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OSCILLATIONS WITH ONE DEGREE OF FREEDOM AND DISCONTINUOUS ENERGY

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ABSTRACT. In 1995 for a linear oscillator, Myshkis imposed a constant impulse to the velocity, each moment the energy reaches a certain level. The main feature of the resulting system is that it defines a nonlinear discontinuous semigroup. In this note we study the orbital stability of a one-parameter family of periodic solutions and state the existence of a period-doubling bifurcation of such solutions.

1. INTRODUCTION

The solutions of the damped linear oscillator

$$\ddot{x} + 2\alpha\dot{x} + \omega^2 x = 0, \quad \omega > \alpha > 0, \tag{1.1}$$

are supposed to undergo a fixed instantaneous increase of velocity whenever they reach a certain level $E_0 > 0$ of energy. More precisely, the following condition is imposed

$$\frac{1}{2}(\dot{x}^2(t) + \omega^2 x^2(t)) = E_0 \Rightarrow \lim_{s\to t+} \dot{x}(s) = \dot{x}(t) + \sigma, \quad \sigma > 0.$$

This note concerns the resulting discontinuous dynamical system in the plane $x\dot{x}$. Motivated by a pioneering work by Myshkis [10], we obtain the existence of orbitally asymptotically stable *simple* periodic solutions, i.e., solutions which have exactly one impulse in the period. We accomplish a period-doubling bifurcation for such solutions.

The main feature of the problem is to be autonomous; that is, besides the involved equation being autonomous, the moments of impulses are not previously known. Therefore the solution operator of the whole system defines a discontinuous semigroup.

Specific references to the subject are Myshkis [12] and Samoilenko-Perestyuk [14]. For a wider class of related poblems see [2, 3, 4, 5, 6, 7, 9, 11, 12, 13] and references therein.

Section 2 aims to build a context for the problem. In Section 3 we state elementary properties of positive simple periodic solutions. In Section 4 we prove

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the existence of orbitally unstable positive simple periodic solutions with small amplitude and of orbitally asymptotically stable with large amplitudes. Finally, in Section 5 we give a sufficient condition for a period-doubling bifurcation of such solutions.

2. Object of study and basic facts

By the time scaling $\tau = \omega t$ and the change of variables $\xi(\tau) = (\omega/\sqrt{2E_0})x(\tau/\omega)$ Equation (1.1) is written as $\xi'' + 2a\xi' + \xi = 0$, where $' = d/d\tau$, $a = \alpha/\omega \in (0,1)$ and the locus of level E_0 of energy is taken to the circle $S : \xi^2 + {\xi'}^2 = 1$ in the plane $\xi\xi'$. Retrieving the original notation and formulating the problem in the $x\dot{x}$ plane we obtain

$$\begin{gathered} \dot{x} = y, \\ \dot{y} = -x - 2ay \end{gathered} \tag{2.1}$$

with the impulsive condition

$$(x(t), y(t)) \in S \;\Rightarrow\; (x(t+), y(t+)) = (x(t), y(t) + v). \tag{2.2}$$

Solutions of (2.1) will be denoted by z and $z(\cdot; t_0, z_0)$, if $z(t_0; t_0, z_0) = z_0$, or briefly $z(\cdot; z_0) = z(\cdot; 0, z_0)$. As the eigenvalues of (2.1) are $-a \pm \delta i$, with $\delta = \sqrt{1 - a^2} > 0$, the origin is a stable focus and the energy decreases strictly along nontrivial solutions, since

$$\dot{E}(z(t)) = -2a(y(t))^2, \quad t \in \mathbb{R}. \tag{2.3}$$

Let $a = \sin b$, $b \in (0, \pi/2)$, so that $\delta = \cos b$. If $\bar{z}(\cdot) = z(\cdot; (0, -1))$,

$$\bar{z}(t) = -\delta^{-1} e^{-at}\big(\sin \delta t, \cos(\delta t + b)\big), \quad t \in \mathbb{R}. \tag{2.4}$$

As $\bar{z}(\cdot)$ crosses the y axis at $(0, -\sigma) = (0, -e^{-2a\pi/\delta})$, completing a lap around the origin, if $\gamma = \bar{z}(\mathbb{R})$, the family $\{\mu\gamma\}_{\mu \in (\sigma, 1]}$ describes all nontrivial orbits of (2.1). That is, the general nontrivial solution is

$$z(\cdot) = \mu \bar{z}(\cdot + \tau), \quad \tau \in \mathbb{R}, \quad \sigma < \mu \le 1.$$

Definition 2.1. A solution of (2.1), (2.2) through $b_0 \in \mathbb{R}^2$ at $t = t_0$ is a function $\phi : [t_0, \infty) \to \mathbb{R}^2$ such that $\phi(t_0) = b_0$ and

(1) $\phi(t-) = \phi(t)$, for all $t \in (t_0, \infty)$;
(2) $\phi \in C^1$ and satisfies (2.1) in $(t, t + \epsilon_t)$, for all $t \in [t_0, \infty)$ and some $\epsilon_t > 0$.
(3) ϕ is continuous in t if $\phi(t) \in \mathbb{R}^2 \setminus S$ and $\phi(t+) = \phi(t) + (0, v)$ if $\phi(t) \in S$.

Remark 2.2. (1) ϕ is denoted by $\phi(\cdot; t_0, b_0)$ or $\phi(\cdot; b_0)$ if $t_0 = 0$.
(2) A function $\psi : (\tau, \infty) \to \mathbb{R}^2$ is solution of (2.1), (2.2) in (τ, ∞) if $\psi\big|_{[t_0, \infty)} = \phi(\cdot; t_0, \psi(t_0))$, for any $t_0 \in (\tau, \infty)$.
(3) The solution $\phi(\cdot; t_0, b_0)$ is unique, but in general there is no uniqueness for backward continuations. If $|b_0| \ge 1$, $\phi(\cdot; t_0, b_0)$ has a continuation to $(-\infty, \infty)$. If $|b_0| < 1$, in general a maximal interval of existence to the left is bounded below.

3. Positive simple solutions

For the dynamics of (2.1), (2.2) the only relevant solutions are $\phi(\cdot;b)$ with $|b| \geq 1$, as they are the only that eventually undergo impulses. There is no loss of generality in taking $|b| = 1$ and we do so. We denote by $\mathfrak{C}$ the class of such solutions.

Definition 3.1. Let $\phi(\cdot;b)$, $|b| = 1$, be a periodic solution of (2.1), (2.2) with minimal period $\omega > 0$. The point $\phi(0;b)$ is called *vertex* of $\gamma = \phi(\cdot;b)$. We say that $\phi(0;b)$ is simple if it has a unique impulse in $[0,\omega)$. If $\phi(\cdot;b) = (x(\cdot), y(\cdot))$, it is positive when $x(t) > 0$ for all t.

We close this section by setting some standing notations. A number β, identified to any $\beta' \equiv \beta \mod 2\pi$, indicates a point $(\cos\beta, \sin\beta) \in S$ or its arc length coordinate in S. The context will clarify the meaning in each case. For $\beta \in S$ we denote $\phi_\beta = \phi(\cdot;\beta)$ and, if $|\beta + (0,v)| > 1$, we set $t_1 = t_1(\beta) > 0$ such that $\phi_\beta(t_1) \in S$ and $\phi_\beta(t) \notin S$ for $0 < t < t_1$.

Definition 3.2. If $D = \{\beta \in S \mid |\beta + (0,v)| > 1\}$, we define the return map $\Phi_v : D \to S$ by $\Phi_v(\beta) = \phi_\beta(t_1(\beta))$ for all $\beta \in D$.

Clearly, if $\beta^* \in D$ is a fixed point of Φ_v, ϕ_{β^*} is a simple periodic solution whose period is $t_1(\beta^*)$ and β^* is the vertex of the simple cycle $\phi_{\beta^*}(\mathbb{R})$. If β^* is an attractor fixed point, ϕ_{β^*} is orbitally asymptotically stable and, if it is repelling, ϕ_{β^*} is orbitally unstable. Here the orbital stability must be in the sense of conditional stability relative to the class $\mathfrak{C}$, see [8], since if $\phi = \phi(\cdot;b)$, $|b| = 1$, there are points b' inside S arbitrarily close to b and therefore $\phi(t;b') \to (0,0)$, as $t \to \infty$.

If $\beta \in S$, let s_β be the vertical line $s_\beta : x = \cos\beta$ and $t_\beta > 0$ such that $z(-t_\beta;\beta) = (\cos\beta, y_\beta) \in s_\beta$ and $z(t;\beta) \notin s_\beta$ for $-t_\beta < t < 0$. We set $v_\beta = y_\beta - \sin\beta$, so that ϕ_β is a positive simple periodic solution of (2.1), (2.2), $v_\beta > 0$. We denote by $\alpha = \alpha_\beta$ the polar angle of $z(-t_\beta;\beta)$, according to Figure 1.

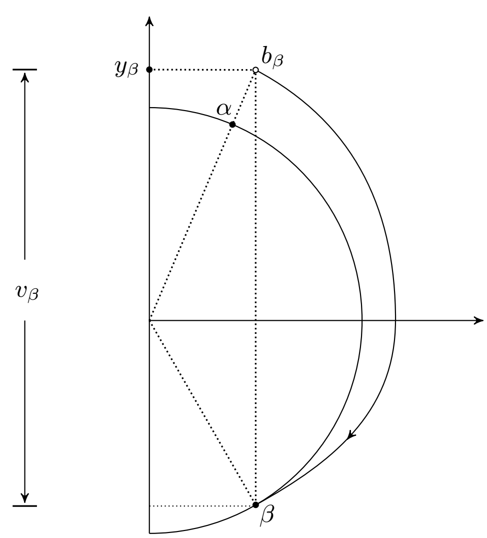

Figure 1. Positive simple cycle.

Remark 3.3. For any $v \in (0, e^{a\pi/\delta}+1)$, there exists exactly one positive simple cycle of (2.1), (2.2) since $\beta \in (-\pi/2, 0) \mapsto v_\beta \in (0, e^{a\pi/\delta}+1)$ is a continuous bijection.

4. Orbital stability

Now we show that, for some $\zeta > 0$, the solution ϕ_β of (2.1), (2.2) is orbitally unstable if $\beta \in (-\zeta, 0)$ and orbitally asymptotically stable if $\beta \in (-\pi/2, -\pi/2+\zeta)$.

Lemma 4.1. *$v_\beta = -2\beta + o(\beta)$ as $\beta \to 0-$.*

Proof. Let $\beta \in (-\pi/2, 0)$. System (2.1) in polar coordinates,

$$\dot{r} = -(2a\sin^2\theta)r,$$
$$\dot{\theta} = -(1 + a\sin 2\theta),$$

yields

$$r' = \big(2a\sin^2\theta/(1+a\sin 2\theta)\big)r, \quad ('= d/d\theta). \tag{4.1}$$

and a parametrization of ϕ_β is

$$r_\beta(\theta) = e^{A_\beta(\theta)} = \exp\Big[2a\int_\beta^\theta \frac{\sin^2 s}{1+a\sin 2s}\,ds\Big], \quad \theta \in \mathbb{R}. \tag{4.2}$$

As the integrand in (4.2) will be a regular participant, we introduce the notation

$$q_a(s) = \frac{\sin^2 s}{1+a\sin 2s}.$$

For any small $\epsilon > 0$ such that $\alpha = -(1+\epsilon)\beta < \pi/2$, the inequality

$$A_\beta(\theta) \le -\frac{2a(2+\epsilon)(1+\epsilon)^2}{1-a}\beta^3, \quad \theta \in [\beta, -(1+\epsilon)\beta],$$

yields

$$r_\beta(-(1+\epsilon)\beta) = e^{A_\beta(-(1+\epsilon)\beta)} = 1 + O(\beta^3) \quad \text{as } \beta \to 0-.$$

If $r^\epsilon = |p_\epsilon|$, p_ϵ being the intersection of the half lines $s_1 : \theta = -(1+\epsilon)\beta$ and $s_2 : r(\theta)\cos\theta = \cos\beta$, $\theta \in (0, \pi/2)$, the similarity of the triangles mnO and $p_\epsilon qO$ seen in Figure 2 yields

$$r^\epsilon = \frac{\cos\beta}{\cos(1+\epsilon)\beta} = 1 + \frac{(2+\epsilon)\epsilon}{2!}\beta^2 + O(\beta^4) \quad \text{as } \beta \to 0-.$$

For $|\beta|$ small enough, the estimates above imply $r_\beta(-(1+\epsilon)\beta) < r^\epsilon$, so that $y_\beta/\cos\beta < -\tan(1+\epsilon)\beta$ and

$$1 < -\frac{y_\beta}{\sin\beta} < \frac{\tan(1+\epsilon)\beta}{\tan\beta}.$$

Taking limits as $\beta \to 0-$,

$$1 \le \liminf_{\beta\to 0-} -\frac{y_\beta}{\sin\beta} \le \limsup_{\beta\to 0-} -\frac{y_\beta}{\sin\beta} \le 1+\epsilon,$$

so that $\lim_{\beta\to 0-} y_\beta/\sin\beta = -1$. Therefore $y_\beta = -\beta + o(\beta)$ and hence $v_\beta = -2\beta + o(\beta)$, as $\beta \to 0-$. □

The theorem below in what concerns orbital instability is a result by Myshkis [10]. We give an alternative approach to extend it.

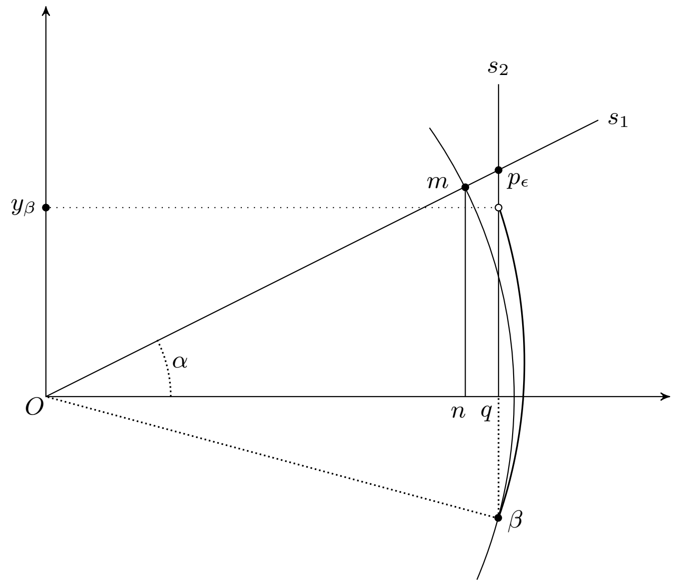

FIGURE 2. $v_\beta = -2\beta + o(\beta)$ as $\beta \to 0-$.

Theorem 4.2. *There is a number $\zeta > 0$ such that if $\beta \in (-\zeta, 0)$, the simple periodic solution ϕ_β of (2.1), (2.2) is orbitally unstable and if $\beta \in (-\pi/2, -\pi/2 + \zeta)$, ϕ_β is orbitally asymptotically stable.*

Proof. Let $\beta \in (-\pi/2, 0)$ and $\epsilon_1 \neq 0$ so that $\beta + \epsilon_1 = \beta_1 \in (-\pi/2, 0)$. We take $|\epsilon_1|$ smaller if necessary to assure the existence of $\Phi_{v_\beta}(\beta_1) = \beta + \epsilon_2 \in (-\pi/2, 0)$, as it is seen in Figure 3 for the case $\epsilon_1 < 0$.

Firstly we notice that ϵ_1 and σ are related by the equation

$$\frac{v_\beta + \sin(\beta + \epsilon_1)}{\cos(\beta + \epsilon_1)} = \tan(\alpha + \sigma),$$

therefore, the implicit function theorem about $(\epsilon_1, \sigma) = (0, 0)$ yields

$$\sigma = \frac{v_\beta \sin\beta + 1}{|b_\beta|^2}\, \epsilon_1 + o(\epsilon_1), \tag{4.3}$$

as $\epsilon_1 \to 0$. By (4.2), if $b_1 = \beta_1 + (0, v_\beta)$, ϵ_2 must satisfy

$$|b_1| \exp\Big[2a \int_{\alpha+\sigma}^{\beta+\epsilon_2} q_a(s)\, ds\Big] = 1.$$

As $|b_1| = \sqrt{(v_\beta + \sin(\beta + \epsilon_1))^2 + \cos^2(\beta + \epsilon_1)}$, we have

$$\big(v_\beta^2 + 2v_\beta \sin(\beta + \epsilon_1) + 1\big) \exp\Big[4a \int_{\alpha+\sigma(\epsilon_1)}^{\beta+\epsilon_2} q_a(s)\, ds\Big] = 1$$

and the implicit function theorem leads to

$$\epsilon_2 = \frac{1}{q_a(\beta)|b_\beta|^2}\big[q_a(\alpha)(1 + v_\beta \sin\beta) - \frac{v_\beta \cos\beta}{2a}\big]\epsilon_1 + o(\epsilon_1), \tag{4.4}$$

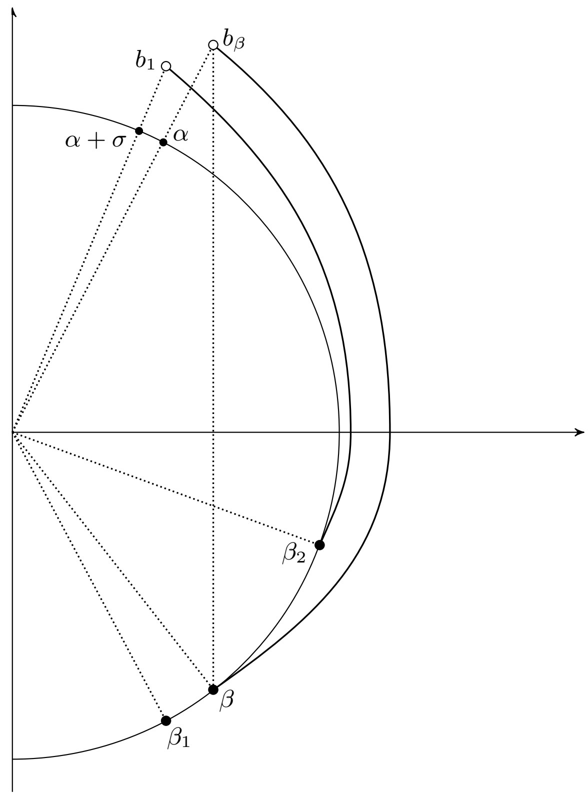

FIGURE 3. $\beta + \epsilon_2 = \Phi_{v_\beta}(\beta + \epsilon_1)$.

as $\epsilon_1 \to 0$. Let

$$F(\beta) = \frac{1}{q_a(\beta)|b_\beta|^2}\big[q_a(\alpha)(1 + v_\beta \sin\beta) - \frac{v_\beta \cos\beta}{2a}\big], \tag{4.5}$$

so that $F(\beta) < 0$ and (4.4) is $\epsilon_2 = F(\beta)\epsilon_1 + o(\epsilon_1)$, as $\epsilon_1 \to 0$, for short. Since $\lim_{\beta\to-\pi/2} |b_\beta| = \lim_{\beta\to-\pi/2} -(1 + v_\beta \sin\beta) = e^{a\pi/\delta}$,

$$|F(\beta)| \to e^{-a\pi/\delta} < 1, \quad \text{as } \beta \to -\pi/2. \tag{4.6}$$

On the other hand, we have $|\sin\beta| < |\sin\alpha| < y_\beta$, see Figure 2, so that by Lemma 4.1, $q_a(\alpha)/q_a(\beta) \to 1$ and $v_\beta = O(\beta)$, as $\beta \to 0$, therefore recalling that $q_a(\beta) = O(\beta^2)$ as $\beta \to 0$,

$$|F(\beta)| \to \infty \quad \text{as } \beta \to 0. \tag{4.7}$$

For some $\zeta > 0$, Eqs. (4.6) and (4.7) imply that $|F(\beta)| < 1$ if $\beta \in (-\pi/2, -\pi/2+\zeta)$ and $|F(\beta)| > 1$ if $\beta \in (-\zeta, 0)$. In other words, any $\beta \in (-\pi/2, -\pi/2 + \zeta)$ is an attractor fixed point of the return map Φ_{v_β} and any $\beta \in (-\zeta, 0)$ is a repelling fixed point of Φ_{v_β}. □

5. Period doubling bifurcation

Solutions ϕ_β of (2.1), (2.2) change from stable to unstable when β varies over $(-\pi/2, 0)$ from left to the right. Therefore it is natural to expect a bifurcation in between. In this section we apply the theorem below [1, Theorem 12.7] to confirm that this indeed occurs at least for small dampings.

Theorem 5.1 (Period doubling bifurcation)**.** *Let $\{f_\lambda\}$ a one-parameter family of real functions and suppose that*

(1) $f_\lambda(0) = 0$ *for all λ in an interval about λ_0;*
(2) $f'_{\lambda_0}(0) = -1$*;*
(3) $\displaystyle\frac{\partial(f^2_\lambda)'}{\partial\lambda}\Big|_{\lambda=\lambda_0}(0) \neq 0$.

Then there is an interval I about 0 and a function $p : I \to \mathbb{R}$ such that

$$f_{p(x)}(x) \neq x \quad \textit{and} \quad f^2_{p(x)}(x) = x.$$

By the proof of Theorem 4.2 there is a $\beta^*_a \in (-\pi/2, 0)$, $0 < a < 1$, such that $F(\beta^*_a) = -1$. Now we show that such a β^*_a is a period doubling bifurcation point of the family of periodic solutions ϕ_β, $-\pi/2 < \beta < 0$, at least if a is small enough.

Theorem 5.2. *If $a \in (0, 1)$ is sufficiently small, then any $\beta^*_a \in (-\pi/2, 0)$ such that $F(\beta^*_a) = -1$ is a period doubling bifurcation point for the family ϕ_β, $-\pi/2 < \beta < 0$.*

Proof. Let us follow (4.4) to define the family of functions f_β, $-\pi/2 < \beta < 0$, in such a way that

$$\epsilon_2 = f_\beta(\epsilon_1) = F(\beta)\epsilon_1 + o(\epsilon_1),$$

as $\epsilon_1 \to 0$. Condition (1) of Theorem 5.1, $f_\beta(0) = 0$ for all $\beta \in (-\pi/2, 0)$, is immediate and, if $'$ denotes for a moment $d/d\epsilon_1$, Condition (2), $f'_{\beta^*_a}(0) = F(\beta^*_a) = -1$, follows from the definition of β^*_a.

Now it remains to show that

$$\Big[\frac{\partial(f^2_\beta)'}{\partial\beta}\Big]_{\beta=\beta^*_a}(0) = \frac{\partial}{\partial\beta}\big[\big(F(\beta)\big)^2\big]_{\beta=\beta^*_a} \neq 0$$

for a small enough. Retaking the notation $' = d/d\beta$ this is equivalent to $F'(\beta^*_a) \neq 0$, since $F(\beta^*_a) \neq 0$. We note that if $\beta = \beta^*_a$,

$$q_a(\beta)|b_\beta|^2 = \frac{v_\beta \cos\beta}{2a} + q_a(\alpha)(-v_\beta \sin\beta - 1);$$

therefore,

$$\begin{aligned} F'(\beta^*_a) &= \Big[\frac{1}{q_a(\beta)|b_\beta|^2}\Big(\frac{v_\beta\cos\beta}{2a} + q_a(\alpha)(-v_\beta\sin\beta - 1)\Big)\Big]'_{\beta=\beta^*_a} \\ &\frac{1}{q_a(\beta^*_a)|b_{\beta^*_a}|^2}\Big[q'_a(\beta)|b_\beta|^2 + 2q_a(\beta)|b_\beta||b_\beta|' + \frac{v'_\beta\cos\beta - v_\beta\sin\beta}{2a} \\ &+ q'_a(\alpha)\alpha'(-v_\beta\sin\beta - 1) + q_a(\alpha)(-v'_\beta\sin\beta - v_\beta\cos\beta)\Big]_{\beta=\beta^*_a}. \end{aligned} \tag{5.1}$$

It suffices to show that the term in the brackets in the right side of (5.1) is nonzero.

Equation (4.2) implies $|b_\beta| = \exp\big[2a\int_\beta^\alpha q_a(s)ds\big] \to 1$ as $a \to 0$, uniformly in $\beta \in (-\pi/2, 0)$. This yields $y_\beta \to -\sin\beta$ and $\alpha \to -\beta$, as $a \to 0$, uniformly in

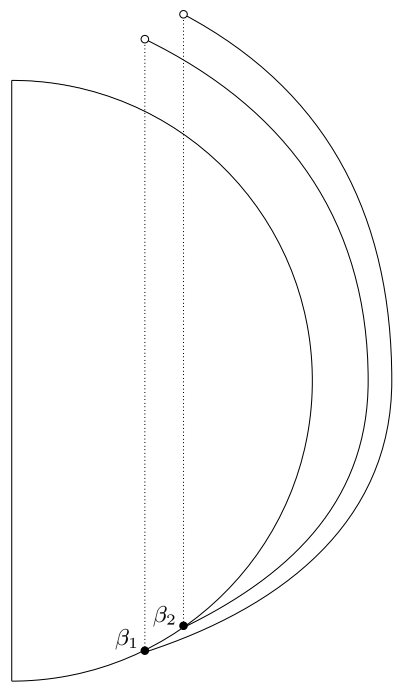

FIGURE 4. $\beta_1 = \Phi_v^2(\beta_1) \neq \Phi_v(\beta_1) = \beta_2$.

$\beta \in (-\pi/2, 0)$. Moreover, the implicit function theorem applied to the equation

$$\exp\Big[2a\int_\beta^\alpha q_a(s)ds\Big]\cos\alpha = \cos\beta,$$

leads to

$$\alpha'(\beta) = \frac{\sin\beta(1+a\sin 2\alpha)}{y_\beta(1+a\sin 2\beta)}.$$

Thus $\alpha' \to -1$ as $a \to 0$, uniformly in $\beta \in (-\pi/2, 0)$. Finally, we note that the following limits, taken as $a \to 0$, are uniform in $\beta \in (-\pi/2, 0)$:

$$\begin{gathered}\lim q_a(\beta) = \sin^2\beta,\\ \lim q_a'(\beta) = \sin 2\beta,\\ \lim v_\beta = -2\sin\beta,\\ \lim v_\beta' = -2\cos\beta,\\ \lim |b_\beta|' = 0.\end{gathered}$$

Therefore, the limit, as $a \to 0$, of the term in the brackets in the right side of (5.1) is

$$\sin 2\beta + \lim_{a\to 0}\frac{v_\beta'\cos\beta - v_\beta\sin\beta}{2a} - \frac{\sin 4\beta}{2}. \tag{5.2}$$

Since $\lim_{a\to 0}(v_\beta'\cos\beta - v_\beta\sin\beta) = -2\cos 2\beta$, in order to assure the expression (5.2) is nonzero, β_a^* must be bounded away from $-\pi/4$ for a small enough. According to

(4.5) $\lim_{a\to 0} -F(-\pi/4) = \infty$; therefore, for some $\eta > 0$, $\beta_a^* \notin (-\pi/4-\eta, -\pi/4+\eta)$. That is, $F'(\beta_a^*) \neq 0$ for $a \in (0,1)$ sufficiently small. □

Figure 4 shows a typical positive periodic orbit emanating from β_a^*.

Final remarks. Smallness of a is a request of our proof of Theorem 5.2, possibly this hypothesis can be weakened or even discarded.

The larger is the coefficient $a \in (0,1)$, the larger is the region of stability in $(-\pi/2, 0)$. In fact, by (4.2), $r_{-\pi/2}(\pi) = e^{a\pi/\delta} \to \infty$ as $a \to 1$. Therefore, for any fixed $\beta \in (-\pi/2, 0)$, one has $|b_\beta| \to \infty$ as $a \to 1$, so that the number ϵ_2 in (4.4) satisfies $\epsilon_2 \to 0$, as $a \to 1$.

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