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EXISTENCE OF POSITIVE SOLUTIONS OF A NONLINEAR SECOND-ORDER BOUNDARY-VALUE PROBLEM WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this article we prove the existence of at least one positive solution for a three-point integral boundary-value problem for a second-order nonlinear differential equation. The existence result is obtained by using Schauder's fixed point theorem. Therefore, we do not need local assumptions such as superlinearity or sublinearity of the involved nonlinear functions.

1. Introduction and preliminary results

Boundary-value problems (BVP) for differential equations have been extensively studied, mainly because they appear in applications in areas such as physics, biology and engineering sciences. See, e.g., the classical monographs [1, 5] and references therein.

BVP with integral boundary conditions constitute a very important class of problems. These BVP include two, three, multipoint and nonlocal BVP as special cases. The study of existence of solutions of multipoint boundary value problems for linear second-order ordinary differential equations was initiated in 1987 by Il'in and Moiseev [3]. The consideration of three-point boundary-value problems for nonlinear ordinary differential equations began in 1992 with the work of Gupta [2].

In 2010, Tariboon and Sitthiwirattham [4], by applying the Krasnoselskii fixed point theorem in cones, proved the existence of positive solutions of a nonlinear three-point integral boundary-value problem whose boundary conditions are related to the area under the curve of the solutions. More precisely, they consider the existence of positive solutions of the BVP

$$
\begin{gathered}
u'' + a(t)f(u) = 0 \\
u(0) = 0, \quad \alpha \int_0^\eta u(s)ds = u(1), \quad \eta \in (0,1).
\end{gathered}
$$

In their analysis they assume that the function f is either superlinear or sublinear. That is, defining

$$
f_0 := \lim_{u\to 0^+} \frac{f(u)}{u}, \quad f_\infty := \lim_{u\to\infty} \frac{f(u)}{u},
$$

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then, $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case.

In 2015 Yao [6], by means of the Leray-Schauder fixed point theorem, relaxed such conditions by showing that the BVP above has at least a positive solution if $f_0 = 0$ (condition $f_\infty = \infty$ being unnecessary), as well as, for $f_\infty = 0$ (condition $f_0 = \infty$ being also unnecessary).

In both works previously mentioned, the fixed point criteria applied to get the corresponding result depends on the local behavior of the related operator. In the analysis of the boundary value problem under study, this fact is reflected in the local growth conditions that have to be imposed on the function f in order to verify the assumptions needed to apply the fixed point argument.

In this article we extend the results in [4, 6] by proving the existence of positive solutions on $C[0,\gamma]$, for the BVP

$$
\begin{gathered}
u'' + a(t)f(u) = 0\\
u(0) = 0, \quad \alpha\int_0^\eta u(s)ds = u(\gamma) \quad \text{with } \eta \in (0,\gamma).
\end{gathered}
$$

More precisely, we do not impose any extra condition on the function f. In this way, for our analysis we use the Schauder's fixed point theorem. Therefore, we only need to prove a global condition (instead of using local arguments): a compactness condition on the involved operators associated to the equation.

For completeness of the presentation we enunciate the classical results that will be used in the sequel.

Theorem 1.1 (Schauder fixed point)**.** *Let K be a closed convex set in a Banach space X and assume that $T : K \to K$ is a continuous mapping such that $T(K)$ is a relatively compact subset of K. Then T has a fixed point in K.*

The classical tool to verify the conditions of the Schauder's fixed point Theorem, in the case when we are dealing with the space of continuous functions $C[a,b]$ is the Arzela-Ascoli's Theorem.

Theorem 1.2 (Arzela-Ascoli)**.** *A necessary and sufficient condition for a family of continuous functions defined on the compact interval $[a,b]$ to be compact in $C[a,b]$ is that this family is uniformly bounded and equicontinuous.*

2. Auxiliary results on a linear BVP

In this section we prove some auxiliary lemmas that are needed in the sequel. In particular, the next result provide conditions for the existence of a unique solution of an auxiliary linear boundary value problem.

Lemma 2.1. *Let $2\gamma \neq \alpha\eta^2$. Then for $y \in C[0,\gamma]$, the problem*

$$
u'' + y(t) = 0 \tag{2.1}
$$

$$
u(0) = 0, \quad \alpha\int_0^\eta u(s)ds = u(\gamma), \quad \eta \in (0,\gamma), \quad \alpha \neq 0, \tag{2.2}
$$

has a unique solution given by

$$\begin{aligned} u(t) = &\frac{2t}{2\gamma - \alpha\eta^2} \int_0^\gamma (\gamma - s) y(s) ds - \frac{\alpha t}{2\gamma - \alpha\eta^2} \int_0^\eta (\eta - s)^2 y(s) ds \\ &- \int_0^t (t - s) y(s) ds. \end{aligned} \tag{2.3}$$

Proof. From equation (2.1) we have $u''(t) = -y(t)$. Then, integrating form 0 to t we obtain

$$u'(t) = u'(0) - \int_0^t y(s) ds, \quad t \in [0, \gamma).$$

For $t \in [0, \gamma]$ we have, by integrating in t and using integration by parts,

$$\begin{aligned} u(t) &= u'(0)t - \int_0^t \Big(\int_0^x y(s) ds \Big) dx \\ &= u'(0)t - \int_0^t (t - s) y(s) ds. \end{aligned} \tag{2.4}$$

Thus, for $t = \gamma$ we find

$$u(\gamma) = u'(0)\gamma - \int_0^\gamma (\gamma - s) y(s) ds. \tag{2.5}$$

Integrating again from 0 to η the expression (2.4), where $\eta \in (0, \gamma)$, we obtain

$$\begin{aligned} \int_0^\eta u(s) ds &= u'(0)\frac{\eta^2}{2} - \int_0^\eta \Big(\int_0^x (x - s) y(s) ds \Big) dx \\ &= u'(0)\frac{\eta^2}{2} - \frac{1}{2} \int_0^\eta (\eta - s)^2 y(s) ds. \end{aligned} \tag{2.6}$$

From (2.2) and (2.5) we have

$$\int_0^\eta u(s) ds = \frac{1}{\alpha} u(\gamma) = u'(0)\frac{\gamma}{\eta} - \frac{1}{\alpha} \int_0^\gamma (\gamma - s) y(s) ds.$$

Then, using (2.6) we see that

$$u'(0)\frac{\gamma}{\alpha} - \frac{1}{\alpha} \int_0^\gamma (\gamma - s) y(s) ds = u'(0)\frac{\eta^2}{2} - \frac{1}{2} \int_0^\eta (\eta - s)^2 y(s) ds.$$

Thus, rearraying terms, we can write

$$u'(0)\Big(\frac{\gamma}{\alpha} - \frac{\eta^2}{2}\Big) = \frac{1}{\alpha} \int_0^\gamma (\gamma - s) y(s) ds - \frac{1}{2} \int_0^\eta (\eta - s)^2 y(s) ds$$

or

$$u'(0) = \frac{2\alpha}{(2\gamma - \alpha\eta^2)\alpha} \int_0^\gamma (\gamma - s) y(s) ds - \frac{2\alpha}{(2\gamma - \alpha\eta^2)2} \int_0^\eta (\eta - s)^2 y(s) ds.$$

Therefore, the boundary-value problem (2.1)–(2.2) has a unique solution

$$u(t) = \frac{2t}{2\gamma - \alpha\eta^2} \int_0^\gamma (\gamma - s) y(s) ds - \frac{\alpha t}{2\gamma - \alpha\eta^2} \int_0^\eta (\eta - s)^2 y(s) ds - \int_0^t (t - s) y(s) ds.$$

□

The existence of positive solutions of the BVP (2.1)–(2.2) is given in the next result.

Lemma 2.2. *Let $0 < \alpha < 2/\eta^2$. If $y \in C(0,\gamma)$ and $y(t) \geq 0$ on $(0,\gamma)$, then the unique solution of the problem (2.1)–(2.2) satisfies $u(t) \geq 0$ for $t \in [0,\gamma]$.*

Proof. First, notice that u is concave. Observe also that if $u(\gamma) \geq 0$, the concavity of u and the fact that $u(0) = 0$ imply that $u(t) \geq 0$ for $t \in (0,\gamma)$. Therefore it is enough to prove that $u(\gamma) \geq 0$. In fact, arguing by contradiction, if we assume that $u(\gamma) < 0$, then, from (2.2) we have

$$\int_0^\eta u(s)ds < 0.$$

The concavity of u and $\int_0^\eta u(s)ds < 0$ imply that $u(\eta) < 0$. Thus, using the fact $0 < \alpha < 2/\eta^2$ and comparing integrals, we conclude

$$u(\gamma) = \alpha \int_0^\eta u(s)ds \geq \frac{\alpha\eta}{2}u(\eta) > \frac{u(\eta)}{\eta}$$

which contradicts the concavity of u. The proof is complete. □

The condition on α is sharp in the sense of the following result.

Lemma 2.3. *Let $\alpha > 2/\eta^2$. If $y \in C(0,\gamma)$ and $y(t) \geq 0$. Then the problem (2.1)–(2.2) has a nonpositive solution.*

Proof. Assume that the problem (2.1)–(2.2) has a positive solution u. If $u(\gamma) > 0$ then $\int_0^\eta u(s)ds > 0$. It implies in particular that $u(\eta) > 0$ and using $\alpha > 2/\eta^2$, we obtain

$$u(\gamma) = \alpha \int_o^\eta u(s)ds \geq \frac{\alpha\eta}{2}u(\eta) > \frac{u(\eta)}{\eta}.$$

This contradicts the concavity of u.

If $u(\gamma) = 0$, then $\int_0^\eta u(s)ds = 0$ and therefore $u(t) = 0$ for all $t \in [0,\eta]$ due to the concavity of u. On the other hand, if there exits $\tau \in (\eta,\gamma)$ such that $u(\tau) > 0$, then $u(0) = u(\eta) < u(\tau)$ which again contradicts the concavity of u. Therefore, no positive solutions exist. □

3. Existence of positive solutions for the nonlinear BVP

From Lemmas 2.1 and 2.2, in particular from expression (2.3), for $0 < \alpha < 2/\eta^2$ with $2\gamma \neq \alpha\eta^2$, the function u is a solution of

$$u'' + a(t)f(u) = 0,$$

under the condition (2.2), for $a : [0,\gamma] \to [0,\infty)$ and $f : [0,\infty) \to [0,\infty)$ continuous functions, if $u(t)$ is a fixed point of the operator

$$\begin{aligned}
Au(t) &:= \frac{2t}{2\gamma - \alpha\eta^2}\int_0^\gamma (\gamma - s)a(s)f(u(s))ds - \frac{\alpha t}{2\gamma - \alpha\eta^2}\int_0^\eta (\eta - s)^2 a(s)f(u(s))ds \\
&\quad - \int_0^t (t-s)a(s)f(u(s))ds \\
&= \frac{(2-\alpha)t}{2\gamma - \alpha\eta^2}\int_0^\gamma [(\gamma - s) - (\eta - s)^2\chi_{(0,\eta)}(s)]a(s)f(u(s))ds \\
&\quad - \int_0^t (t-s)a(s)f(u(s))ds.
\end{aligned}$$

Here $\chi_{(0,\eta)}$ is the characteristic function of the interval $(0,\eta)$.

Let us consider the operators,

$$Fu(t) := \frac{(2-\alpha)t}{2\gamma-\alpha\eta^2}\int_0^\gamma [(\gamma-s)-(\eta-s)^2\chi_{(0,\eta)}(s)]a(s)f(u(s))ds$$
$$Gu(t) := \int_0^t (t-s)a(s)f(u(s))ds.$$

Then, we can write

$$Au(t) = Fu(t) - Gu(t).$$

To use the Schauder's fixed point theorem, first we need to check that the operator A is compact. This fact is establish in the following theorem.

Theorem 3.1. *The operator $A : C[0,\gamma] \to C[0,\gamma]$ is compact.*

Proof. Since $A = F - G$, then we should to prove that the operators F and G are compact. First, we prove that the operator F is compact. Let $u \in C[0,\gamma]$. It is clear that $(Fu)(t)$ is a continuous function, then $F(C[0,\gamma]) \subset C[0,\gamma]$. On the other hand,

$$\begin{aligned} &|(Fu)(t) - (Fu)(w)| \\ &= \Big|\frac{(2-\alpha)t}{2\gamma-\alpha\eta^2}\int_0^\gamma [(\gamma-s)-(\eta-s)^2\chi_{(0,\eta)}(s)]a(s)f(u(s))ds \\ &\quad - \frac{(2-\alpha)w}{2\gamma-\alpha\eta^2}\int_0^\gamma [(\gamma-s)-(\eta-s)^2\chi_{(0,\eta)}(s)]a(s)f(u(s))ds\Big| \\ &= |t-w|\Big|\frac{(2-\alpha)}{2\gamma-\alpha\eta^2}\int_0^\gamma [(\gamma-s)-(\eta-s)^2\chi_{(0,\eta)}(s)]a(s)f(u(s))d\Big| \to 0 \end{aligned} \tag{3.1}$$

uniformly as $|t-w| \to 0$, thus F is continuous. To prove the compactness of F is suffices to check that F satisfies the conditions of the Arzela-Ascoli's Theorem. Let $K = \{u_n : n \in \mathbb{N}\}$ be a uniformly bounded set of $C[0,\gamma]$; that is, there exists a positive constant $M > 0$ such that $|u_n(t)| \le M$ for all $u_n \in K$. Then,

$$\begin{aligned} \|Fu_n\|_\infty &= \Big\|\frac{(2-\alpha)t}{2\gamma-\alpha\eta^2}\int_0^\gamma [(\gamma-s)-(\eta-s)^2\chi_{(0,\eta)}(s)]a(s)f(u_n(s))ds\Big\|_\infty \\ &\le \Big|\frac{(2-\alpha)}{2\gamma-\alpha\eta^2}\Big|\Big\|t\int_0^\gamma (\gamma-s)a(s)f(u_n(s))ds\Big\|_\infty \\ &\le \Big|\frac{(2-\alpha)}{2\gamma-\alpha\eta^2}\Big|\frac{\gamma^3}{2}\|a\|_\infty\|f(u_n)\|_\infty. \end{aligned}$$

Since $f : [0,M] \to [0,\infty)$ is continuous, last inequality is uniformly bounded for all $u_n \in K$. Hence $F(K)$ s uniformly bounded. Replacing u by u_n in (3.1) we show that $F(K)$ is equicontinuous. thus $F : C[0,\gamma] \to C[0,\gamma]$ is completely continuous.

On the other hand, the operator G is the classic Volterra operator which is compact. For completeness we present a proof. Let $B_\infty(1)$ be the unit closed ball of $C[0,\gamma]$ and $u \in B_\infty(1)$. Then

$$|Gu(t) - Gu(w)| = \Big|\int_0^t (t-s)a(s)f(u(s))ds - \int_0^w (w-s)a(s)f(u(s))ds\Big|.$$

The above expression approaches zero when $|t-w| \to 0$ uniformly in $\overline{B}_\infty(1)$. Therefore, from the Arzela-Ascoli Theorem, $G(\overline{B}_\infty(1))$ is relatively compact and then G is compact. This complete the proof of the theorem. $\square$

The existence of positive solutions of the nonlinear second-order boundary-value problem with three-point integral boundary conditions under consideration, is given in the following theorem.

Theorem 3.2. *The boundary-value problem*

$$
\begin{gathered}
u'' + a(t)f(u) = 0 \\
u(0) = 0, \quad \alpha \int_0^\eta u(s)ds = u(\gamma), \quad 0 < \alpha < \frac{2}{\eta^2}, \quad 2\gamma \neq \alpha\eta^2
\end{gathered}
$$

has at least one positive solution on $C[0,\gamma]$.

Proof. From Theorem 3.1, we have that the operator $A : C[0,\gamma] \to C[0,\gamma]$ is compact. Let $R > 0$ be a positive number and consider the closed convex ball on $C[0,\gamma]$, denoted by $B_\infty(R)$. For $u \in B_\infty(R)$ by using the triangle inequality the following estimate holds

$$
\begin{aligned}
&\|Au\|_\infty \\
&= \Big\| \frac{(2-\alpha)t}{2\gamma - \alpha\eta^2} \int_0^\gamma [(\alpha - s) - (\eta - s)^2 \chi_{(0,\eta)}(s)] a(s) f(u(s)) ds \\
&\quad - \int_0^t (t-s) a(s) f(u(s)) ds \Big\|_\infty \\
&\le \Big| \frac{(2-\alpha)\gamma}{2\gamma - \alpha\eta^2} \Big| \int_0^\gamma \| [(\alpha - s) - (\eta - s)^2 \chi_{(0,\eta)}(s)] a(s) f(u(s)) \|_\infty ds \\
&\quad + \Big\| \int_0^t (t-s) a(s) f(u(s)) ds \Big\|_\infty \\
&\le \Big| \frac{(2-\alpha)\gamma}{2\gamma - \alpha\eta^2} \Big| \int_0^\gamma |\eta - \frac{1}{2}| \|a\|_\infty \|f(u)\|_\infty ds + \|a\|_\infty \|f(u)\|_\infty \sup_{t\in[0,\gamma]} \int_0^t |\gamma - s| ds \\
&\le \Big| \frac{(2-\alpha)\gamma^2}{2\gamma - \alpha\eta^2} \Big| |\eta - \frac{1}{2}| \|a\|_\infty \|f(u)\|_\infty + \frac{\gamma^2}{2} \|a\|_\infty \|f(u)\|_\infty .
\end{aligned}
$$

In the inequality above we used that $|\eta - 1/2| = \max_{s\in[0,\eta]} |(\gamma - s) - (\eta - s)^2|$. Since $u \in B_\infty(R)$ and the function $f : [0,R] \to \mathbb{R}$ is bounded and continuous, then $\|f(u)\|_\infty$ is finite. Hence, $A(B_\infty(R)) \subset B_\infty(R)$ whenever

$$
R \ge \Big(\Big| \frac{(2-\alpha)}{2\gamma - \alpha\eta^2} \Big| |\eta - \frac{1}{2}| + \frac{1}{2} \Big) \gamma^2 \|a\|_\infty \|f(u)\|_\infty .
$$

From Theorem 1.1, the operator A has at least a fixed point on $B_\infty(R)$. With this we obtain our result. ☐

To illustrate our result, let us consider the following boundary-value problem defined on $C[0,\pi]$

$$
\begin{gathered}
u''(t) + \frac{10\sin(t)}{e^{10\sin(t)+t}} e^{u(t)} = 0 \\
u(0) = 0, \quad \frac{\pi}{2} \int_0^\eta u(s)ds = \pi, \quad \eta = 0.6.
\end{gathered}
$$

Since $\pi/2 < 2/\eta^2 = 4.1$, from Theorem 3.2 there exists a positive solution of the boundary value problem. In fact, the function $u(t) = 10\sin(t) + t$ is a solution of the problem and it is positive in $[0,\pi]$.

On the other hand, notice that the nonlinear term e^u is neither superlinear nor sublinear, thus this problem cannot be analyze by the results given on [4]. Moreover, the limits

$$\lim_{u\to 0^+} \frac{f(u)}{u} = \lim_{u\to\infty} \frac{f(u)}{u} = \infty,$$

therefore the results on [6] also cannot be applied to show the existence of a positive solution in this example.

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Addendum posted on November 4, 2015

After this article was published, a reader indicated that the condition $\|f\|_\infty < \infty$ is necessary in Theorem 3.2. Under this condition, the example can not be considered, and the results in this article become a particular case of the results on reference [7] below.

Also we want to correct the following misprints.

- A γ was missing in the conditions on the parameter α in our results. That should be, $0 < \alpha < 2\gamma/\eta^2$ in Lemma 2.2 and Theorem 3.2. For the Lemma 2.3 the condition should be $\alpha > 2\gamma/\eta^2$. Note that these changes do not affect any proofs in our results. The only action to be taken is to replace the condition in α by the correct one where it appears.
- In Lemma 2.3. The correct conclusion is: the problem (2.1)-(2.2) has no (strictly) positive solution.
- Theorem 3.2 needs a correction. The correct conclusion is: The boundary-value problem has at least one non-negative solution on $C[0,\gamma]$, assuming that $\|f\|_\infty < \infty$.
- The bound of the radius R in the proof is incorrect: In page 6, line 13 appears $[(\alpha - s) - (\eta - s)^2\chi_{(0,\eta)}(s)]$. Should be $[(\gamma - s) - (\eta - s)^2\chi_{(0,\eta)}(s)]$. This fact affects the lower bound for R, because we claim

$$|\eta - 1/2| = \max_{s\in[0,\eta]} |[(\alpha - s) - (\eta - s)^2|.$$

The correct statement is

$$\max_{s\in[0,\gamma]} |[(\gamma - s) - (\eta - s)^2\chi_{(0,\eta)}(s)]| \le \gamma + \eta^2.$$

Thus, in the proof where appears $|\eta - 1/2|$ should be replace by $\gamma + \eta^2$ (note that the inequality still holds).

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