, \tilde{\mathbf{x}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0 \\ -8 + 2\varphi_0^j + 8(1+\tilde\eta^j) + 2\tilde\varphi_0^j + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0 \\ -16 + 8\varphi_0^j + 16(1+\tilde\eta^j) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0 \\ 32 - 32(1+\tilde\eta^j) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0. \end{gathered} \tag{5.7}$$

For the rest of this article, the terms $\mathcal{O}(r_{\varepsilon,\lambda}^2)$ depend nonlinearly on the variables $\tau^\ell, \tilde x^\ell, \varphi^\ell, \psi^\ell$, $\tilde\varphi^\ell, \tilde\psi^\ell$, but it is bounded (in the appropriate norm) by a constant (independent of ε and κ) time $r_{\varepsilon,\lambda}^2$. Let us comment briefly on how these equations are obtained. These equations simply come from (5.6) when expansions (5.4) and (5.5) are used, together with the expression of H^i and H^e given in Lemma 2.9 and Lemma 2.10, and also the estimates (5.1) and (5.2).

Observe that the projection of the term $\nabla_x E_j(\tilde x^j, \tilde{\mathbf{x}}) \cdot y$ arising in (5.5), as well as the projection of its partial derivative with respect to r, over the set of constant function is equal to 0, while its Laplacian vanishes identically. The system (5.7) can be readily simplified to

$$\begin{gathered} \frac{1}{\log r_{\varepsilon,\lambda}}[4\log\tau^j + E_j(\tilde x^j, \tilde{\mathbf{x}})] = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \tilde\eta^j = \mathcal{O}(r_{\varepsilon,\lambda}^2), \\ \varphi_0^j = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \tilde\varphi_0^j = \mathcal{O}(r_{\varepsilon,\lambda}^2). \end{gathered}$$

We are now in a position to define τ^- and τ^+ since, according to the above, as ε tends to 0 we expect that $\tilde x^j$ will converge to x^j and that τ^j will converge to τ_*^j satisfying

$$4\log\tau_*^j = -E_j(x^j, \mathbf{x})$$

and hence it is enough to choose τ^- and τ^+ in such a way that

$$4\log(\tau^-) < -\sup_j E_j(x^j, \mathbf{x}) \leqslant -\inf_j E_j(x^j, \mathbf{x}) < 4\log(\tau^+).$$

We now consider the L^2-projection of (5.6) over $\mathbb{E}_1$. Given a smooth function f defined in Ω, we identify its gradient $\nabla f = (\partial_{x_1} f, \dots, \partial_{x_4} f)$ with the element of $\mathbb{E}_1$

$$\bar\nabla f = \sum_{i=1}^4 \partial_{x_i} f e_i.$$

With these notation in mind, we obtain the system of equations

$$\begin{gathered} \varphi_1^j - \tilde\varphi_1^j - \bar\nabla E_j(\tilde x^j, \tilde{\mathbf{x}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0 \\ 3\varphi_1^j + 3\tilde\varphi_1^j + \frac{1}{2}\tilde\psi_1^j - \bar\nabla E_j(\tilde x^j, \tilde{\mathbf{x}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0 \\ 12\varphi_1^j - \tilde\varphi_1^j + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0 \\ 12\varphi_1^j + 3\tilde\varphi_1^j + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0. \end{gathered} \tag{5.8}$$

Again, let us comment briefly on how these equations are obtained. This time, the only important observation is that the term $\nabla_x E_j(\tilde{x}^j, \tilde{\mathbf{x}})\cdot y$ projects identically over $\mathbb{E}_1$ as well as its derivative with respect to r.

The system (5.8) simplifies to

$$\varphi_1^j = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \psi_1^j = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \tilde{\psi}_1^j = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \bar{\nabla} E_j(\tilde{x}^j, \tilde{\mathbf{x}}) = \mathcal{O}(r_{\varepsilon,\lambda}^2).$$

Finally, we consider the L^2-projection onto $L^2(S^3)^\perp$. This yields the system

$$\begin{gathered}
\varphi_\perp^j - \tilde{\varphi}_\perp^j + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0 \\
\partial_r \big(H^i_{\varphi_\perp^j, \psi_\perp^j} - H^e_{\tilde{\varphi}_\perp^j, \tilde{\psi}_\perp^j} \big) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0 \\
\psi_\perp^j - \tilde{\psi}_\perp^j + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0 \\
\partial_r \Delta \big(H^i_{\varphi_\perp^j, \psi_\perp^j} - H^e_{\tilde{\varphi}_\perp^j, \tilde{\psi}_\perp^j} \big) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0.
\end{gathered} \tag{5.9}$$

Thanks the Lemma 2.11, this last system can be re-written as

$$\varphi_\perp^j = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \psi_\perp^j = \mathcal{O}(r_{\varepsilon,\lambda}^2).$$

If we define the parameters $\mathbf{t} = (t^j) \in \mathbb{R}^m$ by

$$t^j = \frac{1}{\log r_{\varepsilon,\lambda}} \big[4 \log \tau^j + E_j(\tilde{x}^j, \tilde{x}) \big], \quad \forall 1 \leqslant j \leqslant m.$$

Then the system we have to solve reads

$$\Big(\mathbf{t}, \tilde{\boldsymbol{\eta}}, \boldsymbol{\varphi}_0, \tilde{\varphi}_{\mathbf{0}}, \boldsymbol{\varphi}_1, \tilde{\varphi}_{\mathbf{1}}, \tilde{\psi}_{\mathbf{1}}, \bar{\nabla} E(\tilde{\mathbf{x}}), \varphi_\perp, \tilde{\varphi}_\perp, \psi_\perp, \tilde{\psi}_\perp \Big) = \mathcal{O}(r_{\varepsilon,\lambda}^2), \tag{5.10}$$

where as usual, the terms $\mathcal{O}(r_{\varepsilon,\lambda}^2)$ depend nonlinearly on all the variables on the left side, but is bounded (in the appropriate norm) by a constant (independent of ε and κ) time $r_{\varepsilon,\lambda}^2$, provided $\varepsilon \in (0, \varepsilon_\kappa)$.

We claim, provided that the degree of the mapping

$$\bar{\nabla} E : \tilde{\mathbf{x}} \mapsto (\bar{\nabla} E_1(\tilde{x}^1; \tilde{\mathbf{x}}), \dots, \bar{\nabla} E_m(\tilde{x}^m; \tilde{\mathbf{x}})), \tag{5.11}$$

from a neighborhood of $\mathbf{x} \in \Omega^m$ to a neighborhood of 0 in $\mathbb{E}_1^m$ is equal to 1, this nonlinear system can be solve using Schauder's fixed point theorem in the ball of radius $\kappa r_{\varepsilon,\lambda}^2$ in the product space where the entries live, namely

$$\begin{gathered}
\mathbf{t}, \boldsymbol{\eta} \in \mathbb{R}^m; \quad r_{\varepsilon,\lambda}(\tilde{\mathbf{x}} - \mathbf{x}) \in (\mathbb{R}^4)^m; \quad \boldsymbol{\varphi}_0, \tilde{\boldsymbol{\varphi}}_0 \in \mathbb{R}^m \\
\boldsymbol{\varphi}_1, \tilde{\boldsymbol{\varphi}}_1, \tilde{\boldsymbol{\psi}}_1 \in \mathbb{E}_1^m; \quad \boldsymbol{\varphi}_\perp, \tilde{\boldsymbol{\varphi}}_\perp, \boldsymbol{\psi}_\perp, \tilde{\boldsymbol{\varphi}}_\perp \in (\mathcal{C}^{2,\alpha}(S^3)^\perp)^m.
\end{gathered}$$

Indeed, the nonlinear mapping which appears on the right hand side of (5.10) is continuous, compact. In addition, this nonlinear mapping sends the ball of radius $\kappa r_{\varepsilon,\lambda}^2$ (for the natural product norm) into itself, provided κ is fixed large enough.

To obtain the precise statement of our Theorem, we simply observe that

$$2 \nabla_x E_j(\tilde{x}^j, \tilde{\mathbf{x}}) = \nabla_{\tilde{x}^j} E(\tilde{\mathbf{x}}).$$

where E is the functional defined by (1.4), then a sufficient condition for the mapping (5.11) to have degree 1 is just that the point $\mathbf{x} = (x^1, \dots, x^m)$ is a nondegenerate critical point of the functional E. This completes the proof of our Theorem.

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