Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 263, pp. 1–25.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

RANDOM ATTRACTORS FOR STOCHASTIC LATTICE REVERSIBLE GRAY-SCOTT SYSTEMS WITH ADDITIVE NOISE

HONGYAN LI, JUNYI TU

Abstract. In this article, we prove the existence of a random attractor of the stochastic three-component reversible Gray-Scott system on infinite lattice with additive noise. We use a transformation of addition involved with Ornstein-Uhlenbeck process, for proving the pullback absorbing property and the pullback asymptotic compactness of the reaction diffusion system with cubic nonlinearity.

1. Introduction

In the previous decades, chemical kinetics has produced a variety of phenomenon which had been translated into challenging mathematical problems. A classical example is seen in the waves of the Belousov-Zhabotinskii reaction. Other examples have been produced and require fewer species interactions, but still yield very interesting behavior [26]. The problem of dealing with chemical reactions of systems, together with a number of initial and final products whose concentrations are assumed to be controlled throughout the reaction process is an important one under quite realistic conditions [1] and [26]. It is necessary to consider at least a cubic nonlinearity in the rate equations [28]. These models include the Brusselator system [1, 13, 31], the Gray-Scott system [16, 18, 24, 34, 35, 39] and the Glycolysis model [26, 29, 30].

At the same time, the dynamics of infinite lattice systems has drawn much attention from mathematicians and physicists (e.g., [3, 4, 5, 13, 32]). Lattice dynamical systems (LDSs) are spatiotemporal systems with discretization in some variables including coupled ODEs /PDEs and coupled map lattices and cellular automata [7], which occur in a wide variety of applications, ranging from biology [23, 33] to chemical reaction theory [14, 22], laser system [15], electrical engineering [6], material science [19], image processing and pattern recognition [8], [9]. In this paper, we will deal with the existence of a random attractor for the stochastic three-component reversible Gray-Scott system with additive white noise on an infinite

2010 *Mathematics Subject Classification.* 37H10, 35B40, 35B41.
Key words and phrases. Random attractor; additive white noise; stochastic reversible Gray-Scott system.

Submitted July 25, 2014. Published October 8, 2015.

lattice as follows:

$$\begin{aligned} du_i &= [d_1(u_{i+1} - 2u_i + u_{i-1}) - (F+k)u_i + u_i^2 v_i - u_i^3 + N z_i + f_{1i}]dt \\ &\quad + \alpha_i dw_i, \\ dv_i &= [d_2(v_{i+1} - 2v_i + v_{i-1}) - F v_i - u_i^2 v_i + u_i^3 + f_{2i}]dt + \alpha_i dw_i, \\ dz_i &= [d_3(z_{i+1} - 2z_i + z_{i-1}) + k u_i - (F+N) z_i + f_{3i}]dt + \alpha_i dw_i, \end{aligned} \tag{1.1}$$

for $i \in \mathbb{Z}$ and $t > 0$, with initial conditions

$$u_i(0) = u_{i,0}, \quad v_i(0) = v_{i,0}, \quad z_i(0) = z_{i,0}, \quad i \in \mathbb{Z}, \tag{1.2}$$

where $\mathbb{Z}$ denotes the set of integers, $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$, $v = (v_i)_{i\in\mathbb{Z}} \in \ell^2$ and $z = (z_i)_{i\in\mathbb{Z}} \in \ell^2$, d_1, d_2, $d_3, F, k,$ and N are positive constants, $f_1 = (f_{1i})_{i\in\mathbb{Z}} \in \ell^2$, $f_2 = (f_{2i})_{i\in\mathbb{Z}} \in \ell^2$, $f_3 = (f_{3i})_{i\in\mathbb{Z}} \in \ell^2$, $\alpha = (\alpha_i)_{i\in\mathbb{Z}} \in \ell^2$, $\{w_i | i \in \mathbb{Z}\}$ is independent Brownian motions.

A three-component reversible Gray-Scott model was introduced in [24], which is derived according to the scheme of following two reversible chemical or biochemical reactions:

$$2A + B \leftrightarrows^{k_1}_{k_{-1}} 3A, \quad A \leftrightarrows^{k_2}_{k_{-2}} P.$$

In the reversible Gray-Scott model (1.1), k is the effective production rate for the first reaction, $1/F$ is the mean residence time in the dimensionless unit, N is the normalized rate of the second reverse reaction, and the parameter α measures the proportional strength of the white noise dw/dt to the respective components.

Note the general Gray-Scott equations have the cubic terms $\pm G u^3$, but here we put $G = 1$, since the difficulties arise otherwise in the below estimations (3.9), (4.4) and (4.15). This is one of the key observation in this paper. Equation (1.1) can be regarded as a discrete analogue of the stochastic three-component reversible Gray-Scott system on $\mathbb{R}$:

$$\begin{aligned} u_t &= d_1 \triangle u - (F+k)u + u^2 v - u^3 + N z + f_1 + \alpha w_t, \\ v_t &= d_2 \triangle v - F v - u^2 v + u^3 + f_2 + \alpha w_t, \\ z_t &= d_3 \triangle z + k u - (F+N) z + f_3 + \alpha w_t. \end{aligned} \tag{1.3}$$

The asymptotic dynamics of solutions for the reversible Gray-Scott system (where $\alpha = 0$) have been studied by several authors, e.g. [16, 18, 21, 34, 35, 37, 39]. The established results naturally focus on the existence of global attractors by showing the absorbing property and the asymptotic compactness of the solution semigroups for autonomous system [18, 21, 34, 35, 37, 39] or the skew-product flow for non-autonomous system [16].

For stochastic three-component reversible Gray-Scott system, the solution mapping defines a random dynamical system, which is a parametric dynamical system. Random attractors are the appropriate objects for describing asymptotic dynamics of such a parametric dynamical systems and have been studied by several authors [17, 38]. Note that in these papers, only multiplicative white noise was considered. In this paper we study stochastic reversible Gray-Scott system driven by additive white noise as in (1.1). The impact of these two types of noise on the solutions of stochastic reversible Gray-Scott system is quite different. When dealing with random attractors of stochastic equations, we usually transform the stochastic equation into a pathwise one with random parameters. Additive noises are more challenge than multiplicative noises, there is still open to prove the existence of random attractor for stochastic Brusslator system with additive noise. And if the stochastic three-component reversible Gray-Scott system is driven by additive white noise, then there are several additional terms appearing after the equation is transformed (see (4.1) in Section 4). These terms have great effect on the way to derive uniform estimates of solutions. This is the topic we want to study as well as one of the main contributions in this work. Beside this common difficulty, there exist new challenges in analyzing the second pair of oppositely signed cubic terms $\pm u^3$ in the u-equation and the v-equation, and the occurrence of the third component's z-equation, which do not occur in dealing with the case of Gray-Scott equations in [17]. In addition, for the discrete time models governed by difference equations are more appropriate than the continuous ones. However, very few investigations are on this topic, especially for the stochastic three-component reversible Gray-Scott system with additive white noise on infinite lattice, to the best of our knowledge.

This paper is organized as follows. In Section 2 we recall some basic concepts and results related to random attractor for random dynamical systems. In Section 3 we formulate the problem and make assumptions to define a random dynamical system generated by the stochastic three-component reversible Gray-Scott system in an infinite lattice with additive white noise. In Section 4, we conduct uniform estimate to prove the pullback absorbing property and the pullback asymptotic compactness for the random dynamical system. Then the existence of random attractors for (1.1) on infinite lattices is obtained.

2. Preliminaries

In this section, we introduce the definitions of the random dynamical system and the random attractor, which are taken from [3, 10, 12, 20].

Let (H, d) be a complete separable metric space, $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, $\mathbb{R}^+ = [0, \infty)$.

Definition 2.1. $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta : \mathbb{R}\times\Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$ measurable, $\theta_0 = I, \theta_{s+t} = \theta_s \circ \theta_t$ for all $s, t \in \mathbb{R}$, and $\theta_t\mathcal{P} = \mathcal{P}$ for all $t \in \mathbb{R}$.

Definition 2.2. A continuous random dynamical system (RDS) on H over a metric dynamical system $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t\in\mathbb{R}})$ is a mapping

$$\varphi : \mathbb{R}^+ \times \Omega \times H \to H, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x),$$

which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(H), \mathcal{B}(H))$-measurable and satisfies, for every $\omega \in \Omega$,

- (i) $\varphi(0, \omega, \cdot)$ is the identity on H;
- (ii) Cocycle property: $\varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s\omega, \varphi(s, \omega, \cdot))$ for all $t, s \in \mathbb{R}^+$;
- (iii) $\varphi(\cdot, \omega, \cdot) : \mathbb{R}^+ \times H \to H$ is strongly continuous.

Definition 2.3. The set $A \subset \Omega$ is called invariant with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for all $t \in \mathbb{R}, \theta_t^{-1}A = A$.

Definition 2.4. A random bounded set $B(\omega) \subset X$ is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for every $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{\beta t} d(B(\theta_{-t}\omega)) = 0 \quad \text{for all } \beta > 0,$$

where $d(B) = \sup_{x\in B} \|x\|_X$.

Definition 2.5. A random set $K(\omega)$ is called a pullback absorbing set in $\mathcal{D}$, where $\mathcal{D}$ is a collection of random sets of H, if for all $B \in \mathcal{D}$ and every $\omega \in \Omega$, there exists a $t_B(\omega) > 0$ such that

$$\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega), \quad \text{for all } t \geq t_B(\omega).$$

Definition 2.6. Suppose $\varphi(t, \omega)$ is a RDS, a random set $\mathcal{A}$ is called a random $\mathcal{D}$ attractor if the following hold:

(i) $\mathcal{A}(\omega)$ is a random compact set, i.e., $\omega \to d(x, \mathcal{A}(\omega))$ is measurable for every $x \in H$ and $\mathcal{A}(\omega)$ is compact for every $\omega \in \Omega$;
(ii) $\mathcal{A}(\omega)$ is strictly invariant, i.e., for every $\omega \in \Omega$ and all $t \geq 0$ one has $\varphi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega)$;
(iii) $\mathcal{A}(\omega)$ attracts all sets in $\mathcal{D}$, i.e. for all $B \in \mathcal{D}$ and $\omega \in \Omega$ we have

$$\lim_{t\to\infty} d(\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where $d(X, Y) = \sup_{x\in X} \inf_{y\in Y} \|x - y\|_H$ is the Hausdorff semi-metric (here, $X \subseteq H, Y \subseteq H$).

Theorem 2.7 ([3, Proposition 4.1])**.** *Let $K(\omega) \in \mathcal{D}$ be an absorbing set for the random dynamical system $\varphi(t, \theta_{-t}\omega)_{t\geq 0,\omega\in\Omega}$ which is closed and which satisfies for $\omega \in \Omega$ the following asymptotic compactness condition: each sequence $x_n \in \varphi(t_n, \theta_{-t_n}\omega, K(\theta_{-t_n}\omega))$ with $t_n \to \infty$ has a convergent subsequence in H. Then the random dynamical system φ has a unique global random attractor*

$$\mathcal{A}(\omega) = \cap_{t\geq t_K(\omega)} \overline{\cup_{t\geq\tau} \varphi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}.$$

3. Existence and uniqueness of solutions

In this paper, we have the following standing assumption on the constants in (1.1):

$$N + k \leq 2/7F. \tag{3.1}$$

which will be used in (3.11), (4.6) and (4.20). We shall use C_i to denote generic or specific positive constants which do not depend on the proportional strength α of additive white noise in (1.1).

We first formulate the mathematical setting of our problem (1.1)-(1.2). Set

$$\ell^2 = \{u = (u_i)_{i\in\mathbb{Z}} : u_i \in \mathbb{R}, \sum_{i\in\mathbb{Z}} |u_i|^2 < +\infty\},$$

equipped it with the inner product and norm as follows:

$$\langle u, v\rangle = \sum_{i\in\mathbb{Z}} u_i v_i, \quad \|u\|^2 = \langle u, u\rangle, \quad u = (u_i)_{i\in\mathbb{Z}}, \quad v = (v_i)_{i\in\mathbb{Z}} \in \ell^2.$$

Then $\ell^2 = (\ell^2, \langle\cdot,\cdot\rangle, \|\cdot\|)$ is a Hilbert space. Set $E = \ell^2 \times \ell^2 \times \ell^2$ be the product Hilbert space. In view of the cubic term $\pm u^2 v, \pm u^3$, we need $u \in \ell^6, v \in \ell^6$ to make (1.1) hold in ℓ^2.

Define A, B and B^* to be linear operators from ℓ^2 to ℓ^2 as follows:

$$(Bu)_i = u_{i+1} - u_i, \quad (B^*u)_i = u_{i-1} - u_i,$$
$$(Au)_i = -u_{i+1} + 2u_i - u_{i-1}, \quad \text{for all } i \in \mathbb{Z}.$$

It is easy to show that $A = BB^* = B^*B$, and $(B^*u, v) = (u, Bv)$ for all $u, v \in \ell^2$, which implies that $(Au, u) \geq 0$ for all $u \in \ell^2$. Indeed, A, B and B^* are all bounded operators from ℓ^2 to ℓ^2, because

$$\|Bu\|^2 = \sum_{i\in\mathbb{Z}} |u_{i+1} - u_i|^2 \leq 4\|u\|^2, \quad \forall u = (u_i)_{i\in\mathbb{Z}} \in \ell^2,$$

$$\|B^*u\|^2 \leq 4\|u\|^2, \quad \|Au\|^2 \leq \|BB^*u\| \leq 16\|u\|^2, \quad \forall u = (u_i)_{i\in\mathbb{Z}} \in \ell^2.$$

Let $e^i \in \ell^2$ denote the element having 1 at position i and all the other components 0. Define

$$W(t) \equiv W(t,\omega) = \sum_{i\in\mathbb{Z}} \alpha_i w_i(t) e^i, \quad \text{with } (\alpha_i)_{i\in\mathbb{Z}} \in \ell^2$$

as the white noise with values in ℓ^2 defined on the probabilities space $(\Omega, \mathcal{F}, \mathcal{P})$, where

$$\Omega = \{\omega \in C(\mathbb{R}, \ell^2) : \omega(0) = 0\},$$

the σ-algebra $\mathcal{F}$ is generated by the compact open topology ([2], Appendices A.2 and A.3), and $\mathcal{P}$ is the corresponding Wiener measure on $\mathcal{F}$. Let

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R},$$

then $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system.

We rewrite system (1.1) with initial values $Y_0 = (u_0 = (u_{0,i})_{i\in\mathbb{Z}}, v_0 = (v_{0,i})_{i\in\mathbb{Z}}, z_0 = (z_{0,i})_{i\in\mathbb{Z}})^T$ in a vector form as the integral equations in $E = \ell^2 \times \ell^2 \times \ell^2$:

$$\begin{aligned} u(t) &= u_0 + \int_0^t \Big[- d_1 Au(s) - (F+k)u(s) + u^2(s)v(s) - u^3(s) \\ &\quad + Nz(s) + f_1 \Big] ds + W(t), \\ v(t) &= v_0 + \int_0^t [-d_2 Av(s) - Fv(s) - u^2(s)v(s) + u^3(s) + f_2] ds + W(t), \\ z(t) &= z_0 + \int_0^t [-d_3 Az(s) + ku(s) - (F+N)z(s) + f_3] ds + W(t), \end{aligned} \tag{3.2}$$

for $t \geq 0$ and $\omega \in \Omega$.

Lemma 3.1. *Assume that* (3.1) *holds and $T > 0$. Then the following properties hold:*

(1) *For any initial data $Y_0 = (u_0, v_0, z_0)^T \in E$, there exists a unique solution $Y(t) = (u(t), v(t), z(t))^T \in L^2(\Omega, C([0,T], E))$ of equations* (1.1).
(2) *For all $\omega \in \Omega$, we have the following estimate*

$$\begin{aligned} &\sup_{t\in[0,T]} [\|u(t)\|^2 + \|v(t)\|^2 + \|z(t)\|^2] \\ &\leq 2(\|u_0\|^2 + \|v_0\|^2 + \|z_0\|^2) + 6 \sup_{t\in[0,T]} \|W(t)\|^2 \\ &\quad + 2C_0 \int_0^T (\|W(s)\|^2 + \|f_1\|^2 + \|f_2\|^2 + \|f_3\|^2) ds. \end{aligned}$$

Proof. Equations (3.2) can be written as an abstract first-order ODE in E as follows

$$\begin{gathered} Y(t) = Y_0 + \int_0^t \Theta Y(s) ds + \int_0^t G(Y(s)) ds + \overline{W}(t), \quad t > 0, \\ Y(0) = Y_0 = (u_0, v_0, z_0)^T \in E, \end{gathered} \tag{3.3}$$

where $Y=(u,v,z)^T$, $\overline{W}(t)=(W(t),W(t),W(t))^T$ and

$$\Theta=\begin{pmatrix}-d_1A-(F+k)I & 0 & NI\\ 0 & -d_2A-FI & 0\\ kI & 0 & -d_3A-(F+N)I\end{pmatrix},$$

$$G(Y(s))=\begin{pmatrix}u^2v(s)-u^3(s)+f_1\\ -u^2v(s)+u^3(s)+f_2\\ f_3\end{pmatrix}.$$

It is easy to show that the linear bounded operator Θ maps E into E. Let B be a bounded set in E, we have

$$\begin{aligned}
&\|G(Y^{(1)})-G(Y^{(2)})\|_E^2\\
&=\|((u^{(1)})^2v^{(1)}-(u^{(1)})^3-(u^{(2)})^2v^{(2)}\\
&\quad+(u^{(2)})^3-(u^{(1)})^2v^{(1)}+(u^{(1)})^3+(u^{(2)})^2v^{(2)}-(u^{(2)})^3,0)^T\|_E^2\\
&=2\|(u^{(1)})^2v^{(1)}-(u^{(2)})^2v^{(2)}\|^2+2\|(u^{(1)})^3-(u^{(2)})^3\|^2\\
&=2\|(u^{(1)})^2(v^{(1)}-v^{(2)})+v^{(2)}(u^{(1)})^2-(u^{(2)})^2\|^2+2\|(u^{(1)}-u^{(2)})((u^{(1)})^2\\
&\quad+(u^{(2)})^2+u^{(1)}u^{(2)})\|^2\\
&\le 2(2\|(u^{(1)})^2(v^{(1)}-v^{(2)})\|^2+2\|v^{(2)}(u^{(1)}-u^{(2)})(u^{(1)}+u^{(2)})\|^2)\\
&\quad+3\|(u^{(1)}-u^{(2)})((u^{(1)})^2+(u^{(2)})^2)\|^2\\
&\le 4\|u^{(1)}\|^4\|v^{(1)}-v^{(2)}\|^2+4\|v^{(2)}\|^2\|u^{(1)}-u^{(2)}\|^2\|u^{(1)}+u^{(2)}\|^2\\
&\quad+6\|(u^{(1)}-u^{(2)})\|^2(\|u^{(1)}\|^4+\|u^{(2)}\|^4)\\
&\le 4L^2(B)\|Y^{(1)}-Y^{(2)}\|_E^2+4L(B)4L(B)\|Y^{(1)}-Y^{(2)}\|_E^2\\
&\quad+12L^2(B)\|Y^{(1)}-Y^{(2)}\|_E^2\\
&\le 32L^2(B)\|Y^{(1)}-Y^{(2)}\|_E^2,
\end{aligned}\tag{3.4}$$

where $L(B)$ is a positive constant depending only on B. (3.4) implies that $G(Y)$ is locally (with respect to $Y\in E$) Lipschitz from E to E. We obtain the property (1) by the classical theory of ODEs.

To show the existence of a global solution of equations (3.2), we first transform (3.2) into pathwise equations. Denote

$$\hat{u}(t)=u(t)-W(t),\quad \hat{v}(t)=v(t)-W(t),\quad \hat{z}(t)=z(t)-W(t).$$

Then equations (3.2) are transformed into the equations

$$\begin{aligned}
\hat{u}(t)=u_0&+\int_0^t[-d_1A(\hat{u}(s)+W(s))-(F+k)(\hat{u}(s)+W(s))\\
&+(\hat{u}(s)+W(s))^2(\hat{v}(s)+W(s))]ds\\
&+\int_0^t[-(\hat{u}(s)+W(s))^3+N(\hat{z}(s)+W(s))+f_1]ds,
\end{aligned}\tag{3.5}$$

$$\begin{aligned}
\hat{v}(t)&=v_0+\int_0^t[-d_2A(\hat{v}(s)+W(s))-F(\hat{v}(s)+W(s))\\
&\quad-(\hat{u}(s)+W(s))^2(\hat{v}(s)+W(s))]ds+\int_0^t[(\hat{u}(s)+W(s))^3+f_2]ds,
\end{aligned}\tag{3.6}$$

$$\begin{aligned}\hat{z}(t) = z_0 + \int_0^t [&-d_3 A(\hat{z}(s) + W(s)) + k(\hat{u}(s) + W(s)) \\ &- (F+N)(\hat{z}(s) + W(s)) + f_3] ds.\end{aligned} \tag{3.7}$$

For each fixed $\omega \in \Omega$, equations (3.5)-(3.7) are pathwise equations. By the classical theory of ODEs, it follows that equations (3.5)-(3.7) has a local solution $(\hat{u}(t), \hat{v}(t), \hat{z}(t))^T \in L^2(\Omega, C([0, T_{max}), E))$, where $[0, T_{max})$ is the maximal interval of existence of the solution of (3.5)-(3.7). Next, we show that the local solution is a global solution.

For a fixed $\omega \in \Omega$, taking the inner product of equation (3.5)-(3.7) with $(\hat{u}, \hat{v}, \hat{z})^T$ in E, we obtain

$$\begin{aligned}
&\|\hat{u}(t)\|^2 + \|\hat{v}(t)\|^2 + \|\hat{z}(t)\|^2 \\
&\le \|u_0\|^2 + \|v_0\|^2 + \|z_0\|^2 - 2d_1 \int_0^t \langle A(\hat{u}(s) + W(s)), \hat{u}(s)\rangle ds \\
&\quad - 2(F+k) \int_0^t \langle \hat{u}(s) + W(s), \hat{u}(s)\rangle ds \\
&\quad + 2\int_0^t \langle (\hat{u}(s) + W(s))^2(\hat{v}(s) + W(s)), \hat{u}(s)\rangle ds \\
&\quad - 2\int_0^t \langle (\hat{u}(s) + W(s))^3, \hat{u}(s)\rangle ds + 2\int_0^t \langle f_1, \hat{u}(s)\rangle ds \\
&\quad + 2N \int_0^t \langle \hat{z}(s) + W(s), \hat{u}(s)\rangle ds - 2d_2 \int_0^t \langle A(\hat{v}(s) + W(s)), \hat{v}(s)\rangle ds \\
&\quad - 2F \int_0^t \langle \hat{v}(s) + W(s), \hat{v}(s)\rangle ds - 2\int_0^t \langle (\hat{u}(s) + W(s))^2(\hat{v}(s) + W(s)), \hat{v}(s)\rangle ds \\
&\quad + 2\int_0^t \langle (\hat{u}(s) + W(s))^3, \hat{v}(s)\rangle ds + 2\int_0^t \langle f_2, \hat{v}(s)\rangle ds \\
&\quad - 2d_3 \int_0^t \langle A(\hat{z}(s) + W(s)), \hat{z}(s)\rangle ds + 2k \int_0^t \langle \hat{u}(s) + W(s)), \hat{z}(s)\rangle ds \\
&\quad - 2(F+N) \int_0^t \langle \hat{z}(s) + W(s), \hat{z}(s)\rangle ds + 2\int_0^t \langle f_3, \hat{z}(s)\rangle ds.
\end{aligned} \tag{3.8}$$

Grouping the following terms on the right-hand side of (3.8), we have

$$\begin{aligned}
&2\int_0^t \langle (\hat{u}(s) + W(s))^2(\hat{v}(s) + W(s)), \hat{u}(s)\rangle ds - 2\int_0^t \langle (\hat{u}(s) + W(s))^3, \hat{u}(s)\rangle ds \\
&- 2\int_0^t \langle (\hat{u}(s) + W(s))^2(\hat{v}(s) + W(s)), \hat{v}(s)\rangle ds + 2\int_0^t \langle (\hat{u}(s) + W(s))^3, \hat{v}(s)\rangle ds \\
&= 2\int_0^t \langle (\hat{u}(s) + W(s))^2(\hat{v}(s) + W(s)), \hat{u}(s) - \hat{v}(s)\rangle ds \\
&\quad - 2\int_0^t \langle (\hat{u}(s) + W(s))^3, \hat{u}(s) - \hat{v}(s)\rangle ds \\
&= 2\int_0^t \langle (\hat{u}(s) + W(s))^2(\hat{v}(s) - \hat{u}(s)), \hat{u}(s) - \hat{v}(s)\rangle ds
\end{aligned}$$

$$= -2\int_0^t \sum_{i\in\mathbb{Z}} (\hat{u}_i(s) + W_i(s))^2(\hat{u}_i(s) - \hat{v}_i(s))^2 ds \leq 0. \tag{3.9}$$

From (3.8)-(3.9), we have

$$\begin{aligned}
&\|\hat{u}(t)\|^2 + \|\hat{v}(t)\|^2 + \|\hat{z}(t)\|^2 \\
&\leq \|u_0\|^2 + \|v_0\|^2 + \|z_0\|^2 - 2d_1\int_0^t \langle AW(s), \hat{u}(s)\rangle ds \\
&\quad - 2(F+k)\int_0^t \|\hat{u}(s)\|^2 ds - 2(F+k-N)\int_0^t \langle W(s), \hat{u}(s)\rangle ds \\
&\quad + 2(N+k)\int_0^t \langle \hat{z}(s), \hat{u}(s)\rangle ds - 2d_2\int_0^t \langle AW(s), \hat{v}(s)\rangle ds \\
&\quad - 2F\int_0^t \|\hat{v}(s)\|^2 ds - 2F\int_0^t \langle W(s), \hat{v}(s)\rangle ds - 2d_3\int_0^t \langle AW(s), \hat{z}(s)\rangle ds \\
&\quad - 2(F+N)\int_0^t \|\hat{z}(s)\|^2 ds - 2(F+N-k)\int_0^t \langle W(s), \hat{z}(s)\rangle ds \\
&\quad + 2\int_0^t \langle f_1, \hat{u}(s)\rangle ds + 2\int_0^t \langle f_2, \hat{v}(s)\rangle ds + 2\int_0^t \langle f_3, \hat{z}(s)\rangle ds.
\end{aligned} \tag{3.10}$$

By Young's inequality, we have the following estimates

$$\begin{gathered}
-2d_1\langle AW, \hat{u}\rangle \leq \frac{F+k}{4}\|\hat{u}\|^2 + \frac{4d_1^2}{F+k}\|A\|^2\|W\|^2, \\
-2d_2\langle AW, \hat{v}\rangle \leq \frac{F}{3}\|\hat{v}\|^2 + \frac{3d_2^2}{F}\|A\|^2\|W\|^2, \\
-2d_3\langle AW, \hat{z}\rangle \leq \frac{F+N}{4}\|\hat{z}\|^2 + \frac{4d_3^2}{F+N}\|A\|^2\|W\|^2, \\
-2(F+k-N)\langle W, \hat{u}\rangle \leq \frac{F+k}{4}\|\hat{u}\|^2 + \frac{4(F+k-N)^2}{F+k}\|W\|^2, \\
2(N+k)\langle \hat{u}, \hat{z}\rangle \leq (N+k)\|\hat{u}\|^2 + (N+k)\|\hat{z}\|^2, \\
-2F\langle W, \hat{v}\rangle \leq \frac{F}{3}\|\hat{v}\|^2 + 3F\|W\|^2, \\
-2(F+N-k)\langle W, \hat{z}\rangle \leq \frac{F+N}{4}\|\hat{z}\|^2 + \frac{4(F+N-k)^2}{F+N}\|W\|^2 \\
2\langle f_1, \hat{u}(s)\rangle \leq \frac{F+k}{4}\|\hat{u}\|^2 + \frac{4}{F+k}\|f_1\|^2, \\
2\langle f_2, \hat{v}(s)\rangle ds \leq \frac{F}{3}\|\hat{v}\|^2 + \frac{3}{F}\|f_2\|^2, \\
2\langle f_3, \hat{z}(s)\rangle ds \leq \frac{F+N}{4}\|\hat{z}\|^2 + \frac{4}{F+N}\|f_3\|^2.
\end{gathered}$$

By (3.1), we have

$$N+k \leq \frac{F+k}{4}, \quad N+k \leq \frac{F+N}{4}. \tag{3.11}$$

Thus, we obtain

$$\|\tilde{u}(t)\|^2 + \|\hat{v}(t)\|^2 + \|\hat{z}(t)\|^2$$

$$
\begin{aligned}
&\le \|u_0\|^2+\|v_0\|^2+\|z_0\|^2+\frac{F+k}{4}\int_0^t\|\hat u(s)\|^2ds+\frac{4d_1^2}{F+k}\|A\|^2\|W(s)\|^2ds\\
&\quad -2(F+k)\int_0^t\|\hat u(s)\|^2ds+\frac{F+k}{4}\int_0^t\|\tilde u(s)\|^2ds\\
&\quad +\frac{4(F+k-N)^2}{F+k}\int_0^t\|W(s)\|^2ds+(N+k)\int_0^t\|\hat u(s)\|^2ds\\
&\quad +(N+k)\int_0^t\|\hat z(s)\|^2ds+\frac{F}{3}\int_0^t\|\tilde v(s)\|^2ds+\frac{3d_2^2}{F}\int_0^t\|A\|^2\|W(s)\|^2ds\\
&\quad -2F\int_0^t\|\hat v(s)\|^2ds+\frac{F}{3}\int_0^t\|\tilde v(s)\|^2ds+3F\int_0^t\|W(s)\|^2ds\\
&\quad +\frac{F+N}{4}\int_0^t\|\hat z(s)\|^2ds+\frac{4d_3^2}{F+N}\int_0^t\|A\|^2\|W(s)\|^2ds\\
&\quad -2(F+N)\int_0^t\|\hat z(s)\|^2ds+\frac{F+N}{4}\int_0^t\|\hat z(s)\|^2ds\\
&\quad +\frac{4(F+N-k)^2}{F+N}\int_0^t\|W(s)\|^2ds+\frac{F+k}{4}\int_0^t\|\hat u(s)\|^2ds+\frac{4}{F+k}\int_0^t\|f_1\|^2ds\\
&\quad +\frac{F}{3}\int_0^t\|\hat v(s)\|^2ds+\frac{3}{F}\int_0^t\|f_2\|^2ds+\frac{F+N}{4}\int_0^t\|\hat z(s)\|^2ds\\
&\quad +\frac{4}{F+N}\int_0^t\|f_3\|^2ds\\
&\le \|u_0\|^2+\|v_0\|^2+\|z_0\|^2-(F+k)\int_0^t\|\hat u(s)\|^2ds-F\int_0^t\|\hat v(s)\|^2ds\\
&\quad -(F+N)\int_0^t\|\hat z(s)\|^2ds+\frac{4d_1^2}{F+k}\|A\|^2\|W(s)\|^2ds\\
&\quad +\frac{4(F+k-N)^2}{F+k}\int_0^t\|W(s)\|^2ds+\frac{3d_2^2}{F}\int_0^t\|A\|^2\|W(s)\|^2ds\\
&\quad +3F\int_0^t\|W(s)\|^2ds+\frac{4d_3^2}{F+N}\int_0^t\|A\|^2\|W(s)\|^2ds\\
&\quad +\frac{4(F+N-k)^2}{F+N}\int_0^t\|W(s)\|^2ds+\frac{4}{F+k}\int_0^t\|f_1\|^2ds+\frac{3}{F}\int_0^t\|f_2\|^2ds\\
&\quad +\frac{4}{F+N}\int_0^t\|f_3\|^2ds\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\|\tilde u(t)\|^2+\|\hat v(t)\|^2+\|\hat z(t)\|^2\\
&\le \|u_0\|^2+\|v_0\|^2+\|z_0\|^2-(F+k)\int_0^t\|\hat u(s)\|^2ds-F\int_0^t\|\hat v(s)\|^2ds\\
&\quad -(F+N)\int_0^t\|\hat z(s)\|^2ds+\Big(\frac{4d_1^2}{F+k}+\frac{3d_2^2}{F}+\frac{4d_3^2}{F+N}\Big)\int_0^t\|A\|^2\|W(s)\|^2ds\\
&\quad +\Big(\frac{4(F+k-N)^2}{F+k}+3F+\frac{4(F+N-k)^2}{F+N}\Big)\int_0^t\|W(s)\|^2ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4}{F+k}\int_0^t\|f_1\|^2ds+\frac{3}{F}\int_0^t\|f_2\|^2ds+\frac{4}{F+N}\int_0^t\|f_3\|^2ds\\
&\leq\|u_0\|^2+\|v_0\|^2+\|z_0\|^2-F\int_0^t(\|\hat{u}(s)\|^2+\|\hat{v}(s)\|^2+\|\hat{z}(s)\|^2)ds\\
&+C_1\int_0^t\|A\|^2\|W(s)\|^2ds+C_2\int_0^t\|W(s)\|^2ds\\
&+C_3\int_0^t(\|f_1\|^2+\|f_2\|^2+\|f_3\|^2)ds\\
&\leq\|u_0\|^2+\|v_0\|^2+\|z_0\|^2+C_0\int_0^t(\|W(s)\|^2+\|f_1\|^2+\|f_2\|^2+\|f_3\|^2)ds, \quad (3.12)
\end{aligned}
$$

where

$$
\begin{gathered}
C_1=\frac{4d_1^2}{F+k}+\frac{3d_2^2}{F}+\frac{4d_3^2}{F+N},\\
C_2=\frac{4(F+k-N)^2}{F+k}+3F+\frac{4(F+N-k)^2}{F+N},\\
C_3=\max\{\frac{4}{F+k},\frac{3}{F},\frac{4}{F+N}\},\quad C_0=\max\{C_1\|A\|^2+C_2,C_3\}.
\end{gathered} \tag{3.13}
$$

Hence, from (3.12), we obtain that $\|\hat{u}(t)\|^2+\|\hat{v}(t)\|^2+\|\hat{z}(t)\|^2$ is bounded by a continuous function, which implies the global existence of a solution on interval $[0,T]$. Therefore, for all $\omega\in\Omega$, it follows that

$$
\begin{aligned}
&\sup_{t\in[0,T]}[\|u(t)\|^2+\|v(t)\|^2+\|z(t)\|^2]\\
&=\sup_{t\in[0,T]}[\|\hat{u}(t)+W(t)\|^2+\|\hat{v}(t)+W(t)\|^2+\|\hat{z}(t)+W(t)\|^2]\\
&\leq 2(\|u_0\|^2+\|v_0\|^2+\|z_0\|^2)+6\sup_{t\in[0,T]}\|W(t)\|^2\\
&\quad+2C_0\int_0^T(\|W(s)\|^2+\|f_1\|^2+\|f_2\|^2+\|f_3\|^2)ds.
\end{aligned} \tag{3.14}
$$

Thus, the proof is complete. □

Theorem 3.2. *Equation* (3.2) *generates a continuous RDS* $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ *over* $(\Omega,\mathcal{F},\mathcal{P},(\theta_t)_{t\in\mathbb{R}})$, *where*

$$
\varphi(t,\omega,(u_0,v_0,z_0))=(u(t,\omega,u_0),v(t,\omega,v_0),z(t,\omega,z_0)),\quad \text{for all } t\geq 0,\ \omega\in\Omega.
$$

The proof of the above theorem is similar to that of [3, Theorem 3.2]. We omit it.

4. Existence of random attractors

In this section, we prove the existence of a random attractor for the random dynamical system generated by (1.1). To convert the stochastic wave equation to a pathwise one with random parameters, we introduce an Ornstein-Uhlenbeck process in ℓ^2 on the metric dynamical systems $(\Omega,\mathcal{F},\mathcal{P},(\theta_t)_{t\in\mathbb{R}})$ given by the Wiener process:

$$
y(\theta_t\omega)=-(F+N+k)\int_{-\infty}^0 e^{(F+N+k)s}(\theta_t\omega)(s)ds,\quad t\in\mathbb{R},\ \omega\in\Omega.
$$

The above integral exists for any path ω with a sub-exponential growth, and y solve the following Itô equations respectively:

$$dy + (F + N + k)ydt = dw(t), \quad t > 0.$$

Furthermore, there exists a θ_t-invariant set $\Omega' \subset \Omega$ of full $\mathcal{P}$ measure such that

(1) the mappings $s \to y(\theta_s\omega)$, is continuous for each $\omega \in \Omega$;
(2) the random variables $\|y(\theta_t\omega)\|$ is tempered.

Let

$$\tilde{u}(t) = u(t) - y(\theta_t\omega), \quad \tilde{v}(t) = v(t) - y(\theta_t\omega), \quad \tilde{z}(t) = z(t) - y(\theta_t\omega).$$

Then we obtain

$$\begin{aligned}
\tilde{u}_t &= -d_1A(\tilde{u} + y(\theta_t\omega)) - (F + k)\tilde{u} + (\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega)) \\
&\quad - (\tilde{u} + y(\theta_t\omega))^3 + N\tilde{z} + f_1 + 2Ny(\theta_t\omega), \\
\tilde{v}_t &= -d_2A(\tilde{v} + y(\theta_t\omega)) - F\tilde{v} - (\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega)) \\
&\quad + (\tilde{u} + y(\theta_t\omega))^3 + f_2 + (N + k)y(\theta_t\omega), \\
\tilde{z}_t &= -d_3A(\tilde{z} + y(\theta_t\omega)) + k\tilde{u} - (F + N)\tilde{z} + f_3 + 2ky(\theta_t\omega),
\end{aligned} \tag{4.1}$$

with the initial value conditions

$$\begin{gathered}
\tilde{u}(0, \omega, \tilde{u}_0) = \tilde{u}_0(\omega) = u_0 - y(\omega), \quad \tilde{v}(0, \omega, \tilde{v}_0) = \tilde{v}_0(\omega) = v_0 - y(\omega), \\
\tilde{z}(0, \omega, \tilde{z}_0) = \tilde{z}_0(\omega) = v_0 - y(\omega).
\end{gathered}$$

Lemma 4.1. *Let (3.1) hold. There exists a θ_t-invariant set $\Omega' \subset \Omega$ of full $\mathcal{P}$ measure and an absorbing random set $K(\omega), \omega \in \Omega'$, for the random dynamical system $\varphi(t, \omega)$, i.e. for all $B \in \mathcal{D}$ and all $\omega \in \Omega'$, there exists $T_B(\omega) > 0$ such that*

$$\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega) \quad \text{for all } t \geq T_B(\omega).$$

Moreover, $K \in \mathcal{D}$.

Proof. Taking the inner product of (4.1) with $(\tilde{u}, \tilde{v}, \tilde{z})^T$ in E, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\tilde{u}\|^2 \\
&= -d_1\langle A\tilde{u}, \tilde{u}\rangle - d_1\langle Ay(\theta_t\omega), \tilde{u}\rangle - (F + k)\|\tilde{u}\|^2 + \langle(\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega)), \tilde{u}\rangle \\
&\quad - \langle(\tilde{u} + y(\theta_t\omega))^3, \tilde{u}\rangle + \langle N\tilde{z}, \tilde{u}\rangle + \langle f_1, \tilde{u}\rangle + 2N\langle y(\theta_t\omega), \tilde{u}\rangle, \\
&\frac{1}{2}\frac{d}{dt}\|\tilde{v}\|^2 \\
&= -d_2\langle A\tilde{v}, \tilde{v}\rangle - d_2\langle Ay(\theta_t\omega), \tilde{v}\rangle - F\|\tilde{v}\|^2 - \langle(\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega)), \tilde{v}\rangle \\
&\quad + \langle(\tilde{u} + y(\theta_t\omega))^3, \tilde{v}\rangle + \langle f_2, \tilde{v}\rangle + (N + k)\langle y(\theta_t\omega), \tilde{v}\rangle, \\
&\frac{1}{2}\frac{d}{dt}\|\tilde{z}\|^2 = -d_3\langle A\tilde{z}, \tilde{z}\rangle - d_3\langle Ay(\theta_t\omega)), \tilde{z}\rangle + k\langle\tilde{u}, \tilde{z}\rangle - (F + N)\|\tilde{z}\|^2 \\
&\qquad + \langle f_3, \tilde{z}\rangle + 2k\langle y(\theta_t\omega), \tilde{z}\rangle.
\end{aligned} \tag{4.2}$$

Summing the three equations, we obtain

$$
\begin{aligned}
&\frac{d}{dt}[\|\tilde{u}\|^2+\|\tilde{v}\|^2+\|\tilde{z}\|^2]+2d_1\langle A\tilde{u},\tilde{u}\rangle+2d_2\langle A\tilde{v},\tilde{v}\rangle+2d_3\langle A\tilde{z},\tilde{z}\rangle\\
&\quad+2(F+k)\|\tilde{u}\|^2+2F\|\tilde{v}\|^2+2(F+N)\|\tilde{z}\|^2\\
&=-2d_1\langle Ay(\theta_t\omega),\tilde{u}\rangle-2d_2\langle Ay(\theta_t\omega),\tilde{v}\rangle-2d_3\langle Ay(\theta_t\omega)),\tilde{z}\rangle+2(N+k)\langle\tilde{z},\tilde{u}\rangle\\
&\quad+2\langle f_1,\tilde{u}\rangle+4N\langle y(\theta_t\omega),\tilde{u}\rangle-2\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)),\tilde{v}\rangle\\
&\quad+2\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)),\tilde{u}\rangle-2\langle(\tilde{u}+y(\theta_t\omega))^3,\tilde{u}\rangle\\
&\quad+2\langle(\tilde{u}+y(\theta_t\omega))^3,\tilde{v}\rangle+2\langle f_2,\tilde{v}\rangle+2(N+k)\langle y(\theta_t\omega),\tilde{v}\rangle+2\langle f_3,\tilde{z}\rangle\\
&\quad+4k\langle y(\theta_t\omega),\tilde{z}\rangle.
\end{aligned}
\tag{4.3}
$$

Similar to (3.9), we have

$$
\begin{aligned}
&-2\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)),\tilde{v}\rangle+2\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)),\tilde{u}\rangle\\
&-2\langle(\tilde{u}+y(\theta_t\omega))^3,\tilde{u}\rangle+2\langle(\tilde{u}+y(\theta_t\omega))^3,\tilde{v}\rangle\\
&=2\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)),\tilde{u}-\tilde{v}\rangle-2\langle(\tilde{u}+y(\theta_t\omega))^3,\tilde{u}-\tilde{v}\rangle\\
&=-2\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{u}-\tilde{v}),\tilde{u}-\tilde{v}\rangle\\
&=-2\sum_{i\in\mathbb{Z}}(\tilde{u}_i+y_i(\theta_t\omega))^2(\tilde{u}_i-\tilde{v}_i)^2\le 0.
\end{aligned}
\tag{4.4}
$$

By Young's inequality, we have the estimates

$$
\begin{gathered}
-2d_1\langle Ay(\theta_t\omega),\tilde{u}\rangle\le\frac{F+k}{4}\|\tilde{u}\|^2+\frac{4d_1^2}{F+k}\|Ay(\theta_t\omega)\|^2,\\
-2d_2\langle Ay(\theta_t\omega),\tilde{v}\rangle\le\frac{F}{3}\|\tilde{v}\|^2+\frac{3d_2^2}{F}\|Ay(\theta_t\omega)\|^2,\\
-2d_3\langle Ay(\theta_t\omega),\tilde{z}\rangle\le\frac{F+N}{4}\|\tilde{z}\|^2+\frac{4d_3^2}{F+N}\|Ay(\theta_t\omega)\|^2,\\
4N\langle y(\theta_t\omega),\tilde{u}\rangle\le\frac{F+k}{4}\|\tilde{u}\|^2+\frac{16N^2}{F+k}\|y(\theta_t\omega)\|^2,\\
2(N+k)\langle\tilde{u},\tilde{z}\rangle\le(N+k)\|\tilde{u}\|^2+(N+k)\|\tilde{z}\|^2,\\
2(N+k)\langle y(\theta_t\omega),\tilde{v}\rangle\le\frac{F}{3}\|\tilde{v}\|^2+\frac{3(N+k)^2}{F}\|y(\theta_t\omega)\|^2,\\
4k\langle y(\theta_t\omega),\tilde{z}\rangle\le\frac{F+N}{4}\|\tilde{z}\|^2+\frac{16k^2}{F+N}\|y(\theta_t\omega)\|^2,\\
2\langle f_1,\tilde{u}\rangle\le\frac{F+k}{4}\|\tilde{u}\|^2+\frac{4}{F+k}\|f_1\|^2,\\
2\langle f_2,\tilde{v}\rangle ds\le\frac{F}{3}\|\tilde{v}\|^2+\frac{3}{F}\|f_2\|^2,\\
2\langle f_3,\tilde{z}\rangle ds\le\frac{F+N}{4}\|\tilde{z}\|^2+\frac{4}{F+N}\|f_3\|^2.
\end{gathered}
\tag{4.5}
$$

By (4.3)-(4.5), we obtain

$$
\begin{aligned}
&\frac{d}{dt}[\|\tilde{u}\|^2+\|\tilde{v}\|^2+\|\tilde{z}\|^2]+2(F+k)\|\tilde{u}\|^2+2F\|\tilde{v}\|^2+2(F+N)\|\tilde{z}\|^2\\
&\le\frac{F+k}{4}\|\tilde{u}\|^2+\frac{4d_1^2}{F+k}\|Ay(\theta_t\omega)\|^2+\frac{F}{3}\|\tilde{v}\|^2+\frac{3d_2^2}{F}\|Ay(\theta_t\omega)\|^2+\frac{F+N}{4}\|\tilde{z}\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4d_3^2}{F+N}\|Ay(\theta_t\omega)\|^2+(N+k)\|\tilde{u}\|^2+(N+k)\|\tilde{z}\|^2+\frac{F+k}{4}\|\tilde{u}\|^2\\
&+\frac{4}{F+k}\|f_1\|^2+\frac{F+k}{4}\|\tilde{u}\|^2+\frac{16N^2}{F+k}\|y(\theta_t\omega)\|^2+\frac{F}{3}\|\tilde{v}\|^2\\
&+\frac{3}{F}\|f_2\|^2+\frac{F}{3}\|\tilde{v}\|^2+\frac{3(N+k)^2}{F}\|y(\theta_t\omega)\|^2+\frac{F+N}{4}\|\tilde{z}\|^2\\
&+\frac{4}{F+N}\|f_3\|^2+\frac{F+N}{4}\|\tilde{z}\|^2+\frac{16k^2}{F+N}\|y(\theta_t\omega)\|^2.
\end{aligned}
$$

From (3.1) and (3.11), we have

$$
\begin{aligned}
&\frac{d}{dt}[\|\tilde{u}\|^2+\|\tilde{v}\|^2+\|\tilde{z}\|^2]+(F+k)\|\tilde{u}\|^2+F\|\tilde{v}\|^2+(F+N)\|\tilde{z}\|^2\\
&\le \frac{4d_1^2}{F+k}\|Ay(\theta_t\omega)\|^2+\frac{3d_2^2}{F}\|Ay(\theta_t\omega)\|^2+\frac{4d_3^2}{F+N}\|Ay(\theta_t\omega)\|^2\\
&\quad+\frac{16N^2}{F+k}\|y(\theta_t\omega)\|^2+\frac{3(N+k)^2}{F}\|y(\theta_t\omega)\|^2\\
&\quad+\frac{16k^2}{F+N}\|y(\theta_t\omega)\|^2+\frac{4}{F+k}\|f_1\|^2+\frac{3}{F}\|f_2\|^2+\frac{4}{F+N}\|f_3\|^2\\
&\le C_1\|Ay(\theta_t\omega)\|^2+C_4\|y(\theta_t\omega)\|^2+C_3(\|f_1\|^2+\|f_2\|^2+\|f_3\|^2)\\
&\le C_5(\|y(\theta_t\omega)\|^2+\|Ay(\theta_t\omega)\|^2+\|f_1\|^2+\|f_2\|^2+\|f_3\|^2),
\end{aligned}
\tag{4.6}
$$

where C_1 and C_3 are defined in (3.13), $C_4=16N^2/(F+k)+3(N+k)^2/F+16k^2/(F+N)$, and $C_5=\max\{C_1,C_3,C_4\}$. By Gronwall's inequality, it follows that

$$
\begin{aligned}
&\|\tilde{u}(t,\omega,\tilde{u}_0(\omega))\|^2+\|\tilde{v}(t,\omega,\tilde{v}_0(\omega))\|^2+\|\tilde{z}(t,\omega,\tilde{z}_0(\omega))\|^2\\
&\le e^{-Ft}[\|\tilde{u}_0(\omega)\|^2+\|\tilde{v}_0(\omega)\|^2+\|\tilde{z}_0(\omega)\|^2]+\frac{C_5}{F}(\|f_1\|^2+\|f_2\|^2+\|f_3\|^2)\\
&\quad+C_5\int_0^t e^{-F(t-s)}(\|y(\theta_s\omega)\|^2+\|Ay(\theta_s\omega)\|^2)ds.
\end{aligned}
\tag{4.7}
$$

Note that the random variable $y(\theta_t\omega)$ is tempered and $y(\theta_t\omega)$ is continuous in t. Therefore, it follows from Proposition 4.3.3 in [2] that there exists a tempered function $l(\omega)>0$ such that

$$
\|y(\theta_t\omega)\|^2+\|Ay(\theta_t\omega)\|^2\le l(\theta_t\omega)\le l(\omega)e^{F|t|/2}. \tag{4.8}
$$

Replacing ω by $\theta_{-t}\omega$ in (4.7) and using (4.8), we obtain

$$
\begin{aligned}
&\|\tilde{u}(t,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega))\|^2+\|\tilde{v}(t,\theta_{-t}\omega,\tilde{v}_0(\theta_{-t}\omega))\|^2+\|\tilde{z}(t,\theta_{-t}\omega,\tilde{z}_0(\theta_{-t}\omega))\|^2\\
&\le e^{-Ft}[\|\tilde{u}_0(\theta_{-t}\omega)\|^2+\|\tilde{v}_0(\theta_{-t}\omega)\|^2+\|\tilde{z}_0(\theta_{-t}\omega)\|^2]+\frac{C_5}{F}(\|f_1\|^2+\|f_2\|^2+\|f_3\|^2)\\
&\quad+C_5\int_0^t e^{-F(t-s)}(\|y(\theta_{s-t}\omega)\|^2+\|Ay(\theta_{s-t}\omega)\|^2)ds\\
&\le e^{-Ft}[\|\tilde{u}_0(\theta_{-t}\omega)\|^2+\|\tilde{v}_0(\theta_{-t}\omega)\|^2+\|\tilde{z}_0(\theta_{-t}\omega)\|^2]+\frac{C_5}{F}(\|f_1\|^2+\|f_2\|^2+\|f_3\|^2)\\
&\quad+C_5\int_{-t}^0 e^{F\tau}(\|y(\theta_\tau\omega)\|^2+\|Ay(\theta_\tau\omega)\|^2)d\tau\\
&\le e^{-Ft}[\|\tilde{u}_0(\theta_{-t}\omega)\|^2+\|\tilde{v}_0(\theta_{-t}\omega)\|^2+\|\tilde{z}_0(\theta_{-t}\omega)\|^2]
\end{aligned}
$$

$$+\frac{C_5}{F}(\|f_1\|^2+\|f_2\|^2+\|f_3\|^2)+\frac{2C_5 l(\omega)}{F}. \tag{4.9}$$

Define $R^2(\omega)=2[C_5(\|f_1\|^2+\|f_2\|^2+\|f_3\|^2)+2C_5 l(\omega)]/F$; since $l(\omega)$ is a tempered function, then $R(\omega)$ is also tempered. Define

$$\tilde{K}(\omega)=\{(\tilde{u},\tilde{v},\tilde{z})\in\ell^2\times\ell^2\times\ell^2,\|\tilde{u}\|^2+\|\tilde{v}\|^2+\|\tilde{z}\|^2\le R^2(\omega)\}.$$

Then $\tilde{K}(\omega)$ is an absorbing set for the random dynamical system

$$(\tilde{u}(t,\omega,\tilde{u}_0),\tilde{v}(t,\omega,\tilde{v}_0),\tilde{z}(t,\omega,\tilde{z}_0));$$

that is, for every $B\in\mathcal{D}$ and every $\omega\in\Omega'$, there exists $T_B(\omega)$ such that

$$\Phi(t,\theta_{-t}\omega,B(\theta_{-t}\omega))\subset\tilde{K}(\omega)\quad\text{for } t\ge T_B(\omega).$$

Let

$$K(\omega)=\{(u,v,z)\in\ell^2\times\ell^2\times\ell^2,\|u\|^2+\|v\|^2+\|z\|^2\le R_1^2(\omega)\},$$

where

$$R_1^2(\omega)=2R^2(\omega)+6\|y(\theta_t\omega)\|^2.$$

Then, $K(\omega)$ is an absorbing random set for the random dynamical system $\varphi(t,\omega)$ since

$$\begin{aligned}
&\varphi(t,\omega,(u_0,v_0,z_0))\\
&=\Phi(t,\omega,(u_0-y(\omega),v_0-y(\omega),z_0-y(\omega)))+(y(\theta_t\omega),y(\theta_t\omega),y(\theta_t\omega))\\
&=(\tilde{u}(t,\omega,u_0-y(\omega))+y(\theta_t\omega),\tilde{v}(t,\omega,v_0-y(\omega))+y(\theta_t\omega),\tilde{z}(t,\omega,z_0-y(\omega))\\
&\quad+y(\theta_t\omega))
\end{aligned}$$

and $K\in\mathcal{D}$. This completes the proof. □

To prove the pullback asymptotic compactness for the dynamical system φ, we first prove the following lemma.

Lemma 4.2. *Let* (3.1) *hold. Assume the initial functions* $(u_0(\omega),v_0(\omega),z_0(\omega))\in K(\omega)$, *where* $K(\omega)$ *is the absorbing set in Lemma 4.1. Then for every* $\varepsilon>0$, *there exist* $T(\varepsilon,\omega)>0$ *and* $N(\varepsilon,\omega)>0$ *such that the solution*

$$(u(t,\omega,u_0(\omega)),v(t,\omega,v_0(\omega)),z(t,\omega,z_0(\omega)))$$

of (1.1) *satisfies*

$$\begin{aligned}
\sum_{|i|\ge N(\varepsilon,\omega)}&\Big[\|u(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\|^2+\|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2\\
&+\|z(t,\theta_{-t}\omega,z_0(\theta_{-t}\omega))\|^2\Big]<\varepsilon,
\end{aligned}$$

for all $t\ge T(\varepsilon,\omega)>0$.

Proof. We choose a smooth function ρ such that $0\le\rho\le1$ for all $s\in\mathbb{R}$ and

$$\rho(s)=\begin{cases}0, & \text{if } |s|<1,\\ 1, & \text{if } |s|>2,\end{cases} \tag{4.10}$$

and there exists a positive constant C_6, such that $|\rho'(s)|\le C_6$ for $s\in\mathbb{R}$.

We first consider the random equation (4.1). Let r be a fixed positive integer which will be specified later. Taking the inner product of (4.1) with $\rho(\frac{|i|}{r})\tilde{u}$, $\rho(\frac{|i|}{r})\tilde{v}$ and $\rho(\frac{|i|}{r})\tilde{z}$ in E, respectively, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\rho\big(\frac{|i|}{r}\big)|\tilde{u}_i|^2\\
&= -d_1\langle A\tilde{u},\rho\big(\frac{|i|}{r}\big)\tilde{u}\rangle - d_1\langle Ay(\theta_t\omega),\rho\big(\frac{|i|}{r}\big)\tilde{u}\rangle - (F+k)\sum_{i\in\mathbb{Z}}\rho\big(\frac{|i|}{r}\big)|\tilde{u}_i|^2\\
&\quad + \langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)),\rho\big(\frac{|i|}{r}\big)\tilde{u}\rangle - \langle(\tilde{u}+y(\theta_t\omega))^3,\rho\big(\frac{|i|}{r}\big)\tilde{u}\rangle\\
&\quad + N\langle\tilde{z},\rho\big(\frac{|i|}{r}\big)\tilde{u}\rangle + \langle f_1,\rho\big(\frac{|i|}{r}\big)\tilde{u}\rangle + 2N\langle y(\theta_t\omega),\rho\big(\frac{|i|}{r}\big)\tilde{u}\rangle,\\
&\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\rho\big(\frac{|i|}{r}\big)|\tilde{v}_i|^2\\
&= -d_2\langle A\tilde{v},\rho\big(\frac{|i|}{r}\big)\tilde{v}\rangle - d_2\langle Ay(\theta_t\omega),\rho\big(\frac{|i|}{r}\big)\tilde{v}\rangle - F\sum_{i\in\mathbb{Z}}\rho\big(\frac{|i|}{r}\big)|\tilde{v}_i|^2\\
&\quad - \langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)),\rho\big(\frac{|i|}{r}\big)\tilde{v}\rangle + \langle(\tilde{u}+y(\theta_t\omega))^3,\rho\big(\frac{|i|}{r}\big)\tilde{v}\rangle\\
&\quad + \langle f_2,\rho\big(\frac{|i|}{r}\big)\tilde{v}\rangle + (N+k)\langle y(\theta_t\omega),\rho\big(\frac{|i|}{r}\big)\tilde{v}\rangle,\\
&\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\rho\big(\frac{|i|}{r}\big)|\tilde{z}_i|^2\\
&= -d_3\langle A\tilde{z},\rho\big(\frac{|i|}{r}\big)\tilde{z}\rangle - d_3\langle Ay(\theta_t\omega),\rho\big(\frac{|i|}{r}\big)\tilde{z}\rangle - (F+N)\sum_{i\in\mathbb{Z}}\rho\big(\frac{|i|}{r}\big)|\tilde{z}_i|^2\\
&\quad + k\langle\tilde{u},\rho\big(\frac{|i|}{r}\big)\tilde{z}\rangle + \langle f_3,\rho\big(\frac{|i|}{r}\big)\tilde{z}\rangle + 2k\langle y(\theta_t\omega),\rho\big(\frac{|i|}{r}\big)\tilde{z}\rangle.
\end{aligned}
\tag{4.11}
$$

Recalling the property $|\rho'(s)|\le C_6$, we have

$$
\begin{aligned}
&\langle A\tilde{u},\rho\big(\frac{|i|}{r}\big)\tilde{u}\rangle\\
&= \sum_{i\in\mathbb{Z}}(B\tilde{u})_i\Big(B\rho\big(\frac{|i|}{r}\big)\tilde{u}\Big)_i\\
&= \sum_{i\in\mathbb{Z}}(B\tilde{u})_i\Big[\rho\big(\frac{|i+1|}{r}\big)\tilde{u}_{i+1} - \rho\big(\frac{|i|}{r}\big)\tilde{u}_i\Big]\\
&= \sum_{i\in\mathbb{Z}}(\tilde{u}_{i+1}-\tilde{u}_i)\Big[\rho\big(\frac{|i|}{r}\big)(\tilde{u}_{i+1}-\tilde{u}_i) + \Big(\rho\big(\frac{|i+1|}{r}\big) - \rho\big(\frac{|i|}{r}\big)\Big)\tilde{u}_{i+1}\Big]\\
&\ge \sum_{i\in\mathbb{Z}}\rho\big(\frac{|i|}{r}\big)|\tilde{u}_{i+1}-\tilde{u}_i|^2 - \sum_{i\in\mathbb{Z}}\frac{\rho'(\xi)}{r}(\tilde{u}_{i+1}-\tilde{u}_i)\tilde{u}_{i+1}\\
&\ge -\frac{C_6}{r}\sum_{i\in\mathbb{Z}}(|\tilde{u}_{i+1}|^2 + |\tilde{u}_i||\tilde{u}_{i+1}|) \ge -\frac{2C_6}{r}\|\tilde{u}\|^2.
\end{aligned}
\tag{4.12}
$$

Similarly, we have

$$\begin{aligned}
\langle A\tilde{v}, \rho(\frac{|i|}{r})\tilde{v}\rangle &= \sum_{i\in Z}(B\tilde{v})_i\Big(B\rho(\frac{|i|}{r})\tilde{v}\Big)_i \geq -\frac{2C_6}{r}\|\tilde{v}\|^2, \\
\langle A\tilde{z}, \rho(\frac{|i|}{r})\tilde{z}\rangle &= \sum_{i\in Z}(B\tilde{z})_i\Big(B\rho(\frac{|i|}{r})\tilde{z}\Big)_i \geq -\frac{2C_6}{r}\|\tilde{z}\|^2.
\end{aligned} \tag{4.13}$$

Summing the three equations in (4.11), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\rho(\frac{|i|}{r})\left[|\tilde{u}_i|^2+|\tilde{v}_i|^2+|\tilde{z}_i|^2\right]+(F+k)\sum_{i\in\mathbb{Z}}\rho(\frac{|i|}{r})|\tilde{u}_i|^2 \\
&+F\sum_{i\in\mathbb{Z}}\rho(\frac{|i|}{r})|\tilde{v}_i|^2+(F+N)\sum_{i\in\mathbb{Z}}\rho(\frac{|i|}{r})|\tilde{z}_i|^2 \\
&\leq \frac{2C_6d_1}{r}\|\tilde{u}\|^2+\frac{2C_6d_2}{r}\|\tilde{v}\|^2+\frac{2C_6d_3}{r}\|\tilde{z}\|^2+(N+k)\langle\tilde{z},\rho(\frac{|i|}{r})\tilde{u}\rangle \\
&\quad+\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)),\rho(\frac{|i|}{r})\tilde{u}\rangle-\langle(\tilde{u}+y(\theta_t\omega))^3,\rho(\frac{|i|}{r})\tilde{u}\rangle \\
&\quad-\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)),\rho(\frac{|i|}{r})\tilde{v}\rangle+\langle(\tilde{u}+y(\theta_t\omega))^3,\rho(\frac{|i|}{r})\tilde{v}\rangle \\
&\quad+\langle f_1,\rho(\frac{|i|}{r})\tilde{u}\rangle+\langle f_2,\rho(\frac{|i|}{r})\tilde{v}\rangle+\langle f_3,\rho(\frac{|i|}{r})\tilde{z}\rangle \\
&\quad-d_1\langle Ay(\theta_t\omega),\rho(\frac{|i|}{r})\tilde{u}\rangle-d_2\langle Ay(\theta_t\omega),\rho(\frac{|i|}{r})\tilde{v}\rangle-d_3\langle Ay(\theta_t\omega),\rho(\frac{|i|}{r})\tilde{z}\rangle \\
&\quad+2N\langle y(\theta_t\omega),\rho(\frac{|i|}{r})\tilde{u}\rangle+(N+k)\langle y(\theta_t\omega),\rho(\frac{|i|}{r})\tilde{v}\rangle+2k\langle y(\theta_t\omega),\rho(\frac{|i|}{r})\tilde{z}\rangle.
\end{aligned} \tag{4.14}$$

Grouping from the fifth term to eighth term of the right-hand side of (4.14), we have

$$\begin{aligned}
&\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)),\rho(\frac{|i|}{r})\tilde{u}\rangle-\langle(\tilde{u}+y(\theta_t\omega))^3,\rho(\frac{|i|}{r})\tilde{u}\rangle \\
&-\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)),\rho(\frac{|i|}{r})\tilde{v}\rangle+\langle(\tilde{u}+y(\theta_t\omega))^3,\rho(\frac{|i|}{r})\tilde{v}\rangle \\
&=\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)),\rho(\frac{|i|}{r})(\tilde{u}-\tilde{v})\rangle \\
&\quad-\langle(\tilde{u}+y(\theta_t\omega))^3,\rho(\frac{|i|}{r})(\tilde{u}-\tilde{v})\rangle \\
&=\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}-\tilde{u}),\rho(\frac{|i|}{r})(\tilde{u}-\tilde{v})\rangle \\
&=-\sum_{i\in\mathbb{Z}}\rho(\frac{|i|}{r})(\tilde{u}_i+y_i(\theta_t\omega))^2(\tilde{u}_i-\tilde{v}_i)^2\leq 0.
\end{aligned} \tag{4.15}$$

Then (4.14) can be reduced to

$$\begin{aligned}
&\frac{d}{dt}\sum_{i\in\mathbb{Z}}\rho(\frac{|i|}{r})\left[|\tilde{u}_i|^2+|\tilde{v}_i|^2+|\tilde{z}_i|^2\right]+2(F+k)\sum_{i\in\mathbb{Z}}\rho(\frac{|i|}{r})|\tilde{u}_i|^2 \\
&+2F\sum_{i\in\mathbb{Z}}\rho(\frac{|i|}{r})|\tilde{v}_i|^2+2(F+N)\sum_{i\in\mathbb{Z}}\rho(\frac{|i|}{r})|\tilde{z}_i|^2
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{4C_6 d_1}{r}\|\tilde u\|^2 + \frac{4C_6 d_2}{r}\|\tilde v\|^2 + \frac{4C_6 d_3}{r}\|\tilde z\|^2 + 2(N+k)\langle \tilde z, \rho(\frac{|i|}{r})\tilde u\rangle \\
&\quad + 2\langle f_1, \rho(\frac{|i|}{r})\tilde u\rangle + 2\langle f_2, \rho(\frac{|i|}{r})\tilde v\rangle + 2\langle f_3, \rho(\frac{|i|}{r})\tilde z\rangle \\
&\quad - 2d_1\langle Ay(\theta_t\omega), \rho(\frac{|i|}{r})\tilde u\rangle - 2d_2\langle Ay(\theta_t\omega), \rho(\frac{|i|}{r})\tilde v\rangle - 2d_3\langle Ay(\theta_t\omega), \rho(\frac{|i|}{r})\tilde z\rangle \\
&\quad + 4N\langle y(\theta_t\omega), \rho(\frac{|i|}{r})\tilde u\rangle + 2(N+k)\langle y(\theta_t\omega), \rho(\frac{|i|}{r})\tilde v\rangle + 4k\langle y(\theta_t\omega), \rho(\frac{|i|}{r})\tilde z\rangle.
\end{aligned} \tag{4.16}
$$

For the second term to forth term in the right-hand side of (4.16), we have

$$
\begin{aligned}
\langle f_1, \rho(\frac{|i|}{r})\tilde u\rangle &= \sum_{i\in\mathbb{Z}} \rho(\frac{|i|}{r}) f_{1i}\tilde u_i \\
&= \sum_{|i|\ge r} \rho(\frac{|i|}{r}) f_{1i}\tilde u_i \\
&\le \frac{F+k}{4}\sum_{|i|\ge r}\rho(\frac{|i|}{r})|\tilde u_i|^2 + \frac{4}{F+k}\sum_{|i|\ge r}|f_{1i}|^2,
\end{aligned}
$$

$$
\langle f_2, \rho(\frac{|i|}{r})\tilde v\rangle = \sum_{|i|\ge r}\rho(\frac{|i|}{r}) f_{2i}\tilde v_i \le \frac{F}{3}\sum_{|i|\ge r}\rho(\frac{|i|}{r})|\tilde v_i|^2 + \frac{3}{F}\sum_{|i|\ge r}|f_{2i}|^2, \tag{4.17}
$$

$$
\langle f_3, \rho(\frac{|i|}{r})\tilde z\rangle = \sum_{|i|\ge r}\rho(\frac{|i|}{r}) f_{3i}\tilde z_i \le \frac{F+N}{4}\sum_{|i|\ge r}\rho(\frac{|i|}{r})|\tilde z_i|^2 + \frac{4}{F+N}\sum_{|i|\ge r}|f_{3i}|^2.
$$

For the fifth term to seventh term in the right-hand side of (4.16), we have

$$
\begin{aligned}
&-2d_1\langle Ay, \rho(\frac{|i|}{r})\tilde u\rangle \\
&= -2d_1\langle By, B\Big(\rho(\frac{|i|}{r})\tilde u\Big)\rangle \\
&= -2d_1\sum_{i\in\mathbb{Z}}(y_{i+1}-y_i)\Big(\rho(\frac{|i+1|}{r})\tilde u_{i+1} - \rho(\frac{|i|}{r})\tilde u_i\Big) \\
&= -2d_1\sum_{|i|\ge r-1}\rho(\frac{|i+1|}{r})(y_{i+1}-y_i)\tilde u_{i+1} + 2d_1\sum_{|i|\ge r}\rho(\frac{|i|}{r})(y_{i+1}-y_i)\tilde u_i \\
&\le \frac{F+k}{4}\sum_{|i|\ge r}\rho(\frac{|i|}{r})|\tilde u_i|^2 + C_7\sum_{|i|\ge r-1}|y_i|^2,
\end{aligned}
$$

$$
-2d_2\langle Ay, \rho(\frac{|i|}{r})\tilde v\rangle \le \frac{F}{3}\sum_{|i|\ge r}\rho(\frac{|i|}{r})|\tilde v_i|^2 + C_8\sum_{|i|\ge r-1}|y_i|^2, \tag{4.18}
$$

$$
-2d_3\langle Ay, \rho(\frac{|i|}{r})\tilde z\rangle \le \frac{F+N}{4}\sum_{|i|\ge r}\rho(\frac{|i|}{r})|\tilde z_i|^2 + C_9\sum_{|i|\ge r-1}|y_i|^2,
$$

where $C_j, j = 7,8,9$ are positive constants depending only on d_1, d_2, d_3, F, N, k. Direct computation shows that

$$
2(N+k)\langle \tilde z, \rho(\frac{|i|}{r})\tilde u\rangle \le (N+k)\sum_{i\in Z}\rho(\frac{|i|}{r})|\tilde u_i|^2 + (N+k)\sum_{i\in Z}\rho(\frac{|i|}{r})|\tilde z_i|^2,
$$

$$
\begin{aligned}
4N\langle y(\theta_t\omega),\rho(\frac{|i|}{r})\tilde{u}\rangle &\le \frac{F+k}{4}\sum_{i\in Z}\rho(\frac{|i|}{r})|\tilde{u}_i|^2+\frac{16N^2}{F+k}\sum_{i\in Z}\rho(\frac{|i|}{r})|y_i(\theta_t\omega)|^2, \quad (4.19)\\
2(N+k)\langle y(\theta_t\omega),\rho(\frac{|i|}{r})\tilde{v}\rangle &\le \frac{F}{3}\sum_{i\in Z}\rho(\frac{|i|}{r})|\tilde{v}_i|^2+\frac{3(N+k)^2}{F}\sum_{i\in Z}\rho(\frac{|i|}{r})|y_i(\theta_t\omega)|^2,\\
4k\langle y(\theta_t\omega),\rho(\frac{|i|}{r})\tilde{z}\rangle &\le \frac{F+N}{4}\sum_{i\in Z}\rho(\frac{|i|}{r})|\tilde{z}_i|^2+\frac{16k^2}{F+N}\sum_{i\in Z}\rho(\frac{|i|}{r})|y_i(\theta_t\omega)|^2.
\end{aligned}
$$

From (4.16)-(4.19), (3.1) and (3.11), we have

$$
\begin{aligned}
&\frac{d}{dt}\sum_{i\in\mathbb{Z}}\rho(\frac{|i|}{r})\left[|\tilde{u}_i|^2+|\tilde{v}_i|^2+|\tilde{z}_i|^2\right]+(F+k)\sum_{i\in\mathbb{Z}}\rho(\frac{|i|}{r})|\tilde{u}_i|^2\\
&+F\sum_{i\in\mathbb{Z}}\rho(\frac{|i|}{r})|\tilde{v}_i|^2+(F+N)\sum_{i\in\mathbb{Z}}\rho(\frac{|i|}{r})|\tilde{z}_i|^2\\
&\le \frac{4C_6d_1}{r}\|\tilde{u}\|^2+\frac{4C_6d_2}{r}\|\tilde{v}\|^2+\frac{4C_6d_3}{r}\|\tilde{z}\|^2+\frac{4}{F+k}\sum_{|i|\ge r}|f_{1i}|^2+\frac{3}{F}\sum_{|i|\ge r}|f_{2i}|^2\\
&\quad+\frac{4}{F+N}\sum_{|i|\ge r}|f_{3i}|^2+(C_7+C_8+C_9)\sum_{|i|\ge r-1}|y_i(\theta_t\omega)|^2 \quad (4.20)\\
&\quad+\Big(\frac{16N^2}{F+k}+\frac{3(N+k)^2}{F}+\frac{16k^2}{F+N}\Big)\sum_{|i|\ge r}|y_i(\theta_t\omega)|^2.
\end{aligned}
$$

By Gronwall's inequality, we obtain that for $t\ge T_k=T_k(\omega)\ge 0$,

$$
\begin{aligned}
&\sum_{i\in\mathbb{Z}}\rho(\frac{|i|}{r})\left[|\tilde{u}_i(t,\omega,\tilde{u}_0(\omega))|^2+|\tilde{v}_i(t,\omega,\tilde{v}_0(\omega))|^2+|\tilde{z}_i(t,\omega,\tilde{z}_0(\omega))|^2\right]\\
&\le e^{-F(t-T_k)}\sum_{i\in\mathbb{Z}}\rho(\frac{|i|}{r})\left[|\tilde{u}_i(T_k,\omega,\tilde{u}_0(\omega))|^2+|\tilde{v}_i(T_k,\omega,\tilde{v}_0(\omega))|^2+|\tilde{z}_i(T_k,\omega,\tilde{z}_0(\omega)|^2\right]\\
&\quad+\int_{T_k}^t\Big(\frac{4C_6d_1}{r}\|\tilde{u}\|^2+\frac{4C_6d_2}{r}\|\tilde{v}\|^2+\frac{4C_6d_3}{r}\|\tilde{z}\|^2\Big)e^{F(\tau-t)}d\tau x\\
&\quad+\frac{1}{F}\Big(\frac{4}{F+k}\sum_{|i|\ge r}|f_{1i}|^2+\frac{3}{F}\sum_{|i|\ge r}|f_{2i}|^2+\frac{4}{F+N}\sum_{|i|\ge r}|f_{3i}|^2\Big) \quad (4.21)\\
&\quad+(C_7+C_8+C_9)\int_{T_k}^t e^{F(\tau-t)}\sum_{|i|\ge r-1}|y_i(\theta_\tau\omega)|^2d\tau\\
&\quad+\Big(\frac{16N^2}{F+k}+\frac{3(N+k)^2}{F}+\frac{16k^2}{F+N}\Big)\int_{T_k}^t e^{F(\tau-t)}\sum_{|i|\ge r}|y_i(\theta_\tau\omega)|^2d\tau.
\end{aligned}
$$

Replace ω by $\theta_{-t}\omega$. We then estimate each term on the right-hand side of (4.21). From (4.7)-(4.8) with t replaced by T_k and ω by $\theta_{-t}\omega$, it follows that

$$
\begin{aligned}
&e^{-F(t-T_k)}\sum_{i\in\mathbb{Z}}\rho\big(\frac{|i|}{r}\big)\Big[|\tilde{u}_i(T_k,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega))|^2+|\tilde{v}_i(T_k,\theta_{-t}\omega,\tilde{v}_0(\theta_{-t}\omega))|^2\\
&+|\tilde{z}_i(T_k,\theta_{-t}\omega),\tilde{z}_0(\theta_{-t}\omega)|^2\Big]\\
&\le e^{-F(t-T_k)}\Big(e^{-FT_k}[\|\tilde{u}_0(\theta_{-t}\omega)\|^2+\|\tilde{v}_0(\theta_{-t}\omega)\|^2+\|\tilde{z}_0(\theta_{-t}\omega)\|^2]\\
&\quad+\frac{C_5}{F}(\|f_1\|^2+\|f_2\|^2+\|f_3\|^2)\\
&\quad+C_5\int_0^{T_k}e^{-F(T_k-s)}(\|y(\theta_{s-t}\omega)\|^2+\|Ay(\theta_{s-t}\omega)\|^2)ds\Big)\\
&\le e^{-Ft}[\|\tilde{u}_0(\theta_{-t}\omega)\|^2+\|\tilde{v}_0(\theta_{-t}\omega)\|^2+\|\tilde{z}_0(\theta_{-t}\omega)\|^2]\\
&\quad+e^{-F(t-T_k)}\frac{C_5}{F}(\|f_1\|^2+\|f_2\|^2+\|f_3\|^2)+\frac{2}{F}C_5l(\omega)e^{-\frac{F}{2}(t-T_k)}.
\end{aligned}
\tag{4.22}
$$

Thus, there exists a $T_1(\varepsilon,\omega)>T_k(\omega)$ such that if $t>T_1(\varepsilon,\omega)$, then

$$
\begin{aligned}
&e^{-F(t-T_k)}\sum_{i\in\mathbb{Z}}\rho\big(\frac{|i|}{r}\big)\Big[|\tilde{u}_i(T_k,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega))|^2+|\tilde{v}_i(T_k,\theta_{-t}\omega,\tilde{v}_0(\theta_{-t}\omega))|^2\\
&+|\tilde{z}_i(T_k,\theta_{-t}\omega),\tilde{z}_0(\theta_{-t}\omega)|^2\Big]<\frac{1}{4}\varepsilon.
\end{aligned}
\tag{4.23}
$$

Next we estimate second term on the right-hand side of (4.21),

$$
\begin{aligned}
&\int_{T_k}^t\Big(\frac{4C_6d_1}{r}\|\tilde{u}(\tau,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega))\|^2+\frac{4C_6d_2}{r}\|\tilde{v}(\tau,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega))\|^2\\
&\quad+\frac{4C_6d_3}{r}\|\tilde{z}(\tau,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega))\|^2\Big)e^{F(\tau-t)}d\tau\\
&\le\frac{4C_6d}{r}\int_{T_k}^t e^{F(\tau-t)}\Big(e^{-F\tau}[\|\tilde{u}_0(\theta_{-t}\omega)\|^2+\|\tilde{v}_0(\theta_{-t}\omega)\|^2+\|\tilde{z}_0(\theta_{-t}\omega)\|^2]\\
&\quad+\frac{C_5}{F}(\|f_1\|^2+\|f_2\|^2+\|f_3\|^2)\\
&\quad+C_5\int_0^{\tau}e^{F(s-\tau)}(\|y(\theta_{s-t}\omega)\|^2+\|Ay(\theta_{s-t}\omega)\|^2)ds\Big)d\tau\\
&\le\frac{4C_6d}{r}\Big([\|\tilde{u}_0(\theta_{-t}\omega)\|^2+\|\tilde{v}_0(\theta_{-t}\omega)\|^2+\|\tilde{z}_0(\theta_{-t}\omega)\|^2](t-T_k)e^{-Ft}\\
&\quad+\frac{C_5}{F^2}(\|f_1\|^2+\|f_2\|^2+\|f_3\|^2)+\frac{4}{F}C_5l(\omega)\Big),
\end{aligned}
\tag{4.24}
$$

where $d=\max\{d_1,d_2,d_3\}$. Recall the fact that $(\tilde{u}_0(\theta_{-t}\omega),\tilde{v}_0(\theta_{-t}\omega),\tilde{z}_0(\theta_{-t}\omega))\in K(\theta_{-t}\omega)$, which implies that $\|\tilde{u}_0(\theta_{-t}\omega)\|^2+\|\tilde{v}_0(\theta_{-t}\omega)\|^2+\|\tilde{z}_0(\theta_{-t}\omega)\|^2\le R^2(\theta_{-t}\omega)$, and $R(\omega)$ is tempered. Thus there exists $T_2(\varepsilon,\omega)>T_k(\omega)$ and $N_1(\varepsilon,\omega)>0$ such that for $t>T_2(\varepsilon,\omega)$ and $r>N_1(\varepsilon,\omega)$, we have

$$
\begin{aligned}
&\int_{T_k}^t\Big(\frac{2C_6d_1}{r}\|\tilde{u}(\tau,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega))\|^2+\frac{2C_6d_2}{r}\|\tilde{v}(\tau,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega))\|^2\\
&+\frac{2C_6d_3}{r}\|\tilde{z}(\tau,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega))\|^2\Big)e^{F(\tau-t)}d\tau<\frac{1}{4}\varepsilon.
\end{aligned}
\tag{4.25}
$$

Since $f_1, f_2, f_3 \in \ell^2$, there exists $N_2(\varepsilon,\omega) > 0$ such that for $r > N_2(\varepsilon,\omega)$,

$$\frac{1}{F}\Big(\frac{4}{F+k}\sum_{|i|\ge r}|f_{1i}|^2 + \frac{3}{F}\sum_{|i|\ge r}|f_{2i}|^2 + \frac{4}{F+N}\sum_{|i|\ge r}|f_{3i}|^2\Big) < \frac{1}{4}\varepsilon. \tag{4.26}$$

Finally, we estimate the last term on the right-hand side of (4.21). Let $T^* > 0$ to be determined later. We have for $t > T^* + T_k$

$$\begin{aligned}
&(C_7 + C_8 + C_9)\int_{T_k}^t e^{F(\tau-t)}\sum_{|i|\ge r-1}|y_i(\theta_{\tau-t}\omega)|^2 d\tau\\
&+ \left(\frac{16N^2}{F+k} + \frac{3(N+k)^2}{F} + \frac{16k^2}{F+N}\right)\int_{T_k}^t e^{F(\tau-t)}\sum_{|i|\ge r}|y_i(\theta_{\tau-t}\omega)|^2 d\tau\\
&= \int_{T_k-t}^0 e^{Fs}\Big(C_{10}\sum_{|i|\ge r-1}|y_i(\theta_s\omega)|^2 ds + C_5\sum_{|i|\ge r}|y_i(\theta_s\omega)|^2 ds\Big)\\
&\le \int_{-T^*}^0 e^{Fs}\Big(C_{10}\sum_{|i|\ge r-1}|y_i(\theta_s\omega)|^2 ds + C_5\sum_{|i|\ge r}|y_i(\theta_s\omega)|^2 ds\Big)\\
&\quad + \int_{T_k-t}^{-T^*} e^{Fs}\left(C_{10}\|y(\theta_s\omega)\|^2 ds + C_5\|y(\theta_s\omega)\|^2 ds\right).
\end{aligned} \tag{4.27}$$

Using (4.8), we have

$$\int_{T_k-t}^{-T^*} e^{Fs}\left(C_{10}\|y(\theta_s\omega)\|^2 + C_5\|y(\theta_s\omega)\|^2\right) ds \le \frac{2C_{11}}{F} l(\omega) e^{-\frac{F}{2}T^*}. \tag{4.28}$$

Thus, by choosing

$$T^* > \frac{2}{F}\ln\frac{16C_{11}l(\omega)}{F\varepsilon},$$

for $t > T^* + T_k$, we have

$$\int_{T_k-t}^{-T^*} e^{Fs}\left(C_{10}\|y(\theta_s\omega)\|^2 + C_5\|y(\theta_s\omega)\|^2\right) ds < \frac{\varepsilon}{8}. \tag{4.29}$$

For the fixed T^*, from Lebesgue's theorem there is an $N_3(\varepsilon,\omega)$ such that for $r > N_3(\varepsilon,\omega)$,

$$\int_{-T^*}^0 e^{Fs}\Big(C_{10}\sum_{|i|\ge r-1}|y_i(\theta_s\omega)|^2 ds + C_5\sum_{|i|\ge r}|y_i(\theta_s\omega)|^2 ds\Big) < \frac{\varepsilon}{8}. \tag{4.30}$$

Therefore, by letting

$$\begin{gathered}
T(\varepsilon,\omega) = \max\{T_1(\varepsilon,\omega), T_2(\varepsilon,\omega), T^*(\varepsilon,\omega) + T_k(\omega)\},\\
\tilde{N}(\varepsilon,\omega) = \max\{N_1(\varepsilon,\omega), N_2(\varepsilon,\omega), N_3(\varepsilon,\omega)\},
\end{gathered} \tag{4.31}$$

for $t > T(\varepsilon,\omega)$ and $N > \tilde N(\varepsilon,\omega)$, we have

$$\begin{aligned}
&\sum_{|i|>2r}\big[|\tilde u_i(t,\theta_{-t}\omega,\tilde u_0(\theta_{-t}\omega))|^2+|\tilde v_i(t,\theta_{-t}\omega,\tilde v_0(\theta_{-t}\omega))|^2\\
&\quad+|\tilde z_i(t,\theta_{-t}\omega,\tilde z_0(\theta_{-t}\omega))|^2\big]\\
&\le\sum_{i\in\mathbb{Z}}\rho\big(\frac{|i|}{r}\big)\big[|\tilde u_i(t,\theta_{-t}\omega,\tilde u_0(\theta_{-t}\omega))|^2\\
&\quad+|\tilde v_i(t,\theta_{-t}\omega,\tilde v_0(\theta_{-t}\omega))|^2+|\tilde z_i(t,\theta_{-t}\omega,\tilde z_0(\theta_{-t}\omega))|^2\big]<\varepsilon,
\end{aligned}\tag{4.32}$$

which implies

$$\begin{aligned}
&\sum_{|i|\ge N(\varepsilon,\omega)}\big[|u_i(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))|^2+|v_i(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2\\
&\quad+|z_i(t,\theta_{-t}\omega,z_0(\theta_{-t}\omega))|^2\big]\\
&=\sum_{|i|\ge N(\varepsilon,\omega)}(|\tilde u_i(t,\theta_{-t}\omega,\tilde u_0(\theta_{-t}\omega))+y(\theta_{-t}\omega)|^2+|\tilde v_i(t,\theta_{-t}\omega,\tilde v_0(\theta_{-t}\omega))\\
&\quad+y(\theta_{-t}\omega)|^2+|\tilde z_i(t,\theta_{-t}\omega,z_0(\theta_{-t}\omega))+y(\theta_{-t}\omega)|^2)\\
&\le 2\sum_{|i|\ge N(\varepsilon,\omega)}(|\tilde u_i(t,\theta_{-t}\omega,\tilde u_0(\theta_{-t}\omega))|^2+|\tilde v_i(t,\theta_{-t}\omega,\tilde v_0(\theta_{-t}\omega))|^2\\
&\quad+|\tilde z_i(t,\theta_{-t}\omega,z_0(\theta_{-t}\omega))|^2)+6\sum_{|i|\ge N(\varepsilon,\omega)}|y(\theta_{-t}\omega)|^2<8\varepsilon,
\end{aligned}\tag{4.33}$$

provided $N(\varepsilon,\omega)$ is large enough. This completes the proof of the lemma. □

We are now ready to show the pullback asymptotic compactness of the random set $K(\omega)$.

Lemma 4.3. *For $\omega\in\Omega$, the set $K(\omega)$ is pullback asymptotically compact in the sense of each sequence $(u_n,v_n,z_n)\in\varphi(t_n,\theta_{-t_n}\omega,K(\theta_{-t_n}\omega))$ with $t_n\to\infty$ having a convergent subsequence in $\ell^2\times\ell^2\times\ell^2$.*

Proof. We follow the method in [3]. Let $\omega\in\Omega'$ for each sequence $\{t_n\}_{n=1}^\infty$: $t_1,t_2,\cdots,t_n\to\infty$ as $n\to\infty$, and

$$(u_n(t_n,\theta_{-t_n}\omega,x_n),v_n(t_n,\theta_{-t_n}\omega,y_n),z_n(t_n,\theta_{-t_n}\omega,\varrho_n))\in\varphi(t_n,\theta_{-t_n}\omega,K(\theta_{-t_n}\omega));$$

this implies that there exists $(x_n,y_n,\varrho_n)\in K(\theta_{-t_n}\omega)$ such that

$$(u_n(t_n,\theta_{-t_n}\omega,x_n),v_n(t_n,\theta_{-t_n}\omega,y_n),z_n(t_n,\theta_{-t_n}\omega,\varrho_n))=\varphi(t_n,\theta_{-t_n}\omega,(x_n,y_n,\varrho_n)).$$

Since $K(\omega)$ is a bounded absorbing set, for large n, $\varphi(t_n,\theta_{-t_n}\omega,(x_n,y_n,\varrho_n))\in K(\omega)$; thus there exists $(u,v,z)\in\ell^2\times\ell^2\times\ell^2$, and a subsequence $(u_n',v_n',z_n')=\varphi(t_n,\theta_{-t_n}\omega,(x_n,y_n,\varrho_n))$ such that

$$(u_n'(t_n,\theta_{-t_n}\omega,x_n),v_n'(t_n,\theta_{-t_n}\omega,y_n),z_n'(t_n,\theta_{-t_n}\omega,\varrho_n))\to(u,v,z)\tag{4.34}$$

weak in $\ell^2\times\ell^2\times\ell^2$. Next, we show that (u_n',v_n',z_n') is also strongly convergent in the norm $\|\cdot\|$ in $\ell^2\times\ell^2\times\ell^2$, i.e., for each $\epsilon>0$ there is $N^*(\epsilon,\omega)>0$ such that for $n\ge N^*(\epsilon,\omega)$,

$$\|(u_n'(t_n,\theta_{-t_n}\omega,x_n),v_n'(t_n,\theta_{-t_n}\omega,y_n),z_n'(t_n,\theta_{-t_n}\omega,\varrho_n))-(u,v,z)\|\le\epsilon.$$

In fact, from Lemma 4.2, for any $\epsilon > 0$, there exists an $N^*(\epsilon,\omega)$ and a $K_1(\epsilon,\omega)$ such that for $n \ge N^*(\epsilon,\omega)$,

$$\begin{aligned} &\sum_{|i|\ge K_1(\epsilon,\omega)} \Big(|u'_{ni}(t_n,\theta_{-t_n}\omega,x_n)|^2 + |v'_{ni}(t_n,\theta_{-t_n}\omega,y_n)|^2 \\ &+ |z'_{ni}(t_n,\theta_{-t_n}\omega,\varrho_n)|^2\Big) \le \frac{1}{8}\epsilon^2. \end{aligned} \tag{4.35}$$

Since $(u,v,z) \in \ell^2 \times \ell^2 \times \ell^2$, there exists $K_2(\epsilon) > 0$ such that

$$\sum_{|i|\ge K_2(\epsilon)} \big(|u_i|^2 + |v_i|^2 + |z_i|^2\big) \le \frac{1}{8}\epsilon^2. \tag{4.36}$$

Let $K(\epsilon,\omega) = \max\{K_1(\epsilon,\omega), K_2(\epsilon)\}$. From the weak convergence (4.34), we have for each $|i| \le K(\epsilon,\omega)$ as $n \to \infty$,

$$(u'_{ni}(t_n,\theta_{-t_n}\omega,x_n), v'_{ni}(t_n,\theta_{-t_n}\omega,y_n), z'_{ni}(t_n,\theta_{-t_n}\omega,\varrho_n)) \to (u_i,v_i,z_i),$$

which implies that there exists $N_2^*(\epsilon,\omega) > 0$ such that for $n \ge N_2^*(\epsilon,\omega)$,

$$\begin{aligned} &\sum_{|i|\le K(\epsilon,\omega)} \Big(|u'_{ni}(t_n,\theta_{-t_n}\omega,x_n) - u_i|^2 + |v'_{ni}(t_n,\theta_{-t_n}\omega,y_n) - v_i|^2 \\ &+ |z'_{ni}(t_n,\theta_{-t_n}\omega,\varrho_n) - z_i|^2\Big) \le \frac{1}{2}\epsilon^2. \end{aligned} \tag{4.37}$$

Combining (4.35)-(4.37), we obtain that for $n \ge N^*(\epsilon,\omega)$,

$$\begin{aligned} &\|u'_n(t_n,\theta_{-t_n}\omega,x_n) - u\|^2 + \|v'_n(t_n,\theta_{-t_n}\omega,y_n) - v\|^2 + \|z'_n(t_n,\theta_{-t_n}\omega,\varrho_n) - z\|^2 \\ &\le \sum_{|i|\le K(\epsilon,\omega)} \Big(|u'_{ni}(t_n,\theta_{-t_n}\omega,x_n) - u_i|^2 + |v'_{ni}(t_n,\theta_{-t_n}\omega,y_n) - v_i|^2 \\ &\quad + |z'_{ni}(t_n,\theta_{-t_n}\omega,\varrho_n) - z_i|^2\Big) + \sum_{|i|\ge K(\epsilon,\omega)} \Big(|u'_{ni}(t_n,\theta_{-t_n}\omega,x_n) - u_i|^2 \\ &\quad + |v'_{ni}(t_n,\theta_{-t_n}\omega,y_n) - v_i|^2 + |z'_{ni}(t_n,\theta_{-t_n}\omega,\varrho_n) - z_i|^2\Big) \\ &\le \frac{1}{2}\epsilon^2 + 2\sum_{|i|\ge K(\epsilon,\omega)} \big(|u_i|^2 + |v_i|^2 + |z_i|^2\big) + 2\sum_{|i|\ge K(\epsilon,\omega)} \Big(|u'_{ni}(t_n,\theta_{-t_n}\omega,x_n)|^2 \\ &\quad + |v'_{ni}(t_n,\theta_{-t_n}\omega,y_n)|^2 + |z'_{ni}(t_n,\theta_{-t_n}\omega,\varrho_n)|^2\Big) \\ &\le \frac{1}{2}\epsilon^2 + \frac{1}{4}\epsilon^2 + \frac{1}{4}\epsilon^2 = \epsilon^2. \end{aligned}$$

The proof is complete. □

Combining Lemmas 4.1 and 4.2 with Lemma 4.3, we obtain our main result, which is a direct consequence of Theorem 2.7.

Theorem 4.4. *The random dynamical systems $\{\varphi(t,\omega)\}_{t\ge 0,\omega\in\Omega}$ possess a random attractor in $\ell^2 \times \ell^2 \times \ell^2$.*

Remark 4.5. Equation (1.1) can be reduced to two ordinary differential equations describing kinetics of cubic chemical or biochemical reactions with additive noise on an infinite lattice, such as two following models: stochastic reversible Selkov equations [27, 36]:

$$\begin{aligned} du_i &= (-d_1(Au)_i - au_i + u_i^2 v_i - u_i^3)dt + \alpha_i dw_i, \\ dv_i &= (-d_2(Av)_i - bv_i - u_i^2 v_i + u_i^3)dt + \alpha_i dw_i, \end{aligned} \tag{4.38}$$

stochastic reversible Glocolysis equations [25]:

$$\begin{aligned} du_i &= (-d_1(Au)_i - au_i + bv_i + u_i^2 v_i - u_i^3)dt + \alpha_i dw_i, \\ dv_i &= (-d_2(Av)_i - bv_i - u_i^2 v_i + u_i^3)dt + \alpha_i dw_i, \end{aligned} \tag{4.39}$$

with the conditions (1.2). Equations (4.38)-(4.39) are useful models for study of cooperative processes in chemical kinetics. By introducing an Ornstein-Uhlenbeck process, we transform the stochastic equation into a pathwise one with tempered random variables:

$$y(\theta_t\omega) = -b\int_{-\infty}^{0} e^{bs}(\theta_t\omega)(s)ds, \quad t \in \mathbb{R},\ \omega \in \Omega.$$

Note that y solves the Itô equation

$$dy + bydt = dw(t), \quad t > 0.$$

Let

$$\tilde{u} = u - y(\theta_t\omega), \quad \tilde{v} = v - y(\theta_t\omega).$$

Then, similar to Lemmas 4.1–4.3 and Theorem 4.4, we can prove that

(1) The random dynamical system governed by stochastic reversible Selkov equations (4.38) has a random attractor in E.
(2) If $2b \le a$ holds, the random dynamical system governed by stochastic reversible Glocolysis equations (4.39) has a random attractor in E.

Acknowledgements. We would like to thank the anonymous referee for the carefully reading the manuscript and for making very useful suggestions. The first author is supported by the National Natural Science Foundation of China (11101265) and by the Shanghai Education Research and Innovation Key Project (14ZZ157).

References

[1] G. Adomian; *The diffusion-Brusselator equation*, Computers and Mathematics with Applications, **29** (1995), 1-3.
[2] L. Arnold; *Random Dynamical Systems, Springer Monographs in Mathematics*, Springer-Verlag, Berlin, 1998.
[3] P. W. Bates, H. Lisei, K. Lu; *Attractors for stochastic lattice dynamical systems*, Stochastics and Dynamics, **6** (2006), 1-21.
[4] P. W. Bates, K. Lu, B. Wang; *Attractors for lattice dynamical systems*, International Journal of Bifurcation and Chaos in Applied Sciences and Engineering, **11** (2001), 143-153.
[5] W. J. Beyn, S. Y. Pilyugin; *Attractors of reaction diffusion systems on infinite lattices*, Journal of Dynamics and Differential Equations, **15** (2003), 485-515.
[6] T. L. Carrol, L. M. Pecora; *Synchronization in chaotic systems*, Physical Review Letters, **64** (1990), 821-824.
[7] H. Chate, M. Courbage; *Lattice systems*, Physica D, **103** (1997), 1-612.
[8] S.-N. Chow, J. M. Paret; *Pattern formation and spatial chaos in lattice dynamical systems*, IEEE Transaction on Circuits and Systems, **42** (1995), 746-751.
[9] L. O. Chua, Y. Yang; *Cellular neural networks: theory*, IEEE Transaction on Circuits and Systems, **35** (1988), 1257-1272.
[10] H. Crauel, A. Debussche, F. Flandoli; *Random attractors*, Journal of Dynamics and Differential Equation, **9** (1997), 307-341.

[11] H. Crauel, F. Flandoli; *Attractor for random dynamical systems*, Probability Theory and Related Fields, **100** (1994), 365-393.
[12] H. Crauel, F. Flandoli; *Hausdorff dimensional of invariant sets for random dynamical systems*, Journal of Dynamics and Differential Equations, **10** (1998), 449-474.
[13] J. D. Deuschel; *Central limit theorem for an infinite lattice system of interacting diffusion processes*, The Annals of Probability, **16** (1988), 700-716.
[14] T. Erneux, G. Nicolis; *Propagating waves in discrete bistable reaction diffusion systems*, Physica D, **67** (1993), 237-244.
[15] L. Fabiny, P. Colet, R. Roy; *Coherence and phase dynamics of spatially coupled solid-state lasers*, Physical Review A, **47** (1993), 4287-4296.
[16] A. Gu; *Pullback D-attractor of nonautonomous three-component reversible Gray-Scott system on unbounded domains*, Abstract and Applied Analysis, Volume **2013**, Article ID 719063, 13 pages, 2013.
[17] A. Gu, Z. Wang, S. Zhou; *Random Attractors for stochastic three-component reversible Gray-Scott system on infinite lattices*, Discrete Dynamics in Nature and Society, Volume **2012**, Article ID 340789, 17 pages.
[18] A. Gu, S. Zhou, Z. Wang; *Uniform attractor of nonautonomous three-component reversible Gray-Scott system*, Applied Mathematics and Computation, **219** (2013), 8718-8729.
[19] M. Hillert; *A solid-solution model for inhomogeneous systems*, Acta Metallurgica, **9** (1961), 525-535.
[20] J. Huang; *The random attractor of stochastic FitzHugh-Nagumo equations in an infinite lattice with white noises*, Physica D, **233** (2007), 83-94.
[21] X, Jia, C. Zhao, X. Yang; *Global attractor and Kolmogorov entropy of three component reversible Gray-Scott model on infinite lattices*, Applied Mathematics and Computation, **218** (2012), 9781-9789.
[22] R. Kapval; *Discrete models for chemically reacting systems*, Journal of Mathematical Chemistry, **6** (1991), 113-163.
[23] J. P. Keener; *Propagation and its failure in coupled systems of discret excitable cells*, SIAM Journal on Applied Mathematics, **47** (1987), 556-572.
[24] H. Mahara, N. J. Suematsu, T. Yamaguchi, K. Ohgane, Y. Nishiura, M. Shimomura; *Three-variable reversible Gray-Scott model*, Journal of Chemical Physics, **121** (2004), 8968-8972.
[25] Jeff S. McGough, K. Riley; *Pattern formation in the Gray-Scott model*, Nonlinear Analysis: Real World Applications, **5** (2004), 105-121.
[26] G. Nicolis, I. Prigogine; *Self-Organization in Nonequilibrium Systems*, Wiley-Interscience, 1977.
[27] K. S. Al Noufaey, T. R. Marchant; *Semi-analytical solutions for the reversible Selkov model with feedback delay*, Applied Mathematics and Computation, **232** (2014), 49-59.
[28] I. Prigogine, R. Lefever; *Symmetry breaking instabilities in dissipative systems*, Journal of Chemical Physics, **48** (1968), 1695.
[29] J. Schwender, J. Ohlrogge, Y. Shachar-Hill; *A flux model of glycolysis and the oxidative pentosephosphate pathway in developing brassica napus embryos*, The Journal of Biological Chemistry, **278** (2003), 29442-29453.
[30] Y. Termonia, J. Ross; *Oscillations and control features in glycolysis: Numerical analysis of a comprehensive model*, Proceedings of the National Academy of Sciences of the United States of America, **78** (1981), 2952-2956.
[31] J. Tyson; *Some further studies of nonlinear oscillations in chemical systems*, Journal of Chemical Physics, **58** (1973), 3919.
[32] B. Wang; *Dynamics of systems on infinite lattices*, Journal of Differential Equations, **221** (2006), 224-245.
[33] R. L. Winaow, A. L. Kimball, A. Varghese; *Simulating cardiac sinus and atrial network dynamics on connection machine*, Physica D, **64** (1993), 281-298.
[34] Y. You; *Dynamics of three-component reversible Gray-Scott model*, Discrete and Continuous Dynamical Systems, **14** (2010), 1671-1688.
[35] Y. You; *Dynamics of two-compartment Gray-Scott equations*, Nonlinear Analysis: Theory, Methods and Applications, **74** (2011), 1969-1986.
[36] Y. You; *Global dynamics and robustness of reversible autocatalytic reaction-diffusion systems*, Nonlinear Analysis: Theory, Methods and Applications, **75** (2012), 3049-3071.

[37] Y. You; *Global dynamics of a reaction-diffusion system*, Electronic Journal of Differential Equations, **25** (2011), 1-28.

[38] Y. You; *Random attractors and robustness for stochastic reversible reaction-diffusion systems*, Discrete and continuous dynamical systems, **34** (2014), 301-333.

[39] Y. You; *Robustness of global attractors for reversible Gray-Scott systems*, Journal of Dynamics and Differential Equations, **24** (2012), 495-520.

Hongyan Li
College of Management, Shanghai University of Engineering Science, Shanghai 201620, China
E-mail address: hongyanlishu@163.com

Junyi Tu
Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA
E-mail address: junyi@mail.usf.edu