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ASYMPTOTICALLY LINEAR SCHRÖDINGER EQUATION WITH ZERO ON THE BOUNDARY OF THE SPECTRUM

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Abstract. This article concerns the Schrödinger equation

$$\begin{gathered}-\Delta u+V(x)u=f(x,u),\quad \text{for } x\in\mathbb{R}^N,\\ u(x)\to 0,\quad \text{as } |x|\to\infty,\end{gathered}$$

where V and f are periodic in x, and 0 is a boundary point of the spectrum $\sigma(-\Delta+V)$. Assuming that $f(x,u)$ is asymptotically linear as $|u|\to\infty$, existence of a ground state solution is established using some new techniques.

1. Introduction and statement of main results

In this article, we consider the Schrödinger equation

$$\begin{gathered}-\Delta u+V(x)u=f(x,u),\quad \text{for } x\in\mathbb{R}^N,\\ u(x)\to 0,\quad \text{as } |x|\to\infty,\end{gathered}\tag{1.1}$$

where $V:\mathbb{R}^N\to\mathbb{R}$ is a potential and being 1-periodic in x_i, $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a nonlinear coupling which is asymptotically linear as $|u|\to\infty$, i.e. the nonlinearity f satisfies the assumption

(A1) $f(x,t)-V_\infty(x)t=o(|t|)$, as $|t|\to\infty$, uniformly in $x\in\mathbb{R}^N$, where $f\in C(\mathbb{R}^N\times\mathbb{R}), V_\infty\in C(\mathbb{R}^N)$ is 1-periodic in x_i, $i=1,2,\dots,N$, and $\inf V_\infty(x)>\bar\Lambda:=\inf[\sigma(-\Delta+V)\cap(0,\infty)]$.

This equation arise in applications from mathematical physics, and solutions of (1.1) can be interpreted as stationary states of the corresponding reaction-diffusion equation which models phenomena from chemical dynamics. It is known that for periodic potential, the operator $\mathcal{A}:=-\Delta+V$ has purely continuous spectrum $\sigma(\mathcal{A})$ which is bounded below and consists of closed disjoint intervals (see [27, Theorem XIII.100]). Problem (1.1) with periodic potentials and asymptotically linear nonlinearities has been widely investigated in the literature over the past several decades, see [5, 9, 10, 11, 19, 20, 14, 16, 22, 30, 33, 36, 37, 38, 42] and the references therein. Here, we recall some results on existence and multiplicity of solutions of (1.1) depending on the location of 0 in $\sigma(\mathcal{A})$.

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Case 1: $\inf \sigma(\mathcal{A}) > 0$. Since the operator $\mathcal{A}$ is strictly positive definite, techniques based on the mountain pass theorem have been well applied. For example, using the 'monotonicity trick' introduced by Struwe [28], Jeanjean [14] (see also [16]) proved a positive solution for (1.1) under (A1), $V(x) \equiv K > 0$ and the following growth and technical assumptions:

- (A2') $F(x,t) := \int_0^t f(x,s)\mathrm{d}s \geq 0$, and $f(x,t) = o(|t|)$ as $|t| \to 0$ uniformly in $x \in \mathbb{R}^N$;
- (A3) $\mathcal{F}(x,t) := \frac{1}{2}tf(x,t) - F(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, and there exists a $\delta_0 \in (0, \bar{\Lambda})$ such that

$$\frac{f(x,t)}{t} \geq \bar{\Lambda} - \delta_0 \Longrightarrow \mathcal{F}(x,t) \geq \delta_0. \tag{1.2}$$

Ding and Luan [10] obtained infinitely many geometrically distinct solutions with (A1), (A2') and (A3) (in particular, $V \in \mathcal{C}^1$, $f \in \mathcal{C}^2$). Similar results can be found in [37] with f being independent of x and $V_\infty \equiv a > \bar{\Lambda}$. Under assumption that $V(x) = \lambda g(x) + 1$ provided that $\lambda \geq 0$ and $g(x) \geq 0$ has a potential well, multiple solutions are obtained by Heerden [36] (see also [38]). For asymptotically periodic nonlinearities, we refer readers to [20] where a nontrivial solution was obtained by using a version of the mountain pass theorem and comparing with appropriate solutions of a periodic problem associated with (1.1).

Case 2: 0 lies in a spectral gap of $\sigma(\mathcal{A})$, i.e.

$$\sup[\sigma(\mathcal{A}) \cap (-\infty, 0)] := \underline{\Lambda} < 0 < \bar{\Lambda} = \inf[\sigma(\mathcal{A}) \cap (0, \infty)]. \tag{1.3}$$

In this case, Szulkin and Zou [30] first proved the existence of a nontrivial solution for (1.1) with (A1), (A2') and a modified version of (A3):

- (A3') $\mathcal{F}(x,t) := \frac{1}{2}tf(x,t) - F(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, and there exists a $\delta_0 \in (0, \lambda_0)$ such that: if $f(x,t)/t \geq \lambda_0 - \delta_0$ then $\mathcal{F}(x,t) \geq \delta_0$, where $\lambda_0 := \min\{-\underline{\Lambda}, \bar{\Lambda}\}$.

Under assumptions (A1), (A2') and (A3'), moreover $f(x,t)$ is odd in t, Ding and Lee [9] proved that (1.1) has infinitely many geometrically distinct solutions. In recent paper, the author [33] developed a much more direct approach to find a ground state solution of Nehari-Pankov type for (1.1) with (A2'), a slightly stronger version of (A1) and the following monotone assumption:

- (A4) $t \mapsto \frac{f(x,t)}{|t|}$ is non-decreasing on $(-\infty, 0) \cup (0, \infty)$.

Note that, it follows from (A4) that

$$\mathcal{F}(x,t) = \frac{1}{2}tf(x,t) - F(x,t) = \int_0^t \Big(\frac{f(x,t)}{t} - \frac{f(x,s)}{s}\Big)s\mathrm{d}s \geq 0, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

and $\mathcal{F}$ is non-decreasing on $t \in [0, \infty)$ and non-increasing on $t \in (-\infty, 0]$, which together with (A1) and $f(x,t) = o(|t|)$ as $|t| \to 0$ uniformly in x, imply that (A3) and (A3') hold (see [16, Remark 1.3] or [19]). For asymptotically periodic nonlinearities, Li and Szulkin [19] obtained a nontrivial solution with (A1), (A3') and some assumptions on the asymptotic behaviour of f as $|x| \to \infty$.

Case 3: 0 is a boundary point of the spectrum $\sigma(\mathcal{A})$, i.e. the potential $V(x)$ satisfies

- (A5) $V \in C(\mathbb{R}^N)$ is 1-periodic in x_i, $i = 1, 2, \dots, N$, $0 \in \sigma(\mathcal{A})$, and there exists $b_0 > 0$ such that $(0, b_0] \cap \sigma(\mathcal{A}) = \emptyset$.

Clearly, $\bar{\Lambda} \geq b_0$ by (A1). To the author's best knowledge, no previous study has focused on this situation (Even for superlinear nonlinearities, there are few papers [2, 21, 23, 24, 25, 32, 40, 41] in the literature). The main difficulties to overcome are the lack of a priori bounds for Cerami sequences and the working space for this case is only a Banach space, not a Hilbert space which is different from [30, 9]. Unlike Case 1, strongly indefinite problem (1.1) can not be reformulated in terms of a functional having the mountain pass geometry. Moreover, the methods used in [2, 40, 41] are no more applicable, and even though techniques used in [30, 9] can be adapted, the condition (A3') does not hold in this case since $\lambda_0 = 0$. Inspired by above works and using a generalized linking theorem established in [32], we are going to consider this case in the present paper. To conquer difficulties mentioned above, the concentration compactness arguments introduced by P.L. Lions [18] and developed by Jeanjean [14] are adapted, a new variational framework which is more suitable for this case is introduced. Additionally, some new techniques and (A3) instead of (A3') are used in this paper. Before presenting our main results, we introduce the following mild assumptions:

(A2) there exist constants c_1, $c_2 > 0$, $\varrho \in (2, 2^*)$ such that

$$c_1 \min\{|t|^{\varrho}, |t|^2\} \leq tf(x,t) \leq c_2 |t|^{\varrho}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{1.4}$$

Let E be the Banach space defined in Section 2. Under assumptions (A5), (A1) and (A2), the functional

$$\Phi(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x, \tag{1.5}$$

is well defined for all $u \in E$, moreover $\Phi \in C^1(E, \mathbb{R})$ (see Lemma 2.2). Denote the critical set by

$$\mathcal{M} = \{u \in E \setminus \{0\} : \langle \Phi'(u), v\rangle = 0, \ \ \forall\, v \in E\}. \tag{1.6}$$

Now, we are ready to state the main results of this article.

Theorem 1.1. *Let* (A1)–(A3) *and* (A5) *be satisfied. Then problem* (1.1) *has a ground state solution, i.e. a nontrivial solution* $u_0 \in E$ *satisfying* $\Phi(u_0) = inf_{\mathcal{M}} \Phi$.

Corollary 1.2. *Let* (A1)–(A2) *and* (A4)–(A5) *be satisfied. Then problem* (1.1) *has a ground state solution, i.e. a nontrivial solution* $u_0 \in E$ *satisfying* $\Phi(u_0) = inf_{\mathcal{M}} \Phi$.

The following three functions satisfy all assumptions of Corollary 1.2:

$f(x,t) = V_\infty(x) \min\{|t|^\nu, 1\}\, t$, where $\nu \in (0, 2^* - 2)$ and $V_\infty \in C(\mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$ and $\inf V_\infty > \overline{\Lambda}$.

$f(x,t) = V_\infty(x) \left[1 - \frac{1}{\ln(e+|t|^\nu)}\right] t$, where $\nu \in (0, 2^* - 2)$, $V_\infty \in C(\mathbb{R}^N)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$ and $\inf V_\infty > \overline{\Lambda}$.

$f(x,t) = h(x,|t|)t$, where $h(x,s)$ is non-decreasing on $s \in [0,\infty)$ and 1-periodic in each of $x_1, x_2, \dots, x_N$, $h(x,s) = O(|s|^\nu)$ as $s \to 0$ with $\nu \in (0, 2^* - 2)$, and $h(x,s) \to V_\infty(x)$ as $s \to \infty$ with $\inf V_\infty > \overline{\Lambda}$ uniformly in x.

We point out that Jeanjean [15] considered a related problem of (1.1) by using the dual approach and constraint method without periodicity assumption on f. Clearly, there is no more translational invariance of the equation. As pointed out by the referee, it is very interesting to investigate further problem (1.1) without the translational invariance, and this is work under consideration. The remaining of this paper is organized as follows. In Section 2, we introduce the variational framework setting established in author's recent paper [32] which is more suitable for the case that 0 is a boundary point of the spectrum $\sigma(-\Delta+V)$. The proof of main results will be given in the last Section.

2. Variational setting and preliminaries

In this section, as in [32], we introduce the variational framework associated with problem (1.1). Throughout this paper, we denote by $\|\cdot\|_s$ the usual $L^s(\mathbb{R}^N)$ norm for $s\in[1,\infty)$ and $C_i,\ i\in\mathbb{N}$ for different positive constants. Let $\mathcal{A}=-\Delta+V$, then $\mathcal{A}$ is self-adjoint in $L^2(\mathbb{R}^N)$ with domain $\mathfrak{D}(\mathcal{A})=H^2(\mathbb{R}^N)$. Let $\{\mathcal{E}(\lambda):-\infty<\lambda<+\infty\}$ be the spectral family of $\mathcal{A}$, and $|\mathcal{A}|^{1/2}$ be the square root of $|\mathcal{A}|$. Set $\mathcal{U}=id-\mathcal{E}(0)-\mathcal{E}(0-)$. Then $\mathcal{U}$ commutes with $\mathcal{A}$, $|\mathcal{A}|$ and $|\mathcal{A}|^{1/2}$, and $\mathcal{A}=\mathcal{U}|\mathcal{A}|$ is the polar decomposition of $\mathcal{A}$ (see [12, Theorem 4.3.3]). Let $E_*=\mathfrak{D}(|\mathcal{A}|^{1/2})$, the domain of $|\mathcal{A}|^{1/2}$, then $\mathcal{E}(\lambda)E_*\subset E_*$ for all $\lambda\in\mathbb{R}$. On E_* define an inner product

$$(u,v)_0=\Big(|\mathcal{A}|^{1/2}u,|\mathcal{A}|^{1/2}v\Big)_{L^2}+(u,v)_{L^2},\quad \forall u,v\in E_*,$$

and the norm

$$\|u\|_0=\sqrt{(u,v)_0},\quad \forall u\in E_*,$$

where and in the sequel, $(\cdot,\cdot)_{L^2}$ denotes the usual $L^2(\mathbb{R}^N)$ inner product.

By (A5), we can choose $a_0>0$ such that

$$V(x)+a_0>0,\quad \forall x\in\mathbb{R}^N. \tag{2.1}$$

For $u\in C_0^\infty(\mathbb{R}^N)$, one has

$$\begin{aligned}\|u\|_0^2&=(|\mathcal{A}|u,u)_{L^2}+\|u\|_2^2=((\mathcal{A}+a_0)\mathcal{U}u,u)_{L^2}-a_0(\mathcal{U}u,u)_{L^2}+\|u\|_2^2\\&\le\|\mathcal{U}(\mathcal{A}+a_0)^{1/2}u\|_2\|(\mathcal{A}+a_0)^{1/2}u\|_2+a_0\|\mathcal{U}u\|_2\|u\|_2+\|u\|_2^2\\&\le\|(\mathcal{A}+a_0)^{1/2}u\|_2^2+(a_0+1)\|u\|_2^2\\&\le(1+2a_0+M)\|u\|_{H^1(\mathbb{R}^N)}^2\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\|u\|_{H^1(\mathbb{R}^N)}^2&\le((\mathcal{A}+a_0+1)u,u)_{L^2}\\&=(\mathcal{A}u,u)_{L^2}+(a_0+1)\|u\|_2^2\\&=\Big(\mathcal{U}|\mathcal{A}|^{1/2}u,|\mathcal{A}|^{1/2}u\Big)_{L^2}+(a_0+1)\|u\|_2^2\\&\le\||\mathcal{A}|^{1/2}u\|_2^2+(a_0+1)\|u\|_2^2\le(1+a_0)\|u\|_0^2,\end{aligned} \tag{2.3}$$

where $M=\sup_{x\in\mathbb{R}^N}|V(x)|$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $(E_*,\|\cdot\|_0)$ and $H^1(\mathbb{R}^N)$, thus

$$\frac{1}{1+a_0}\|u\|_{H^1(\mathbb{R}^N)}^2\le\|u\|_0^2\le(1+2a_0+M)\|u\|_{H^1(\mathbb{R}^N)}^2, \tag{2.4}$$

for all $u\in E_*=H^1(\mathbb{R}^N)$.

Denote

$$E_*^-=\mathcal{E}(0)E_*,\quad E^+=[\mathcal{E}(+\infty)-\mathcal{E}(0)]E_*,$$

and

$$(u,v)_*=\Big(|\mathcal{A}|^{1/2}u,|\mathcal{A}|^{1/2}v\Big)_{L^2},\quad \|u\|_*=\sqrt{(u,u)_*},\quad \forall\, u,v\in E_*. \tag{2.5}$$

Lemma 2.1 ([32, Lemma 3.1])). *Suppose that* (A5) *is satisfied. Then* $E_* = E_*^- \oplus E^+$,

$$(u,v)_* = (u,v)_{L^2} = 0, \quad \forall u \in E_*^-,\ v \in E^+, \tag{2.6}$$

and

$$\|u^+\|_*^2 \geq \bar{\Lambda}\|u^+\|_2^2, \quad \|u^-\|_*^2 \leq a_0\|u^-\|_2^2, \quad \forall u = u^- + u^+ \in E_* = E_*^- \oplus E^+, \tag{2.7}$$

where a_0 *is given by* (2.1).

It is easy to see that $\|\cdot\|_*$ and $\|\cdot\|_{H^1(\mathbb{R}^N)}$ are equivalent norms on E^+, and if $u \in E_*$ then $u \in E^+ \Leftrightarrow \mathcal{E}(0)u = 0$. Thus E^+ is a closed subset of $(E_*, \|\cdot\|_0) = H^1(\mathbb{R}^N)$. We introduce a new norm on E_*^- by setting

$$\|u\|_- = \left(\|u\|_*^2 + \|u\|_\varrho^2\right)^{1/2}, \quad \forall u \in E_*^-. \tag{2.8}$$

Let E^- be the completion of E_*^- with respect to $\|\cdot\|_-$. Then E^- is separable and reflexive, $E^- \cap E^+ = \{0\}$ and $(u,v)_* = 0$ for all $u \in E^-$, $v \in E^+$. Let $E = E^- \oplus E^+$ and define norm $\|\cdot\|$ as follows

$$\|u\| = \left(\|u^-\|_-^2 + \|u^+\|_*^2\right)^{1/2}, \quad \forall u = u^- + u^+ \in E = E^- \oplus E^+. \tag{2.9}$$

It is easy to verify that $(E, \|\cdot\|)$ is a Banach space, and

$$\sqrt{\bar{\Lambda}}\|u^+\|_2 \leq \|u^+\|_* = \|u^+\|, \quad \|u^+\|_s \leq \gamma_s\|u^+\|, \quad \forall u \in E,\ s \in [2, 2^*], \tag{2.10}$$

where $\gamma_s \in (0, +\infty)$ is imbedding constant.

Lemma 2.2 ([32, Lemma 3.2]). *Suppose that* (A5) *is satisfied. Then the following statements hold.*

- (i) $E^- \hookrightarrow L^s(\mathbb{R}^N)$ *for* $\varrho \leq s \leq 2^*$;
- (ii) $E^- \hookrightarrow H^1_{\rm loc}(\mathbb{R}^N)$ *and* $E^- \hookrightarrow\hookrightarrow L^s_{\rm loc}(\mathbb{R}^N)$ *for* $2 \leq s < 2^*$;
- (iii) *For* $\varrho \leq s \leq 2^*$, *there exists a constant* $C_s > 0$ *such that*

$$\|u\|_s^s \leq C_s\Big[\|u\|_*^s + \Big(\int_\Omega |u|^\varrho \,\mathrm{d}x\Big)^{s/\varrho} + \Big(\int_{\Omega^c} |u|^2 \,\mathrm{d}x\Big)^{s/2}\Big], \tag{2.11}$$

for all $u \in E^-$, *where* $\Omega \subset \mathbb{R}^N$ *is any measurable set,* $\Omega^c = \mathbb{R}^N \setminus \Omega$.

The following linking theorem is an extension of [17] (see also [3] and [39, Theorem 6.10]), which plays an important role in proving our main results.

Theorem 2.3 ([32, Theorem 2.4]). *Let* X *be real Banach space with* $X = Y \oplus Z$, *where* Y *and* Z *are subspaces of* X, Y *is separable and reflexive, and there exists a constant* $\zeta_0 > 0$ *such that the following inequality holds*

$$\|P_1 u\| + \|P_2 u\| \leq \zeta_0\|u\|, \quad \forall u \in X, \tag{2.12}$$

where $P_1 : X \to Y$, $P_2 : X \to Z$ *are the projections. Let* $\{\mathfrak{f}_k\}_{k\in\mathbb{N}} \subset Y^*$ *be the dense subset with* $\|\mathfrak{f}_k\|_{Y^*} = 1$, *and the* τ*-topology on* X *be generated by the norm*

$$\|u\|_\tau := \max\Big\{\|P_2 u\|, \sum_{k=1}^\infty \frac{1}{2^k}|\langle \mathfrak{f}_k, P_1 u\rangle|\Big\}, \quad \forall u \in X. \tag{2.13}$$

Suppose that the following assumptions are satisfied:

- (A6) $\varphi \in C^1(X, \mathbb{R})$ *is* τ*-upper semi-continuous and* $\varphi' : (\varphi_a, \|\cdot\|_\tau) \to (X^*, \mathcal{T}_{w^*})$ *is continuous for every* $a \in \mathbb{R}$;

(A7) *there exists* $r > \rho > 0$ *and* $e \in Z$ *with* $\|e\| = 1$ *such that*

$$\kappa := \inf \varphi(S_\rho) > 0 \geq \sup \varphi(\partial Q),$$

where

$$S_\rho = \{u \in Z : \|u\| = \rho\}, \quad Q = \{v + se : v \in Y,\, s \geq 0,\, \|v + se\| \leq r\}.$$

Then there exist $c \in [\kappa,\ \sup_Q \varphi]$ *and a sequence* $\{u_n\} \subset X$ *satisfying*

$$\varphi(u_n) \to c, \quad \|\varphi'(u_n)\|_{X^*}(1 + \|u_n\|) \to 0. \tag{2.14}$$

Such a sequence is called a Cerami sequence on the level c, or a $(C)_c$*-sequence.*

Let $X = E$, $Y = E^-$ and $Z = E^+$. Then (2.12) is obviously true by (2.9). Since E^- is separable and reflective subspace of E, then $(E^-)^*$ is also separable. Thus we can choose a dense subset $\{\mathfrak{f}_k\}_{k\in\mathbb{N}} \subset (E^-)^*$ with $\|\mathfrak{f}_k\|_{(E^-)^*} = 1$. Hence, it follows from (2.13) that

$$\|u\|_\tau := \max\big\{\|u^+\|, \sum_{k=1}^{\infty} \frac{1}{2^k} |\langle \mathfrak{f}_k, u^-\rangle|\big\}, \quad \forall u \in E. \tag{2.15}$$

It is clear that

$$\|u^+\| \leq \|u\|_\tau \leq \|u\|, \quad \forall u \in E. \tag{2.16}$$

By Lemma 2.2, it is easy to see that $\Phi \in C^1(E, \mathbb{R})$, moreover

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)\, \mathrm{d}x - \int_{\mathbb{R}^N} f(x,u)v\, \mathrm{d}x, \tag{2.17}$$

for all $u, v \in E$. This shows that critical points of Φ are the solutions of (1.1). Furthermore,

$$\Phi(u) = \frac{1}{2}(\|u^+\|_*^2 - \|u^-\|_*^2) - \int_{\mathbb{R}^N} F(x,u)\, \mathrm{d}x, \tag{2.18}$$

for all $u = u^+ + u^- \in E^- \oplus E^+ = E$, and

$$\langle \Phi'(u), v\rangle = (u^+, v)_* - (u^-, v)_* - \int_{\mathbb{R}^N} f(x,u)v\, \mathrm{d}x, \quad \forall u, v \in E. \tag{2.19}$$

Lemma 2.4 ([32, Lemma 3.3])**.** *Suppose that* (A1)–(A2), (A5) *are satisfied. Then* $\Phi \in C^1(E, \mathbb{R})$ *is* τ*-upper semi-continuous and* $\Phi' : (\Phi_a, \|\cdot\|_\tau) \to (E^*, \mathcal{T}_{w^*})$ *is continuous for every* $a \in \mathbb{R}$*.*

3. Proof of main results

Lemma 3.1. *Suppose that* (A1)–(A2), (A5) *are satisfied. Then there exists a constant* $\rho > 0$ *such that* $\kappa := \inf \Phi(S_\rho^+) > 0$*, where* $S_\rho^+ = \partial B_\rho \cap E^+$*.*

The proof of the above lemma is standard, and we omit it. Observe that, (A1) implies the existence of a constant $\mu > 0$ such that

$$\bar{\Lambda} < \mu < \inf V_\infty. \tag{3.1}$$

Let

$$E_0 := [\mathcal{E}(\mu) - \mathcal{E}(0)]L^2(\mathbb{R}^N).$$

Then $E_0 \subset E^+$ is nonempty and

$$\bar{\Lambda}\|u\|_2^2 \leq \|u\|^2 \leq \mu\|u\|_2^2 \quad \text{for all } u \in E_0 \tag{3.2}$$

Lemma 3.2. *Suppose that* (A1)–(A2), (A5) *are satisfied. Let* $e \in E_0 \subset E^+$ *with* $\|e\| = 1$. *Then there is a* $r_1 > 0$ *such that* $\sup \Phi(\partial Q) \leq 0$, *where*

$$Q = \{w + se : w \in E^-,\ s \geq 0,\ \|w + se\| \leq r_1\}. \tag{3.3}$$

Proof. (A2) yields that $F(x,t) \geq 0$ for any $(x,t) \in \mathbb{R}^{N+1}$, so we have $\Phi(u) \leq 0$ for $u \in E^-$. Next, it is sufficient to show $\Phi(u) \to -\infty$ as $u \in E^- \oplus \mathbb{R}e$, $\|u\| \to \infty$. Arguing indirectly, assume that for some sequence $\{w_n + t_n e\} \subset E^- \oplus \mathbb{R}e$ with $\|w_n + t_n e\| \to \infty$, there is $M > 0$ such that $\Phi(w_n + t_n e) \geq -M$ for all $n \in \mathbb{N}$. Set $v_n = \frac{w_n + t_n e}{\|w_n + t_n e\|} = v_n^- + s_n e$, then $\|v_n\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v = v^- + se$ in E, $s_n \to s$ and $v_n \to v$ a.e. on $\mathbb{R}^N$. By (A2) and (2.18), we have

$$\begin{aligned}
-2M &\leq 2\Phi(w_n + t_n e) \\
&= t_n^2 \|e\|_*^2 - \|w_n\|_*^2 - 2\int_{\mathbb{R}^N} F(x, w_n + t_n e)\,\mathrm{d}x \\
&\leq t_n^2 - \|w_n\|_*^2 - \frac{2c_1}{\varrho}\Big(\int_{|w_n + t_n e| < 1} |w_n + t_n e|^{\varrho}\,\mathrm{d}x \\
&\quad + \int_{|w_n + t_n e| \geq 1} |w_n + t_n e|^2\,\mathrm{d}x\Big).
\end{aligned} \tag{3.4}$$

From (2.10), (2.11) and (3.4), we have

$$\begin{aligned}
\|w_n\|_{\varrho}^{\varrho} &\leq C_1\Big[\|w_n\|_*^{\varrho} + \int_{|w_n + t_n e| < 1} |w_n|^{\varrho}\,\mathrm{d}x + \Big(\int_{|w_n + t_n e| \geq 1} |w_n|^2\,\mathrm{d}x\Big)^{\varrho/2}\Big] \\
&\leq C_1\|w_n\|_*^{\varrho} + C_2\Big(|t_n|^{\varrho}\int_{|w_n + t_n e| < 1} |e|^{\varrho}\,\mathrm{d}x + \int_{|w_n + t_n e| < 1} |w_n + t_n e|^{\varrho}\,\mathrm{d}x\Big) \\
&\quad + C_2\Big(t_n^2\int_{|w_n + t_n e| \geq 1} |e|^2\,\mathrm{d}x + \int_{|w_n + t_n e| \geq 1} |w_n + t_n e|^2\,\mathrm{d}x\Big)^{\varrho/2} \\
&\leq C_1\|w_n\|_*^{\varrho} + C_3\left(|t_n|^{\varrho} + t_n^2 + 2M\right) + C_4\left(t_n^2 + 2M\right)^{\varrho/2} \\
&\leq C_5\left(1 + |t_n|^{\varrho} + t_n^2\right),
\end{aligned}$$

which, together with (2.8), (2.9) and (3.4), implies that

$$\|w_n + t_n e\|^2 = t_n^2 + \|w_n\|_*^2 + \|w_n\|_{\varrho}^2 \leq 2t_n^2 + 2M + C_6\left(1 + |t_n|^{\varrho} + t_n^2\right)^{2/\varrho}. \tag{3.5}$$

Since $\|w_n + t_n e\|^2 \to \infty$, it follows that $|t_n| \to \infty$ and

$$s_n^2 = \frac{t_n^2}{\|w_n + t_n e\|^2} \geq \frac{t_n^2}{2t_n^2 + 2M + C_6\left(1 + |t_n|^{\varrho} + t_n^2\right)^{2/\varrho}} \geq \frac{1}{2(1 + C_7)}.$$

This shows that $s > 0$, and so $v \neq 0$. By (3.1), (3.2) and the fact $e \in E_0$, one has

$$\begin{aligned}
&s^2 - \|v^-\|_*^2 - \int_{\mathbb{R}^N} V_\infty(x) v^2\,\mathrm{d}x \\
&\leq s^2\|e\|_*^2 - \|v^-\|_*^2 - \inf V_\infty \|v\|_2^2 \\
&\leq -\left[\left(\inf V_\infty - \mu\right) s^2 \|e\|_2^2 + \|v^-\|_*^2 + \inf V_\infty \|v^-\|_2^2\right] < 0.
\end{aligned}$$

Hence, there is a bounded domain $\Omega \subset \mathbb{R}^N$ such that

$$s^2 - \|v^-\|_*^2 - \int_{\Omega} V_\infty(x) v^2\,\mathrm{d}x < 0. \tag{3.6}$$

Let

$$f_1(x,t) := f(x,t) - V_\infty(x)t, \quad \text{and} \quad F_1(x,t) = \int_0^t f_1(x,s)\,\mathrm{d}s. \tag{3.7}$$

By (A1) and (A2), there exists a positive constant C such that

$$F_1(x,t) \le Ct^2, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, \quad \text{and} \quad \lim_{|t|\to\infty} \frac{F_1(x,t)}{t^2} \to 0 \quad \text{uniformly in } x. \tag{3.8}$$

It follows from Lebesgue's dominated convergence theorem and the fact $\|v_n - v\|_{L^2(\Omega)} \to 0$ that

$$\lim_{n\to\infty} \int_\Omega \frac{F_1(x, w_n + t_n e)}{\|w_n + t_n e\|^2}\,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega \frac{F_1(x, w_n + t_n e)}{|w_n + t_n e|^2} |v_n|^2\,\mathrm{d}x = 0. \tag{3.9}$$

By (3.4), (3.7) and (3.9), we have

$$\begin{aligned}
0 &\le \lim_{n\to\infty} \Big(s_n^2 \|e\|_*^2 - \|v_n^-\|_*^2 - 2\int_\Omega \frac{F(x, w_n + t_n e)}{\|w_n + t_n e\|^2}\,\mathrm{d}x \Big) \\
&= \lim_{n\to\infty} \Big[s_n^2 - \|v_n^-\|_*^2 - 2\int_\Omega \Big(\frac{F_1(x, w_n + t_n e)}{\|w_n + t_n e\|^2} + \frac{1}{2} V_\infty(x) v_n^2 \Big)\,\mathrm{d}x \Big] \\
&\le s^2 - \|v^-\|_*^2 - \int_\Omega V_\infty(x) v^2\,\mathrm{d}x,
\end{aligned}$$

a contradiction to (3.6). □

Lemma 3.3. *Suppose that* (A1)–(A2), (A5) *are satisfied. Then there exist a constant* $c_* \in \big[\kappa, \sup_Q \Phi\big]$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c_*, \quad \|\Phi'(u_n)\|_{E^*}(1 + \|u_n\|) \to 0. \tag{3.10}$$

where Q *is defined by* (3.3).

The above lemma is a direct corollary of Theorem 2.3 and Lemmas 2.4, 3.1 and 3.2.

Lemma 3.4. *Suppose that* (A1)–(A2), (A5) *are satisfied. Then*

$$\|u\|_*^2 \le \langle \Phi'(u), u^+ - u^- \rangle + \int_{u\neq 0} \frac{f(x,u)}{u} |u^+|^2 \mathrm{d}x, \quad \forall u \in E. \tag{3.11}$$

Proof. By (A1), (A2) and (2.19), for any $u \in E$, one has

$$\begin{aligned}
\langle \Phi'(u), u^+ - u^- \rangle &= \|u\|_*^2 - \int_{\mathbb{R}^N} f(x,u)(u^+ - u^-)\mathrm{d}x \\
&= \|u\|_*^2 - \int_{u\neq 0} \frac{f(x,u)}{u} \big[(u^+)^2 - (u^-)^2\big]\,\mathrm{d}x \\
&\ge \|u\|_*^2 - \int_{u\neq 0} \frac{f(x,u)}{u} |u^+|^2 \mathrm{d}x.
\end{aligned}$$

This shows that (3.11) holds. □

Lemma 3.5. *Suppose that* (A1)–(A), (A5) *are satisfied. Then any sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c \ge 0, \quad \|\Phi'(u_n)\|_{E^*}(1 + \|u_n\|) \to 0 \tag{3.12}$$

is boundeded in E.

Proof. First we prove that $\{\|u_n\|_*\}$ is bounded. To this end, arguing by contradiction, suppose that $\|u_n\|_* \to \infty$. Let $v_n = u_n/\|u_n\|_*$, then $\|v_n\|_* = 1$. If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |v_n^+|^2 \,\mathrm{d}x = 0,$$

by Lions's concentration compactness principle ([18] or [39, Lemma 1.21]), then $v_n^+ \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. Denote

$$\Omega_n := \big\{x \in \mathbb{R}^N : \frac{f(x,u_n)}{u_n} \le \bar\Lambda - \delta_0\big\}.$$

By (2.10), one gets

$$\begin{aligned}\int_{\Omega_n} \frac{f(x,u_n)}{u_n}|v_n^+|^2\mathrm{d}x &\le \big(\bar\Lambda - \delta_0\big)\int_{\Omega_n}|v_n^+|^2\mathrm{d}x\\ &\le \big(\bar\Lambda - \delta_0\big)\,\|v_n^+\|_2^2\\ &\le \big(1 - \frac{\delta_0}{\bar\Lambda}\big)\|v_n^+\|_*^2 \le 1 - \frac{\delta_0}{\bar\Lambda}.\end{aligned} \tag{3.13}$$

From (A3) and (3.12), one has

$$c + o(1) = \Phi(u_n) - \frac{1}{2}\langle\Phi(u_n), u_n\rangle = \int_{\mathbb{R}^N}\mathcal{F}(x,u_n)\mathrm{d}x \ge \int_{\mathbb{R}^N\setminus\Omega_n}\delta_0\mathrm{d}x. \tag{3.14}$$

It follows from (A1), (A2), (3.14) and Hölder inequality that

$$\begin{aligned}\int_{\mathbb{R}^N\setminus\Omega_n}\frac{f(x,u_n)}{u_n}|v_n^+|^2\mathrm{d}x &\le C_1\int_{\mathbb{R}^N\setminus\Omega_n}|v_n^+|^2\mathrm{d}x\\ &\le C_1\Big(\int_{\mathbb{R}^N\setminus\Omega_n}1\mathrm{d}x\Big)^{(\varrho-2)/\varrho}\Big(\int_{\mathbb{R}^N\setminus\Omega_n}|v_n^+|^\varrho\mathrm{d}x\Big)^{2/\varrho}\\ &\le C_2\|v_n^+\|_\varrho^2 = o(1).\end{aligned} \tag{3.15}$$

By (3.11), (3.12), (3.13) and (3.15), we have

$$\begin{aligned}1 &\le \frac{1}{\|u_n\|_*^2}\langle\Phi'(u_n), u_n^+ - u_n^-\rangle + \int_{u_n\neq 0}\frac{f(x,u_n)}{u_n}|v_n^+|^2\mathrm{d}x\\ &= \int_{u_n\neq 0}\frac{f(x,u_n)}{u_n}|v_n^+|^2\mathrm{d}x + o(1) \le 1 - \frac{\delta_0}{\bar\Lambda} + o(1),\end{aligned} \tag{3.16}$$

which is a contradiction. Thus $\delta > 0$.

Going to a subsequence, if necessary, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that $\int_{B(k_n,1+\sqrt{N})}|v_n^+|^2\,\mathrm{d}x > \frac{\delta}{2}$. Let $w_n(x) = v_n(x+k_n)$. Then

$$\int_{B(0,1+\sqrt{N})}|w_n^+|^2\,\mathrm{d}x > \frac{\delta}{2}. \tag{3.17}$$

Since $V(x)$ is periodic, we have $\|w_n^+\| = \|v_n^+\| \le 1$. Passing to a subsequence, we have $w_n^+ \rightharpoonup w^{(1)}$ in E, $w_n^+ \to w^{(1)}$ in $L^2_{\rm loc}(\mathbb{R}^N)$ and $w_n^+ \to w^{(1)}$ a.e. on $\mathbb{R}^N$.

Obviously, (3.17) implies that $w^{(1)} \neq 0$. By (A2), (2.19) and (3.12), one has

$$\begin{aligned} &\|u_n^+\|_*^2 - \|u_n^-\|_*^2 + o(1) \\ &= \int_{\mathbb{R}^N} f(x, u_n)u_n \, \mathrm{d}x = \int_{\mathbb{R}^N} f(x, \|u_n\|_* w_n)\|u_n\|_* w_n \, \mathrm{d}x \\ &\geq c_1 \|u_n\|_*^\varrho \int_{\|u_n\|_*|w_n|<1} |w_n|^\varrho \, \mathrm{d}x + c_1\|u_n\|_*^2 \int_{\|u_n\|_*|w_n|\geq 1} |w_n|^2 \, \mathrm{d}x. \end{aligned} \tag{3.18}$$

From (3.18), we have

$$\int_{\|u_n\|_*|w_n|<1} |w_n|^\varrho \, \mathrm{d}x \leq \frac{\|u_n^+\|_*^2}{c_1\|u_n\|_*^\varrho} + o(1) = o(1), \tag{3.19}$$

$$\int_{\|u_n\|_*|w_n|\geq 1} |w_n|^2 \, \mathrm{d}x \leq \frac{\|u_n^+\|_*^2}{c_1\|u_n\|_*^2} + o(1) \leq C_3, \tag{3.20}$$

By (2.10), (2.11), (3.18), (3.19) and (3.20), we have

$$\begin{aligned} &\|w_n^-\|_*^2 + \|w_n^-\|_\varrho^\varrho \\ &\leq \|w_n^-\|_*^2 + C_4\Big[\|w_n^-\|_*^\varrho + \int_{\|u_n\|_*|w_n|<1} |w_n^-|^\varrho \, \mathrm{d}x + \Big(\int_{\|u_n\|_*|w_n|\geq 1} |w_n^-|^2 \, \mathrm{d}x\Big)^{\varrho/2}\Big] \\ &\leq 1 + C_4 + C_5\Big(\int_{\|u_n\|_*|w_n|<1} |w_n^+|^\varrho \, \mathrm{d}x + \int_{\|u_n\|_*|w_n|<1} |w_n|^\varrho \, \mathrm{d}x\Big) \\ &\quad + C_6\Big(\int_{\|u_n\|_*|w_n|\geq 1} |w_n^+|^2 \, \mathrm{d}x + \int_{\|u_n\|_*|w_n|\geq 1} |w_n|^2 \, \mathrm{d}x\Big)^{\varrho/2} \leq C_7. \end{aligned} \tag{3.21}$$

This shows that $\{w_n^-\}$ is bounded in E and so $w_n^- \rightharpoonup w^{(2)}$ in E and $w_n^- \to w^{(2)}$ a.e. on $\mathbb{R}^N$. Let $w_0 = w^{(1)} + w^{(2)}$. It is clear that $w_0^+ = w^{(1)} \neq 0$ and $w_n \to w_0$ a.e. on $\mathbb{R}^N$.

Now we define $\tilde{u}_n(x) = u_n(x + k_n)$, then $\tilde{u}_n/\|u_n\|_* = w_n \to w_0$ a.e. on $\mathbb{R}^N$ and $w_0 \neq 0$. For a.e. $x \in \Omega := \{y \in \mathbb{R}^N : \ w(y) \neq 0\}$, we have $\lim_{n\to\infty} |\tilde{u}_n(x)| = \infty$. For any $\psi \in C_0^\infty(\mathbb{R}^N)$, set $\psi_n(x) = \psi(x - k_n)$. By (A5), (A2), (2.19) and (3.7), then we have

$$\begin{aligned} \langle \Phi'(u_n), \psi_n\rangle &= (u_n^+ - u_n^-, \psi_n)_* - (V_\infty u_n, \psi_n)_{L^2} - \int_{\mathbb{R}^N} f_1(x, u_n)\psi_n \, \mathrm{d}x \\ &= \|u_n\|_*\Big[(v_n^+ - v_n^-, \psi_n)_* - (V_\infty v_n, \psi_n)_{L^2} - \int_{\mathbb{R}^N} \frac{f_1(x, u_n)}{|u_n|}|v_n|\psi_n \, \mathrm{d}x\Big] \\ &= \|u_n\|_*\Big[(w_n^+ - w_n^-, \psi)_* - (V_\infty w_n, \psi)_{L^2} - \int_{\mathbb{R}^N} \frac{f_1(x, \tilde{u}_n)}{|\tilde{u}_n|}|w_n|\psi \, \mathrm{d}x\Big], \end{aligned}$$

which, together with (3.12), yields that

$$(w_n^+ - w_n^-, \psi)_* - (V_\infty w_n, \psi)_{L^2} - \int_{\mathbb{R}^N} \frac{f_1(x, \tilde{u}_n)}{|\tilde{u}_n|}|w_n|\psi \, \mathrm{d}x = o(1). \tag{3.22}$$

Note that $\lim_{|t|\to\infty} f_1(x, t)/|t| = 0$ uniformly in x, then

$$\begin{aligned} &\Big|\int_{\mathbb{R}^N} \frac{f_1(x, \tilde{u}_n)}{|\tilde{u}_n|}|w_n|\psi \mathrm{d}x\Big| \\ &\leq \int_{\mathbb{R}^N} \Big|\frac{f_1(x, \tilde{u}_n)}{\tilde{u}_n}\Big||w_n - w_0||\psi| \mathrm{d}x + \int_{\mathbb{R}^N} \Big|\frac{f_1(x, \tilde{u}_n)}{\tilde{u}_n}\Big||w_0||\psi| \mathrm{d}x \end{aligned}$$

$$\leq C_8 \int_{\text{supp } \psi} |w_n - w_0||\psi| \mathrm{d}x + \int_\Omega \Big| \frac{f_1(x, \tilde{u}_n)}{\tilde{u}_n} \Big| |w_0||\psi| \mathrm{d}x = o(1).$$

Hence,

$$(w_0^+ - w_0^-, \psi)_* - (V_\infty w_0, \psi)_{L^2} = 0.$$

Thus w_0 is an eigenfunction of the operator $\mathcal{B} := -\Delta + (V - V_\infty)$ contradicting with the fact that $\mathcal{B}$ has only continuous spectrum. This contradiction shows that $\{\|u_n\|_*\}$ is bounded. By (A2), (2.19) and (3.12), we have

$$\begin{aligned} \|u_n^+\|_*^2 - \|u_n^-\|_*^2 + o(1) &= \int_{\mathbb{R}^N} f(x, u_n) u_n \, \mathrm{d}x \\ &\geq c_1 \Big(\int_{|u_n|<1} |u_n|^\varrho \, \mathrm{d}x + \int_{|u_n|\geq 1} |u_n|^2 \, \mathrm{d}x \Big). \end{aligned} \tag{3.23}$$

From (2.10), (2.11) and (3.23), we have

$$\begin{aligned} \|u_n^-\|_\varrho^\varrho &\leq C_9 \Big[\|u_n^-\|_*^\varrho + \int_{|u_n|<1} |u_n^-|^\varrho \mathrm{d}x + \Big(\int_{|u_n|\geq 1} |u_n^-|^2 \mathrm{d}x \Big)^{\varrho/2} \Big] \\ &\leq C_{10} \Big[\|u_n^-\|_*^\varrho + \int_{|u_n|<1} |u_n^+|^\varrho \mathrm{d}x + \int_{|u_n|<1} |u_n|^\varrho \mathrm{d}x \\ &\quad + \Big(\int_{|u_n|\geq 1} |u_n^+|^2 \mathrm{d}x + \int_{|u_n|\geq 1} |u_n|^2 \mathrm{d}x \Big)^{\varrho/2} \Big] \leq C_{11}. \end{aligned} \tag{3.24}$$

This shows that $\{\|u_n^-\|_\varrho\}_n$ is also bounded and so $\{u_n\}$ is bounded in E. $\square$

Lemma 3.6 ([2, Corollary 2.3])**.** *Suppose that* (A5) *is satisfied. If $u \subset E$ is a weak solution of the Schrödinger equations*

$$-\Delta u + V(x) u = f(x, u), \quad x \in \mathbb{R}^N, \tag{3.25}$$

i.e.

$$\int_{\mathbb{R}^N} (\nabla u \nabla \psi + V(x) u \psi) \, \mathrm{d}x = \int_{\mathbb{R}^N} f(x, u) \psi \, \mathrm{d}x, \quad \forall \psi \in C_0^\infty(\mathbb{R}^N), \tag{3.26}$$

then $u_n \to 0$ as $|x| \to \infty$.

Lemma 3.7. *Suppose that* (A5), (A1)–(A3), (A5) *are satisfied. Then $\mathcal{M} \neq \emptyset$, i.e., problem* (1.1) *has a nontrivial solution.*

Proof. Lemma 3.3 implies the existence of a sequence $\{u_n\} \subset E$ satisfying (3.10). By Lemma 3.5, $\{u_n\}$ is bounded in E. Thus $\|u_n\|_\varrho^\varrho$ is also bounded. If

$$\delta := \limsup_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |u_n^+|^2 \, \mathrm{d}x = 0,$$

then by Lions's concentration compactness principle, $u_n^+ \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. From (A2), (2.18), (2.19) and (3.10), one has

$$\begin{aligned} 2c_* + o(1) &= \|u_n^+\|_*^2 - \|u_n^-\|_*^2 - 2 \int_{\mathbb{R}^N} F(x, u_n) \, \mathrm{d}x \\ &\leq \|u_n^+\|_*^2 = \int_{\mathbb{R}^N} f(x, u_n) u_n^+ \, \mathrm{d}x + \langle \Phi'(u_n), u_n^+ \rangle \\ &\leq c_2 \int_{\mathbb{R}^N} |u_n|^{\varrho-1} |u_n^+| \, \mathrm{d}x + o(1) \\ &\leq c_2 \|u_n\|_\varrho^{\varrho-1} \|u_n^+\|_\varrho + o(1) = o(1) \end{aligned}$$

which is a contradiction. Thus $\delta > 0$.

Going to a subsequence, if necessary, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that

$$\int_{B(k_n,1+\sqrt{N})} |u_n^+|^2 \,\mathrm{d}x > \frac{\delta}{2}.$$

Let us define $v_n(x) = u_n(x + k_n)$ so that

$$\int_{B(0,1+\sqrt{N})} |v_n^+|^2 \,\mathrm{d}x > \frac{\delta}{2}. \tag{3.27}$$

Since $V(x)$ and $f(x,t)$ are periodic in x, we have $\|v_n\| = \|u_n\|$ and

$$\Phi(v_n) \to c_* \in [\kappa, \sup_Q \Phi], \quad \|\Phi'(v_n)\|_{E^*}(1 + \|v_n\|) \to 0. \tag{3.28}$$

Passing to a subsequence, we have $v_n \rightharpoonup v_0$ in E, $v_n \to v_0$ in $L^s_{\rm loc}(\mathbb{R}^N)$ for $2 \le s < 2^*$ and $v_n \to v_0$ a.e. on $\mathbb{R}^N$. Then (3.27) implies that $v_0 \neq 0$. For any $\psi \in C_0^\infty(\mathbb{R}^N)$, there exists a $R_\psi > 0$ such that supp$\psi \subset B(0, R_\psi)$. By (A2) and [39, Theorem A.2], we have

$$\lim_{n\to\infty} \int_{B(0,R_\psi)} |f(x,u_n) - f(x,u)||\psi| \,\mathrm{d}x = 0. \tag{3.29}$$

Note that

$$\big(v_n^+ - v_0^+, \psi\big)_* - \big(v_n^- - v_0^-, \psi\big)_* \to 0. \tag{3.30}$$

Hence, it follows from (2.19), (3.28), (3.29) and (3.30) that

$$\begin{aligned} |\langle \Phi'(v_0), \psi\rangle| &= \Big|\langle \Phi'(v_n), \psi\rangle - \big[\big(v_n^+ - v_0^+, \psi\big)_* - \big(v_n^- - v_0^-, \psi\big)_*\big] \\ &\quad + \int_{\mathbb{R}^N} [f(x,u_n) - f(x,u)]\psi \,\mathrm{d}x\Big| \\ &\le o(1) + \int_{\mathbb{R}^N} |f(x,u_n) - f(x,u)||\psi| \,\mathrm{d}x = o(1). \end{aligned}$$

This shows that $\langle \Phi'(v_0), \psi\rangle = 0$ for all $\psi \in C_0^\infty(\mathbb{R}^N)$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in E, we can conclude that $\Phi'(v_0) = 0$. This shows that $v_0 \in \mathcal{M}$ and so $\mathcal{M} \neq \emptyset$. Lemma 3.6 implies that v_0 is a nontrivial solution of (1.1). $\square$

Proof of Theorem 1.1. Lemma 3.7 shows that $\mathcal{M}$ is not an empty set. Let $c_0 := \inf_{\mathcal{M}} \Phi$. Since $\mathcal{F}(x,t) \ge 0$ for all $(x,t) \in \mathbb{R}^{N+1}$, one has $\Phi(u) \ge 0$ for all $u \in \mathcal{M}$. Thus $c_0 \ge 0$. Let $\{u_n\} \subset \mathcal{M}$ such that $\Phi(u_n) \to c_0$. Then $\langle \Phi'(u_n), v\rangle = 0$ for any $v \in E$. In view of the proof of Lemma 3.5, we can show that $\{u_n\}$ is bounded in E. By (A2) and (2.19),

$$0 = \langle \Phi'(u_n), u_n^+\rangle = \|u_n^+\|_*^2 - \int_{\mathbb{R}^N} f(x,u_n)u_n^+ \,\mathrm{d}x, \tag{3.31}$$

and

$$\begin{aligned} \|u_n^+\|_*^2 - \|u_n^-\|_*^2 &= \int_{\mathbb{R}^N} f(x,u_n)u_n \,\mathrm{d}x \\ &\ge c_1\Big(\int_{|u_n|<1} |u_n|^\varrho \,\mathrm{d}x + \int_{|u_n|\ge 1} |u_n|^2 \,\mathrm{d}x\Big). \end{aligned} \tag{3.32}$$

From (2.10), (2.11) and (3.32), we have

$$
\begin{aligned}
\|u_n\|_\varrho^\varrho &\le C_1(\|u_n^+\|_\varrho^\varrho + \|u_n^-\|_\varrho^\varrho) \\
&\le C_2\Big[\|u_n^+\|_\varrho^\varrho + \|u_n^-\|_*^\varrho + \int_{|u_n|<1} |u_n^-|^\varrho \mathrm{d}x + \Big(\int_{|u_n|\ge 1} |u_n^-|^2 \mathrm{d}x\Big)^{\varrho/2}\Big] \\
&\le C_3\Big[\|u_n^+\|_*^\varrho + \int_{|u_n|<1} |u_n^+|^\varrho \mathrm{d}x + \int_{|u_n|<1} |u_n|^\varrho \mathrm{d}x \\
&\quad + \Big(\int_{|u_n|\ge 1} |u_n^+|^2 \mathrm{d}x + \int_{|u_n|\ge 1} |u_n|^2 \mathrm{d}x\Big)^{\varrho/2}\Big] \\
&\le C_4\left(\|u_n^+\|_*^\varrho + \|u_n^+\|_*^2\right).
\end{aligned}
\tag{3.33}
$$

By (A2), (2.10), (3.31) and (3.33), one has

$$
\begin{aligned}
\|u_n^+\|_*^2 &= \int_{\mathbb{R}^N} f(x,u_n)u_n^+ \,\mathrm{d}x \le c_2 \int_{\mathbb{R}^N} |u_n|^{\varrho-1}|u_n^+| \,\mathrm{d}x \\
&\le C_5\left(\|u_n^+\|_*^\varrho + \|u_n^+\|_*^2\right)^{1-1/\varrho} \|u_n^+\|_*,
\end{aligned}
$$

which implies that

$$
C_5^{-\varrho/(\varrho-1)} \le \|u_n^+\|_*^{\varrho(\varrho-2)/(\varrho-1)} + \|u_n^+\|_*^{(\varrho-2)/(\varrho-1)}.
$$

This shows that $\|u_n^+\|_* \ge \alpha_0$ for some $\alpha_0 > 0$. If

$$
\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |u_n^+|^2 \,\mathrm{d}x = 0,
$$

then by Lions's concentration compactness principle, $u_n^+ \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. From (A2) and (3.31), one has

$$
\|u_n^+\|_*^2 = \int_{\mathbb{R}^N} f(x,u_n)u_n^+ \,\mathrm{d}x \le c_2 \int_{\mathbb{R}^N} |u_n|^{\varrho-1}|u_n^+| \,\mathrm{d}x \le c_2\|u_n\|_\varrho^{\varrho-1}\|u_n^+\|_\varrho = o(1),
$$

a contradiction. Thus $\delta > 0$.

By a similar argument as in the proof of lemma 3.7, we can show that there exist a sequence $\{v_n\} \subset E$ and $v_0 \in E \setminus \{0\}$ such that $\|v_n\| = \|u_n\|$, $v_n \to v_0$ a.e. on $\mathbb{R}^N$ and

$$
\Phi(v_0) \to c_0, \quad \Phi'(v_0) = 0. \tag{3.34}
$$

This shows that $v_0 \in \mathcal{M}$, and so $\Phi(v_0) \ge c_0$. On the other hand, by (A3), (2.18), (2.19), (3.34) and Fatou's Lemma, we have

$$
\begin{aligned}
c_0 &= \lim_{n\to\infty} \big[\Phi(v_n) - \frac{1}{2}\langle\Phi'(v_n), v_n\rangle\big] = \lim_{n\to\infty} \int_{\mathbb{R}^N} \big[\frac{1}{2}f(x,v_n) - F(x,v_n)\big] \,\mathrm{d}x \\
&\ge \int_{\mathbb{R}^N} \lim_{n\to\infty} \big[\frac{1}{2}f(x,v_n) - F(x,v_n)\big] \,\mathrm{d}x = \int_{\mathbb{R}^N} \big[\frac{1}{2}f(x,v_0) - F(x,v_0)\big] \,\mathrm{d}x \\
&= \Phi(v_0) - \frac{1}{2}\langle\Phi'(v_0), v_0\rangle = \Phi(v_0).
\end{aligned}
$$

This shows that $\Phi(v_0) \le c_0$ and so $\Phi(v_0) = \inf_{\mathcal{M}} \Phi$, which together with lemma 3.6, implies that v_0 is a ground state solution of (1.1). $\square$

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