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MULTIPLE SOLUTIONS FOR FRACTIONAL SCHRÖDINGER EQUATIONS

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Abstract. In this article we study the fractional Schrödinger equations

$$(-\Delta)^\alpha u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N,$$

where $0 < \alpha < 1$, $N \geq 2$, $(-\Delta)^\alpha$ stands for the fractional Laplacian of order α. First by using Morse theory in combination with local linking arguments, we prove the existence of at least two nontrivial solutions. Next we prove that the problem has k distinct pairs of solutions by using the Clark theorem.

1. Introduction and main results

In this article, we consider the fractional Schrödinger equation

$$(-\Delta)^\alpha u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $0 < \alpha < 1$, $N \geq 2$, $(-\Delta)^\alpha$ stands for the fractional Laplacian of order α, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

When $\alpha = 1$, (1.1) becomes the classical Schrödinger equation

$$-\Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

In recent years, the existence and multiplicity of standing wave solutions of (1.2) have been widely studied, we refer the readers to [12, 18, 25, 26, 27, 28] and the references therein.

When $0 < \alpha < 1$, (1.1) is a nonlocal model known as nonlinear fractional Schrödinger equation. The nonlocal model has attracted much attentions recently. For the case of a bounded domain, Ricceri [19] established a theorem tailor-made for a class of nonlocal problems involving nonlinearities with bounded primitive. In [9], Molica Bisci and Repovš studied a class of nonlocal fractional Laplacian equations depending on two real parameters and obtained the existence of three weak solutions by exploiting the result established by Ricceri in [19]. For more related results, we refer the readers to [7, 8] and the references therein.

Equations of the form (1.1) in the whole space $\mathbb{R}^N$ were studied by a number of authors. See, for instance, [2, 4, 6, 20, 21, 22, 29] and the references therein. Felmer, Quaas and Tan [6] studied the existence and regularity of positive solution of (1.1) with $V(x) \equiv 1$ for general $s \in (0,1)$ when f has subcritical growth and satisfies

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the Ambrosetti-Rabinowitz((AR) for short) condition. Secchi [20] obtained the existence of ground state solutions of (1.1) for general $s \in (0,1)$ when $V(x) \to +\infty$ as $|x| \to +\infty$ and (AR) condition holds. In [4], the authors looked for radially symmetric solutions of (1.1) when V and f do not depend explicitly on the space variable x. In [29], the authors obtained the existence of infinitely many weak solutions for (1.1) by variant fountain theorem established by Zou in [30] when f has subcritical growth.

On the other hand, Morse theory and local linking theorem are powerful tools in modern nonlinear analysis [3, 10, 24], especially for the problems with resonance [11, 23]. However, there are no existed papers dealing with the existence of solutions for fractional Schrödinger equations by using Morse theory.

Motivated by the above facts, the goal of this paper is to consider the multiplicity of nontrivial solutions for problem (1.1). Under some natural assumptions, by using Morse theory in combination with local linking arguments, the existence results of at least two nontrivial solutions are obtained. Next we prove that the problems have k distinct pairs of solutions by using the Clark theorem. It is worthy stressing that we will use a more general assumption on $V(x)$, which extend some recent results from the literature.

Next we state our main results, using the following assumptions:

- (V1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\beta := \inf_{\mathbb{R}^N} V(x) > 0$.
- (F1) There exist constants $1 < \gamma_1 < \gamma_2 < \cdots < \gamma_m < 2$ and positive functions $\xi_1(x) \in L^{\frac{2}{2-\gamma_1}}(\mathbb{R}^N, \mathbb{R}), \dots, \xi_m(x) \in L^{\frac{2}{2-\gamma_m}}(\mathbb{R}^N, \mathbb{R})$ such that
$$|f(x,u)| \le \gamma_1 \xi_1(x)|u|^{\gamma_1 - 1} + \cdots + \gamma_m \xi_m(x)|u|^{\gamma_m - 1}, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$
- (F2) There exist $c_1 > 0$, $0 < c_2 < \frac{1}{2S_2^2}$, $1 < \gamma < 2$ and small constants $0 < r < r_0$, such that
$$c_2|u|^2 > F(x,u) \ge c_1|u|^\gamma, \quad r \le |u| \le r_0 \quad \text{a.e.} \quad x \in \mathbb{R}^N,$$
where S_2 is the Sobolev constant from $H^\alpha(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$; furthermore, in the sequel $F(x,u) = \int_0^u f(x,s)ds$.
- (F3) $f(x,-u) = -f(x,u)$.

Theorem 1.1. *Assume that the potential $V(x)$ and the nonlinearity $f(x,u)$ satisfy* (V1), (F1)–(F2). *Then problem* (1.1) *has at least two nontrivial solutions..*

Theorem 1.2. *Assume that* (V1), (F1)–(F3) *are satisfied. Then problem* (1.1) *has at least k distinct pairs of solutions.*

The remainder of this article is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proof of our main results.

2. Variational setting and preliminaries

In this section, we collect some information to be used later. We will denote either by $\hat{u}$ or by $\mathcal{F}u$ the usual Fourier transform of u.

Sobolev spaces of fractional order are the convenient setting for our equations. A complete introduction to fractional Sobolev spaces can be found in [5], we offer below a short review. We recall that the fractional Sobolev space $W^{\alpha,p}(\mathbb{R}^N)$ is defined for any $p \in [1, +\infty)$ and $\alpha \in (0,1)$ as

$$W^{\alpha,p}(\mathbb{R}^N) = \big\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{\alpha p + N}} \, dx \, dy < \infty \big\}.$$

This space is endowed with the Gagliardo norm

$$\|u\|_{W^{\alpha,p}} = \Big(\int_{\mathbb{R}^N} |u|^p \, dx + \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p+N}} \, dx \, dy \Big)^{1/p}.$$

When $p = 2$, these spaces are also denoted by $H^\alpha(\mathbb{R}^N)$.

If p =2, an equivalent definition of fractional Sobolev spaces is possible, based on Fourier analysis. Indeed, it turns out that

$$H^\alpha(\mathbb{R}^N) = \big\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1+|\xi|^{2\alpha})|\hat{u}|^2 d\xi < \infty \big\},$$

and the norm can be equivalently written by

$$\|u\|_{H^\alpha} = \Big(\|u\|_2^2 + \int_{\mathbb{R}^N} |\xi|^{2\alpha} |\hat{u}|^2 d\xi \Big)^{1/2}.$$

Furthermore, we know that $\|\cdot\|_{H^\alpha}$ is equivalent to the norm

$$\|u\|_{H^\alpha} = \Big(\int_{\mathbb{R}^N} (|(-\Delta)^{\alpha/2} u|^2 + u^2) \, dx \Big)^{1/2}.$$

In this article, in view of the potential $V(x)$, we consider its subspace

$$E = \big\{ u \in H^\alpha(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) u^2 \, dx < \infty \big\}.$$

Then, by [20], E is a Hilbert space with the inner product

$$(u,v)_E = \int_{\mathbb{R}^N} (|\xi|^{2\alpha} \hat{u}(\xi)\hat{v}(\xi) + \hat{u}(\xi)\hat{v}(\xi)) d\xi + \int_{\mathbb{R}^N} V(x) u(x) v(x) \, dx, \quad \forall u, v \in E,$$

and the norm

$$\|u\|_E = \Big(\int_{\mathbb{R}^N} (|\xi|^{2\alpha} \hat{u}^2 + \hat{u}^2) d\xi + \int_{\mathbb{R}^N} V(x) u^2 \, dx \Big)^{1/2}.$$

Furthermore, we know that $\|\cdot\|_E$ is equivalent to the norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} (|(-\Delta)^{\alpha/2} u|^2 + V(x) u^2) \, dx \Big)^{1/2}.$$

The corresponding inner product is

$$(u,v) = \int_{\mathbb{R}^N} ((-\Delta)^{\alpha/2} u(x) (-\Delta)^{\alpha/2} v(x) + V(x) u(x) v(x)) \, dx.$$

Throughout out this paper, we use the norm $\|\cdot\|$ in E.

As usual, for $1 \le p < +\infty$, we let

$$\|u\|_p = \Big(\int_{\mathbb{R}^N} |u(x)|^p \, dx \Big)^{1/p}, \quad u \in L^p(\mathbb{R}^N),$$

$$\|u\|_\infty = \text{ess sup}_{x \in \mathbb{R}^N} |u(x)|, \quad u \in L^\infty(\mathbb{R}^N).$$

To prove our results, the following compactness result is necessary.

Lemma 2.1 ([13]). *E is continuously embedded into $L^p(\mathbb{R}^N)$ for $2 \le p \le 2_\alpha^*$ and compactly embedded into $L^p_{loc}(\mathbb{R}^N)$ for $2 \le p < 2_\alpha^*$ with $2_\alpha^* = \frac{2N}{N-2\alpha}$.*

It follows directly from the Lemma 2.1 that there are constants $S_p > 0$ such that

$$\|u\|_p \le S_p \|u\|, \quad \forall u \in E, \ p \in [2, 2_\alpha^*].$$

Lemma 2.2. *Assume that* (V1), (F1) *hold. Then the functional* $\Phi : E \to \mathbb{R}$ *defined by*

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x, u(x))\, dx \tag{2.1}$$

is well defined and of class $C^1(E, \mathbb{R})$ *and*

$$(\Phi'(u), v) = (u, v) - \int_{\mathbb{R}^N} f(x, u(x))v(x)\, dx. \tag{2.2}$$

Furthermore, the critical points of Φ *in* E *are solutions of problem* (1.1).

Proof. From (F1), one has

$$|F(x, u)| \le \xi_1(x)|u(x)|^{\gamma_1} + \dots + \xi_m(x)|u(x)|^{\gamma_m}, \quad \forall (x, u) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.3}$$

For any $u \in E$, from (V1), (2.3) and the Hölder inequality, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^N} |F(x, u)|\, dx &\le \int_{\mathbb{R}^N} [\xi_1(x)|u(x)|^{\gamma_1} + \dots + \xi_m(x)|u(x)|^{\gamma_m}]\, dx \\
&\le \sum_{i=1}^m \beta^{-\gamma_i/2}\Big(\int_{\mathbb{R}^N} |\xi_i(x)|^{\frac{2}{2-\gamma_i}}\, dx\Big)^{\frac{2-\gamma_i}{2}}\Big(\int_{\mathbb{R}^N} V(x)|u(x)|^2\, dx\Big)^{\gamma_i/2} \\
&\le \sum_{i=1}^m \beta^{-\gamma_i/2}\|\xi_i\|_{\frac{2}{2-\gamma_i}}\|u\|^{\gamma_i},
\end{aligned}$$

and so Φ defined by (2.1) is well defined on E.

Next, we prove that (2.2) holds. For any function $\theta : \mathbb{R} \to (0, 1)$, by (F1) and the Hölder inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \max_{t\in[0,1]} |f(x, u(x) + t\theta(x)v(x))v(x)|\, dx \\
&\le \int_{\mathbb{R}^N} \max_{t\in[0,1]} |f(x, u(x) + t\theta(x)v(x))||v(x)|\, dx \\
&\le \sum_{i=1}^m \gamma_i \int_{\mathbb{R}^N} \xi_i(x)(|u(x)| + |v(x)|)^{\gamma_i-1}|v(x)|\, dx \\
&\le \sum_{i=1}^m \gamma_i \int_{\mathbb{R}^N} \xi_i(x)(|u(x)|^{\gamma_i-1} + |v(x)|^{\gamma_i-1})|v(x)|\, dx \\
&\le \sum_{i=1}^m \gamma_i\beta^{-\gamma_i/2}\Big(\int_{\mathbb{R}^N} |\xi_i(x)|^{\frac{2}{2-\gamma_i}}\, dx\Big)^{\frac{2-\gamma_i}{2}}\Big(\int_{\mathbb{R}^N} V(x)|u(x)|^2\, dx\Big)^{\frac{\gamma_i-1}{2}} \\
&\quad \times \Big(\int_{\mathbb{R}^N} V(x)|v(x)|^2\, dx\Big)^{1/2} \\
&\quad + \sum_{i=1}^m \gamma_i\beta^{-\gamma_i/2}\Big(\int_{\mathbb{R}^N} |\xi_i(x)|^{\frac{2}{2-\gamma_i}}\, dx\Big)^{\frac{2-\gamma_i}{2}}\Big(\int_{\mathbb{R}^N} V(x)|u(x)|^2\, dx\Big)^{\gamma_i/2} \\
&\le \sum_{i=1}^m \gamma_i\beta^{-\gamma_i/2}\|\xi_i\|_{\frac{2}{2-\gamma_i}}(\|u\|^{\gamma_i-1} + \|v\|^{\gamma_i-1})\|v\| < +\infty.
\end{aligned} \tag{2.4}$$

Then by the above inequality, (2.1) and the Lebesgue's Dominated Convergence Theorem, we have

$$
\begin{aligned}
(\Phi'(u), v) &= \lim_{t\to 0^+} \frac{\Phi(u+tv)-\Phi(u)}{t} \\
&= \lim_{t\to 0^+} \frac{1}{t}\Big\{ \frac{\|u+tv\|^2-\|u\|^2}{2} - \int_{\mathbb{R}^N} [F(x,u(x)+tv(x)) - F(x,u(x))]\,dx \Big\} \\
&= \lim_{t\to 0^+} \Big[(u,v) + \frac{t\|v\|^2}{2} - \int_{\mathbb{R}^N} f(x,u(x)+t\theta(x)v(x))v(x)\,dx \Big] \\
&= (u,v) - \int_{\mathbb{R}^N} f(x,u(x))v(x)\,dx.
\end{aligned}
$$

This shows that (2.2) holds. Furthermore, by a standard argument, it is easy to show that the critical points of Φ in E are solutions of problem (1.1) (see[16]).

Let us prove that Φ' is continuous. Let $u_k \to u$ in E, then $u_k \to u$ in $L^2(\mathbb{R}^N)$, and so

$$
\lim_{k\to\infty} u_k(x) = u(x), \quad \text{a.e. } x \in \mathbb{R}^N. \tag{2.5}
$$

We claim that

$$
\lim_{k\to\infty} \int_{\mathbb{R}^N} |f(x,u_k(x)) - f(x,u(x))|^2\,dx = 0. \tag{2.6}
$$

Indeed, if it is not true, then there exists a constant $\varepsilon > 0$ and a subsequence u_{k_i} such that

$$
\int_{\mathbb{R}^N} |f(x,u_{k_i}(x)) - f(x,u(x))|^2\,dx \geq \varepsilon, \quad \forall i \in \mathbb{N}. \tag{2.7}
$$

Since $u_k \to u$ in $L^2(\mathbb{R}^N)$, passing to a subsequence if necessary, it can be assumed that $\sum_{i=1}^\infty \|u_{k_i} - u\|_2^2 < +\infty$. Set $\omega(x) = \big[\sum_{i=1}^\infty |u_{k_i}(x) - u(x)|^2\big]^{1/2}$, $x \in \mathbb{R}^N$. Then $\omega \in L^2(\mathbb{R}^N)$. Note that

$$
\begin{aligned}
&|f(x,u_{k_i}(x)) - f(x,u(x))|^2 \\
&\leq 2|f(x,u_{k_i}(x))|^2 + 2|f(x,u(x))|^2 \\
&\quad \leq 4\gamma_1^2|\xi_1(x)|^2\big[|u_{k_i}(x)|^{2(\gamma_1-1)} + |u(x)|^{2(\gamma_1-1)}\big] \\
&\quad + \dots + 4\gamma_m^2|\xi_m(x)|^2\big[|u_{k_i}(x)|^{2(\gamma_m-1)} + |u(x)|^{2(\gamma_m-1)}\big] \\
&\leq \sum_{j=1}^m (4^{\gamma_j}+4)\gamma_j^2|\xi_j(x)|^2\big[|u_{k_i}(x)-u(x)|^{2(\gamma_j-1)} + |u(x)|^{2(\gamma_j-1)}\big] \\
&\leq \sum_{j=1}^m (4^{\gamma_j}+4)\gamma_j^2|\xi_j(x)|^2\big[|\omega(x)|^{2(\gamma_j-1)} + |u(x)|^{2(\gamma_j-1)}\big] \\
&:= g(x), \quad \forall i \in \mathbb{N}, x \in \mathbb{R}^N
\end{aligned} \tag{2.8}
$$

and

$$
\begin{aligned}
\int_{\mathbb{R}^N} g(x)\,dx &= \sum_{j=1}^m (4^{\gamma_j}+4)\gamma_j^2 \int_{\mathbb{R}^N} |\xi_j(x)|^2\big[|\omega(x)|^{2(\gamma_j-1)} + |u(x)|^{2(\gamma_j-1)}\big]\,dx \\
&\leq \sum_{j=1}^m (4^{\gamma_j}+4)\gamma_j^2 \|\xi_j\|^2_{\frac{2}{2-\gamma_j}} \Big(\|\omega\|_2^{2(\gamma_j-1)} + \|u\|_2^{2(\gamma_j-1)}\Big) < +\infty.
\end{aligned} \tag{2.9}
$$

Then by (2.5), (2.8), (2.9) and the Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{i\to\infty}\int_{\mathbb{R}^N}|f(x,u_{k_i}(x))-f(x,u(x))|^2\,dx=0,$$

which contradicts (2.7). Hence (2.6) holds. From (2.2), (2.6) and the Hölder inequality, we have

$$\begin{aligned}
&|(\Phi'(u_k)-\Phi'(u),v)|\\
&=\Big|(u_k-u,v)-\int_{\mathbb{R}^N}[f(x,u_k(x))-f(x,u(x))]v(x)\,dx\Big|\\
&\le\|u_k-u\|\|v\|+\int_{\mathbb{R}^N}|f(x,u_k(x))-f(x,u(x))||v(x)|\,dx\\
&\le\|u_k-u\|\|v\|+\beta^{\frac{-1}{2}}\Big(\int_{\mathbb{R}^N}|f(x,u_k(x))-f(x,u(x))|^2\,dx\Big)^{1/2}\|v\|=o(1),
\end{aligned}$$

as $k\to+\infty$, which implies the continuity of Φ'. The proof is complete. □

We will use Morse theory in combination with local linking arguments to obtain the critical points of Φ. Now, it is necessary to recall the following definitions and results.

Definition 2.3. Let E be a real reflexive Banach space. We say that Φ satisfies the (PS)-condition, i.e. every sequence $\{u_n\}\subset E$ satisfying $\Phi(u_n)$ bounded and $\lim_{n\to\infty}\Phi'(u_n)=0$ contains a convergent subsequence.

Let E be a real Banach space and $\Phi\in C^1(E,\mathbb{R})$. $K=\{u\in E:\Phi'(u)=0\}$, then the qth critical group of Φ at an isolated critical point $u\in K$ with $\Phi(u)=c$ is defined by

$$C_q(\Phi,u):=H_q(\Phi^c\cap U,\Phi^c\cap U\setminus\{u\}),\quad q\in\mathbb{N}:=\{0,1,2,\dots\},$$

where $\Phi^c=\{u\in E:\Phi(u)\le c\}$, U is a neighborhood of u, containing the unique critical point, H_* is the singular relative homology with coefficient in an Abelian group G.

We say that $u\in E$ is a homological nontrivial critical point of Φ if at least one of its critical groups is nontrivial. Now, we present the following propositions which will be used later.

Proposition 2.4 ([15, Proposition 2.1])**.** *Assume that Φ has a critical point $u=0$ with $\Phi(0)=0$. Suppose that Φ has a local linking at 0 with respect to $E=V\oplus W$, $k=\dim V<\infty$; that is, there exists $\rho>0$ small such that*

$$\begin{gathered}
\Phi(u)\le 0,\quad u\in V,\ \|u\|\le\rho;\\
\Phi(u)>0,\quad u\in W,\ 0<\|u\|\le\rho.
\end{gathered}$$

Then $C_k(\Phi,0)\ncong 0$, hence 0 is a homological nontrivial critical point of Φ.

Proposition 2.5 ([15, Theorem 2.1])**.** *Let E be a real Banach space and let $\Phi\in C^1(E,\mathbb{R})$ satisfy the (PS)-condition and is bounded from below. If Φ has a critical point that is homological nontrivial and is not a minimizer of Φ, then Φ has at least three critical points.*

Proposition 2.6 ([17, Theorem 9.1]). *Let E be a real Banach space, $\Phi \in C^1(E, \mathbb{R})$ with Φ even, bounded from below, and satisfying (PS)-condition. Suppose $\Phi(0) = 0$, there is a set $K \subset E$ such that K is homeomorphic to S^{j-1} by an odd map, and $\sup_K \Phi < 0$. Then Φ possesses at least j distinct pairs of critical points.*

3. Proof of main results

In this section, we prove Theorems 1.1 and 1.2. To this end we need the following lemmas.

Lemma 3.1. *Suppose that Φ satisfies* (V1) *and* (F1)*, then Φ satisfies the (PS)-condition.*

Proof. We first prove that Φ is coercive. By (2.1), (2.3) and the Hölder inequality, we have

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x, u(x))\,dx \\ &\geq \frac{1}{2}\|u\|^2 - \sum_{i=1}^{m} \int_{\mathbb{R}^N} \xi_i(x)|u(x)|^{\gamma_i}\,dx \\ &\geq \frac{1}{2}\|u\|^2 - \sum_{i=1}^{m} \beta^{-\gamma_i/2}\Big(\int_{\mathbb{R}^N} |\xi_i(x)|^{\frac{2}{2-\gamma_i}}\,dx\Big)^{\frac{2-\gamma_i}{2}}\Big(\int_{\mathbb{R}^N} V(x)|u(x)|^2\,dx\Big)^{\gamma_i/2} \\ &\geq \frac{1}{2}\|u\|^2 - \sum_{i=1}^{m} \beta^{-\gamma_i/2}\|\xi_i\|_{\frac{2}{2-\gamma_i}}\|u\|^{\gamma_i}. \end{aligned} \tag{3.1}$$

Since $1 < \gamma_1 < \dots < \gamma_m < 2$, (3.1) implies that $\Phi(u) \to +\infty$ as $\|u\| \to +\infty$.

Next, we prove that Φ satisfies the (PS)-condition. Assume that $\{u_k\} \subset E$ is a sequence such that $\{\Phi(u_k)\}$ is bounded and $\Phi'(u_k) \to 0$ as $k \to +\infty$. Then by (3.1), there exists a constant $M > 0$ such that

$$\|u_k\| \leq M, \quad \forall k \in \mathbb{N}. \tag{3.2}$$

Going if necessary to a subsequence we can assume that $u_k \rightharpoonup u_0$ in E. For any given number $\varepsilon > 0$, by (F1), we can choose $R_\varepsilon > 0$ such that

$$\Big(\int_{|x|>R_\varepsilon} |\xi_i(x)|^{\frac{2}{2-\gamma_i}}\Big)^{\frac{2-\gamma_i}{2}} < \varepsilon, \quad i = 1, 2, \dots, n. \tag{3.3}$$

Since the embedding of $E \hookrightarrow L^2_{loc}(\mathbb{R}^N)$ is compact, then

$$u_k \to u_0, \quad \text{in } L^2_{loc}(\mathbb{R}^N),$$

and hence,

$$\lim_{k\to\infty} \int_{|x|\leq R_\varepsilon} |u_k(x) - u_0(x)|^2\,dx = 0. \tag{3.4}$$

By (3.4), there exists $k_0 \in \mathbb{N}$ such that

$$\int_{|x|\leq R_\varepsilon} |u_k(x) - u_0(x)|^2\,dx < \varepsilon^2, \quad \text{for } k \geq k_0. \tag{3.5}$$

Hence, by (F1), (3.2), (3.5) and the Hölder inequality, for any $k \geq k_0$ we have

$$
\begin{aligned}
&\int_{|x|\leq R_\varepsilon} |f(x,u_k(x)) - f(x,u_0(x))||u_k(x)-u_0(x)|\,dx \\
&\leq \Big(\int_{|x|\leq R_\varepsilon} |f(x,u_k(x)) - f(x,u_0(x))|^2\,dx\Big)^{1/2}\Big(\int_{|x|\leq R_\varepsilon} |u_k(x)-u_0(x)|^2\,dx\Big)^{1/2} \\
&\leq \Big[\int_{|x|\leq R_\varepsilon} 2(|f(x,u_k(x))|^2 + |f(x,u_0(x))|^2)\,dx\Big]^{1/2}\varepsilon \\
&\leq 2\Big[\sum_{i=1}^m \gamma_i^2 \int_{|x|\leq R_\varepsilon} |\xi_i(x)|^2(|u_k(x)|^{2(\gamma_i-1)} + |u_0(x)|^{2(\gamma_i-1)})\,dx\Big]^{1/2}\varepsilon \\
&\leq 2\Big[\sum_{i=1}^m \gamma_i^2\|\xi_i\|^2_{\frac{2}{2-\gamma_i}}(\|u_k\|_2^{2(\gamma_i-1)} + \|u_0\|_2^{2(\gamma_i-1)})\Big]^{1/2}\varepsilon \\
&\leq 2\Big[\sum_{i=1}^m \gamma_i^2\|\xi_i\|^2_{\frac{2}{2-\gamma_i}}(M^{2(\gamma_i-1)} + \|u_0\|_2^{2(\gamma_i-1)})\Big]^{1/2}\varepsilon.
\end{aligned}
\tag{3.6}
$$

On the other hand, for $k \in \mathbb{N}$, it follows from (F1), (3.2), (3.3) and the Hölder inequality that

$$
\begin{aligned}
&\int_{|x|> R_\varepsilon} |f(x,u_k(x)) - f(x,u_0(x))||u_k(x)-u_0(x)|\,dx \\
&\leq \sum_{i=1}^m \gamma_i \int_{|x|>R_\varepsilon} |\xi_i(x)|(|u_k(x)|^{\gamma_i-1} + |u_0(x)|^{\gamma_i-1})(|u_k(x)|+|u_0(x)|)\,dx \\
&\leq 2\sum_{i=1}^m \gamma_i \int_{|x|>R_\varepsilon} |\xi_i(x)|(|u_k(x)|^{\gamma_i} + |u_0(x)|^{\gamma_i})\,dx \\
&\leq 2\sum_{i=1}^m \gamma_i \Big(\int_{|x|>R_\varepsilon} |\xi_i(x)|^{\frac{2}{2-\gamma_i}}\,dx\Big)^{\frac{2-\gamma_i}{2}}(\|u_k\|_2^{\gamma_i} + \|u_0\|_2^{\gamma_i}) \\
&\leq 2\sum_{i=1}^m \gamma_i \Big(\int_{|x|>R_\varepsilon} |\xi_i(x)|^{\frac{2}{2-\gamma_i}}\,dx\Big)^{\frac{2-\gamma_i}{2}}(M^{\gamma_i} + \|u_0\|_2^{\gamma_i}) \\
&\leq 2\sum_{i=1}^m \gamma_i (M^{\gamma_i} + \|u_0\|_2^{\gamma_i})\varepsilon.
\end{aligned}
\tag{3.7}
$$

Since ε is arbitrary, combining (3.6) with (3.7), one has

$$
\int_{\mathbb{R}^N} [f(x,u_k(x)) - f(x,u_0(x))][u_k(x)-u_0(x)]\,dx \to 0 \tag{3.8}
$$

as $k \to \infty$. It follows from (2.2) that

$$
\begin{aligned}
&(\Phi'(u_k) - \Phi'(u_0), u_k - u_0) \\
&= \|u_k - u_0\|^2 - \int_{\mathbb{R}^N} [f(x,u_k(x)) - f(x,u_0(x))][u_k(x)-u_0(x)]\,dx.
\end{aligned}
\tag{3.9}
$$

In view of the definition of weak convergence, we have

$$
(\Phi'(u_k) - \Phi'(u_0), u_k - u_0) = 0. \tag{3.10}
$$

It follows from (3.8)-(3.10) that
$$u_k \to u_0 \quad \text{in } E.$$
Hence, Φ satisfies the (PS)-condition. $\square$

We choose an orthogonal basis $\{e_j\}$ of E and define $X_j := \operatorname{span}\{e_j\}, j = 1, 2, \dots$, $Y_k := \oplus_{j=1}^{k} X_j$, $Z_k = \overline{\oplus_{j=k+1}^{\infty} X_j}$, then $E = Y_k \oplus Z_k$.

Lemma 3.2. *Suppose that the conditions of Theorem 1.1 are satisfied, then there exists $k_0 \in \mathbb{N}$ such that $C_{k_0}(\Phi, 0) \ncong 0$.*

Proof. It follows from (F1) that the zero function is a critical point of Φ. So we only need to prove that Φ has a local linking at 0 with respect to $E = Y_k \oplus Z_k$.

Step 1: Take $u \in Y_k$, since Y_k is finite dimensional, we have that for given r_0, there exists $0 < \rho < 1$ small such that
$$u \in Y_k,\ \|u\| \le \rho \Rightarrow |u| < r_0, \quad x \in \mathbb{R}^N$$
For $0 < r < r_0$, let $\Omega_1 = \{x \in \mathbb{R}^N : |u(x)| < r\}$, $\Omega_2 = \{x \in \mathbb{R}^N : r \le |u(x)| \le r_0\}$, $\Omega_3 = \{x \in \mathbb{R}^N : |u(x)| > r_0\}$, then $\mathbb{R}^N = \bigcup_{i=1}^{3} \Omega_i$. For the sake of simplicity, let $G(x, u) = F(x, u) - c_1|u|^\gamma$. Therefore, form (F2) it follows that
$$\begin{aligned} \Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} c_1|u|^\gamma\,dx - \Big(\int_{\Omega_1} + \int_{\Omega_2} + \int_{\Omega_3}\Big) G(x, u)\,dx \\ &\le \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} c_1|u|^\gamma\,dx - \int_{\Omega_1} G(x, u)\,dx. \end{aligned}$$

Note that the norms on Y_k are equivalent to each other, $\|u\|_\gamma$ is equivalent to $\|u\|$ and $\int_{\Omega_1} G(x, u)\,dx \to 0$ as $r \to 0$. Since $0 < \gamma < 2$, then $\Phi(u) \le 0$, for all $u \in Y_k$ with $\|u\| \le \rho$.

Step 2: Take $u \in Z_k$, since the embedding $E \hookrightarrow L^p$ is continuous, we have that for given r_0, there exists $0 < \rho < 1$ small such that
$$u \in Z_k,\ \|u\| \le \rho \Rightarrow |u| < r_0, \quad x \in \mathbb{R}^N.$$
Therefore, it follows from (F2) that
$$\Phi(u) \ge \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} c_2|u|^2\,dx > \frac{1}{2}\|u\|^2 - \frac{1}{2}\|u\|^2 = 0.$$
Therefore, by Proposition 2.4, the proof is complete. $\square$

Proof of Theorem 1.1. By Lemma 3.1, Φ satisfies the (PS)-condition and is bounded from below. By Lemma 3.2 and Proposition 2.4, the trivial solution $u = 0$ is homological nontrivial and is not a minimizer. Then Theorem 1.1 follows immediately from Proposition 2.5. $\square$

Proof of Theorem 1.2. By (F3), we can easily check that the functional Φ is even. Lemma 3.1 shows that Φ satisfies the (PS)-condition and is bounded from below. For $\rho > 0$, let $K = S_\rho = \{u \in Y_k : \|u\| = \rho\}$. Thus, just as shown in the proof of Lemma 3.2, if $\rho > 0$ is small enough, we have that
$$\sup_K \Phi(u) \le 0.$$
By the definition of Y_k, we have $\dim Y_k = k$, then by Proposition 2.6, we have that Φ has at least k distinct pairs of critical points. Therefore, problem (1.1) has at least k distinct pairs of solutions. $\square$

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