Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 208, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

QUENCHING OF A SEMILINEAR DIFFUSION EQUATION WITH CONVECTION AND REACTION

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Abstract. This article concerns the quenching phenomenon of the solution to the Dirichlet problem of a semilinear diffusion equation with convection and reaction. It is shown that there exists a critical length for the spatial interval in the sense that the solution exists globally in time if the length of the spatial interval is less than this number while the solution quenches if the length is greater than this number. For the solution quenching at a finite time, we study the location of the quenching points and the blowing up of the derivative of the solution with respect to the time.

1. Introduction

In this article, we consider the problem

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+b(x)\frac{\partial u}{\partial x}=f(u),\quad (x,t)\in(0,a)\times(0,T), \tag{1.1}$$

$$u(0,t)=0=u(a,t),\quad t\in(0,T), \tag{1.2}$$

$$u(x,0)=0,\quad x\in(0,a), \tag{1.3}$$

where $a>0$, $b\in C^1([0,+\infty))\cap L^\infty([0,+\infty))$ and $f\in C^1([0,c))$ with $c>0$ satisfies

$$f(0)>0,\quad f'(s)>0\quad \text{for } 0<s<c,\quad \lim_{s\to c^-}f(s)=+\infty. \tag{1.4}$$

By the properties of f, the solution u to the problem (1.1)–(1.3) may quench, i.e., there exists a time $0<T_*\le+\infty$ such that

$$\sup_{(0,a)}u(\cdot,t)<c\quad \text{for each } 0<t<T_*\quad\text{and}\quad \lim_{t\to T_*^-}\sup_{(0,a)}u(\cdot,t)=c.$$

It is called that u quenches at a finite time if $T_*<+\infty$, while u quenches at the infinite time if $T_*=+\infty$.

Quenching phenomena were introduced by Kawarada [10] in 1975 for the problem (1.1)–(1.3) in the case that $b\equiv0$ and $f(s)=(1-s)^{-1}$ $(0\le s<1)$, where Kawarada proved the existence of the critical length (which is $2\sqrt{2}$). That is to say, the solution exists globally in time if a is less than the critical length, while it quenches if a is greater than the critical length. For the quenching case, Kawarada also showed that $a/2$ is the quenching point and the derivative of the solution with respect to

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2010 *Mathematics Subject Classification.* 35K20, 35B40.
Key words and phrases. Quenching; critical length.

Submitted April 16, 2015. Published August 10, 2015.

the time blows up at the quenching time. However, it was unknown what happens when a is equal to the critical length in [10]. For the special case that $f(s) = (c-s)^{-\beta}$ $(0 \le s < c, \beta > 0)$, Levine ([12]) in 1989 proved that the solution can not quench in infinite time by finding the explicit form of the minimum steady-state soluton. Since [10], there are many interesting results on quenching phenomena for semilinear uniformly parabolic equations (see, e.g., [1, 2, 7, 11, 13, 14]), singular or degenerate semilinear parabolic equations (see, e.g., [3, 4, 5, 6, 9]) and quasilinear diffusion equations ([8, 15, 16, 17]).

In this article, we study the quenching phenomenon of the solution to (1.1)–(1.3). Since there is a convection term in (1.1), it can describe more diffusion phenomena. By constructing suitable super and sub solutions, we prove the existence of the critical length. For the solution quenching at a finite time, we also study the location of the quenching points and the blowing up of the derivative of the solution with respect to the time at the quenching time by energy estimates and many kinds of super and sub solutions. Due to the existence of the convection term in (1.1), we have to overcome some technical difficulties when doing estimates and constructing super and sub solutions.

This paper is arranged as follows. The existence of the critical length is proved in §2. Subsequently, in §3 we study the quenching properties for the quenching solution, including the location of the quenching points and the blowing up of the derivative of the solution with respect to the time at the quenching time.

2. Critical length

Thanks to the classical theory on parabolic equations, problem (1.1)–(1.3) is well-posed locally in time. Denote

$$\begin{aligned} T_* = \sup \Big\{ & T > 0 : \text{problem (1.1)–(1.3) admits a solution} \\ & u \in C^{2,1}((0,a)\times(0,T)) \cap C([0,a]\times[0,T]) \text{ and } \sup_{(0,a)\times(0,T)} u < c \Big\}. \end{aligned}$$

T_* is called the life span of the solution to (1.1)–(1.3).

Proposition 2.1. *Problem* (1.1)–(1.3) *admits uniquely a solution u in $(0,T_*)$. Furthermore, $u \in C^{2,1}((0,a)\times(0,T_*)) \cap C([0,a]\times[0,T_*))$ and satisfies $u > 0$ and $\frac{\partial u}{\partial t} > 0$ in $(0,a)\times(0,T_*)$.*

Proof. Clearly, the existence and uniqueness follow from the local well-posedness and a standard extension process. Set

$$v(x,t) = \frac{\partial u}{\partial t}(x,t), \quad (x,t) \in [0,a]\times[0,T_*).$$

Then v solves

$$\begin{gathered} \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + b(x)\frac{\partial v}{\partial x} = f'(u(x,t))v, \quad (x,t) \in (0,a)\times(0,T_*), \\ v(0,t) = 0 = v(a,t), \quad t \in (0,T_*), \\ v(x,0) = f(0), \quad x \in (0,a). \end{gathered}$$

The strong maximal principles for u and v show that $u > 0$ and $v > 0$ in $(0,a)\times(0,T_*)$. □

If $T_* = +\infty$, then u exists globally in time. If $T_* < +\infty$, then u must quench at a finite time. Let us study the relation between T_* and a below. For convenience, we denote u_a to be the solution to (1.1)–(1.3) and $T_*(a)$ to be its life span.

Lemma 2.2. *If $a > 0$ is sufficiently small, then $T_*(a) = +\infty$, and*

$$\sup_{(0,a)\times(0,+\infty)} u_a < c.$$

Proof. Fix $0 < c_0 < c$ and

$$0 < a \le \min\Big\{\Big(\frac{4c_0}{f(c_0)}\Big)^{1/2}, \frac{1}{\|b\|_{L^\infty([0,+\infty))}+1}\Big\}.$$

Set

$$\bar u_a(x,t) = f(c_0)x(a-x), \quad (x,t)\in[0,a]\times[0,+\infty).$$

Then, $\bar u_a$ satisfies

$$0 \le \bar u_a(x,t) \le \frac14 f(c_0)a^2 \le c_0, \quad (x,t)\in[0,a]\times[0,+\infty),$$

$$\frac{\partial \bar u_a}{\partial t} - \frac{\partial^2 \bar u_a}{\partial x^2} + b(x)\frac{\partial \bar u_a}{\partial x} = 2f(c_0) + f(c_0)b(x)(a-2x) \ge f(c_0) \ge f(\bar u_a),$$
$$(x,t)\in(0,a)\times(0,+\infty).$$

The comparison principle shows that $u_a \le \bar u_a \le c_0$ in $(0,a)\times(0,+\infty)$. □

Lemma 2.3. *If $a > 0$ is sufficiently large, then $T_*(a) < +\infty$.*

Proof. Set

$$\underline{u}_a(x,t) = \frac{t}{4T} f(0)x(a-x), \quad (x,t)\in[0,a]\times[0,T]$$

with $T = \max\big\{\frac14 a^2, a(\|b\|_{L^\infty([0,+\infty))}+1)\big\}$. Then, $\underline{u}_a$ satisfies

$$\frac{\partial \underline{u}_a}{\partial t} - \frac{\partial^2 \underline{u}_a}{\partial x^2} + b(x)\frac{\partial \underline{u}_a}{\partial x} = \frac{1}{4T}f(0)x(a-x) + \frac{t}{2T}f(0) + \frac{t}{4T}f(0)b(x)(a-2x)$$
$$\le f(0) \le f(\underline{u}_a), \quad (x,t)\in(0,a)\times(0,T).$$

The comparison principle shows $u_a \ge \underline{u}_a$ in $(0,a)\times(0,T)$. Particularly, $u_a(a/2,T) \ge \frac{1}{16}f(0)a^2$, which yields $T_*(a) < +\infty$ if $a \ge 4\sqrt{c}/\sqrt{f(0)}$. □

Lemma 2.4. *For any $0 < a_1 < a_2$, we have $u_{a_1} < u_{a_2}$ in $(0,a_1)\times(0,T_*(a_2))$ and $\frac{\partial u_{a_1}}{\partial x}(0,\cdot) < \frac{\partial u_{a_2}}{\partial x}(0,\cdot)$ in $(0,T_*(a_2))$.*

Proof. Proposition 2.1 shows $T_*(a_1) \ge T_*(a_2)$ and $u_{a_2}(a_1,t) > 0$ for each $t \in (0,T_*(a_2))$. Set

$$w(x,t) = u_{a_1}(x,t) - u_{a_2}(x,t), \quad (x,t)\in[0,a_1]\times[0,T_*(a_2)).$$

Then w solves

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} + b(x)\frac{\partial w}{\partial x} = h(x,t)w, \quad (x,t)\in(0,a_1)\times(0,T_*(a_2)),$$
$$w(0,t)=0, \quad w(a_1,t) = u_{a_2}(a_1,t) > 0, \quad t\in(0,T_*(a_2)),$$
$$w(x,0)=0, \quad x\in(0,a_1),$$

where

$$h(x,t) = \int_0^1 f'(\sigma u_{a_1}(x,t) + (1-\sigma)u_{a_2}(x,t))d\sigma, \quad (x,t)\in(0,a_1)\times(0,T_*(a_2)).$$

The strong maximum principle leads to $w < 0$ in $(0, a_1) \times (0, T_*(a_2))$, and thus the Hopf Lemma yields $\frac{\partial w}{\partial x}(0, \cdot) < 0$ in $(0, T_*(a_2))$. □

Lemma 2.5. *There exists at most one $a > 0$ such that u_a quenches at the infinite time.*

Proof. Assume that u_{a_0} quenches at the infinite time for some $a_0 > 0$. For each $a > a_0$, let us show that u_a quenches at a finite time by contradiction. Otherwise, Lemma 2.4 shows that u_a must quench at the infinite time. Proposition 2.1 and Lemma 2.4 yield

$$u_a(a_0, t) > u_a(a_0, 1) > 0, \quad t \in (1, +\infty), \tag{2.1}$$

$$u_a(x, 1) > u_{a_0}(x, 1), \quad x \in (0, a_0) \text{ and } \frac{\partial u_a}{\partial x}(0, t) > \frac{\partial u_{a_0}}{\partial x}(0, t), \quad t \in (1, +\infty). \tag{2.2}$$

Let

$$\underline{u}_a(x, t) = u_{a_0}(x, t) + \delta \int_0^x \exp\Big\{ \int_0^y b(s)ds \Big\} dy, \quad (x, t) \in [0, a_0] \times [1, +\infty).$$

By (2.1) and (2.2), there exists $\delta > 0$ such that

$$u_a(a_0, t) > \underline{u}_a, \quad t \in (1, +\infty), \quad u_a(x, 1) > \underline{u}_a(x, 1), \quad x \in (0, a_0). \tag{2.3}$$

Note that $\underline{u}_a$ satisfies

$$\frac{\partial \underline{u}_a}{\partial t} - \frac{\partial^2 \underline{u}_a}{\partial x^2} + b(x)\ pd\underline{u}_a x = f(u_{a_0}) < f(\underline{u}_a), \quad (x, t) \in (0, a_0) \times (1, +\infty). \tag{2.4}$$

Owing to (2.3) and (2.4), the comparison principle gives

$$u_a(x, t) \ge \underline{u}_a(x, t) = u_{a_0}(x, t) + \delta \int_0^x \exp\Big\{ \int_0^y b(s)ds \Big\} dy, \quad (x, t) \in [0, a_0] \times [1, +\infty),$$

which contradicts that both u_{a_0} and u_a quench at the infinite time. □

Theorem 2.6. *Assume that $f \in C^1([0, c))$ satisfies (1.4). Then there exists $a_* > 0$ such that*

(i) $T_*(a) = +\infty$ *and* $\sup_{(0,a)\times(0,+\infty)} u_a < c$ *if* $0 < a < a_*$,
(ii) $T_*(a) < +\infty$ *if* $a > a_*$.

Proof. Set

$$S = \big\{ a > 0 : T_*(a) = +\infty \text{ and } \sup_{(0,a)\times(0,+\infty)} u_a < c \big\}.$$

From Lemmas 2.2 and 2.3, S is a bounded set. Denote $a_* = \sup S$. By Lemma 2.4, $a \in S$ for each $0 < a < a_*$. For $a > a_*$, the definition of S shows that $T_*(a) < +\infty$ or u_a quenches at the infinite time. Let us prove that the latter case is impossible by contradiction. Otherwise, assume that u_{a_0} quenches at the infinite time for some $a_0 > a_*$. From the definition of S and Lemma 2.4, $u_{\tilde{a}}$ must quench at the infinite time for each $a_* < \tilde{a} < a_0$, which contradicts Lemma 2.5. □

Remark 2.7. $T_*(a_*) = +\infty$. However, it is unknown whether u_{a_*} quenches or not at the infinite time.

Remark 2.8. To consider the classical solution to problem (1.1)–(1.3), we need $b \in C^1([0, +\infty)) \cap L^\infty([0, +\infty))$. While $b \in L^\infty([0, +\infty))$, we can investigate the weak solution to (1.1)–(1.3), and it is not hard to show that Lemmas 2.2–2.5 also hold. Therefore, Theorem 2.6 still holds if $b \in L^\infty([0, +\infty))$.

3. Quenching properties

Definition 3.1. *Assume that the solution u to (1.1)–(1.3) quenches at $0 < T_* < +\infty$. A point $x \in [0,a]$ is said to be a quenching point if there exist two sequences $\{t_n\}_{n=1}^{\infty} \subset (0,T_*)$ and $\{x_n\}_{n=1}^{\infty} \subset (0,a)$ such that*

$$\lim_{n\to\infty} t_n = T_*, \quad \lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} u(x_n,t_n) = c.$$

Theorem 3.2. *Assume that $f \in C^1([0,c))$ satisfies (1.4) and $M = \int_0^c f(s)ds < +\infty$. Let u be the solution to (1.1)–(1.3) quenching at a finite time T_*. Then the quenching points belong to $[\delta, a-\delta]$ with*

$$\delta = \frac{c^2}{2aM} \exp\Big\{ -\frac{1}{2}\|b\|_{L^\infty([0,+\infty))} T_* \Big\}.$$

Proof. For each $0 < s < T_*$, multiplying (1.1) by $\frac{\partial u}{\partial t}$ and then integrating over $(0,a)\times(0,s)$ by parts with (1.2), one gets

$$\begin{aligned} &\int_0^s \int_0^a \Big(\frac{\partial u}{\partial t}\Big)^2 dx\,dt + \frac{1}{2}\int_0^s \int_0^a \frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial x}\Big)^2 dx\,dt + \int_0^s \int_0^a b(x)\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\,dx\,dt \\ &= \int_0^s \int_0^a \frac{\partial}{\partial t} F(u)\,dx\,dt \end{aligned}$$

with

$$F(\omega) = \int_0^\omega f(y)\,dy \quad (\omega \ge 0),$$

which, together with (1.3), the Young inequality and the Schwarz inequality, lead to

$$\begin{aligned} \int_0^a \Big(\frac{\partial u}{\partial x}(x,s)\Big)^2 dx &\le 2\int_0^a F(u(x,s))dx + \frac{1}{2}\|b\|_{L^\infty([0,+\infty))}\int_0^s \int_0^a \Big(\frac{\partial u}{\partial x}(x,t)\Big)^2 dx\,dt \\ &\le 2aM + \frac{1}{2}\|b\|_{L^\infty([0,+\infty))}\int_0^s \int_0^a \Big(\frac{\partial u}{\partial x}(x,t)\Big)^2 dx\,dt. \end{aligned}$$

Then, the Gronwall inequality shows

$$\int_0^a \Big(\frac{\partial u}{\partial x}(x,s)\Big)^2 dx \le 2aM \exp\Big\{\frac{1}{2}\|b\|_{L^\infty([0,+\infty))} T_*\Big\}, \quad t \in (0,T_*). \tag{3.1}$$

By (3.1), (1.2) and the Schwarz inequality, one gets

$$\begin{aligned} u(x,t) &= \int_0^x \frac{\partial u}{\partial x}(y,t)dy \\ &\le x^{1/2}\Big(\int_0^a \Big(\frac{\partial u}{\partial x}(y,t)\Big)^2 dy\Big)^{1/2} \\ &\le (2aMx)^{1/2}\exp\Big\{\frac{1}{4}\|b\|_{L^\infty([0,+\infty))} T_*\Big\}, \quad (x,t) \in [0,a/2]\times(0,T_*) \end{aligned}$$

and

$$\begin{aligned} u(x,t) = -\int_x^a \frac{\partial u}{\partial x}(y,t)dy &\le (a-x)^{1/2}\Big(\int_0^a \Big(\frac{\partial u}{\partial x}(y,t)\Big)^2 dy\Big)^{1/2} \\ &\le (2aM(a-x))^{1/2}\exp\Big\{\frac{1}{4}\|b\|_{L^\infty([0,+\infty))} T_*\Big\}, \quad (x,t) \in [a/2,a]\times(0,T_*), \end{aligned}$$

which show that there is no quenching point in $[0,\delta) \cup (a-\delta, a]$. □

Theorem 3.3. *Assume that* $f \in C^2([0,c))$ *satisfies* (1.4), $\int_0^c f(s)ds < +\infty$ *and* $f'' \geq 0$ *in* $(0,c)$. *Let* u *be the solution to the problem* (1.1)–(1.3) *quenching at a finite time* T_*. *Then the solution* u *to* (1.1)–(1.3) *satisfies* $\lim_{t\to T_*^-} \sup_{(0,a)} \frac{\partial u}{\partial t}(\cdot,t) = +\infty$.

Proof. From Theorem 3.2, there exist $0 < x_1 < x_2 < x_3 < x_4 < a$ such that

$$\lim_{t\to T_*^-} \sup_{(x_2,x_3)} u(\cdot,t) = c, \qquad \sup_{(0,x_2)\times(0,T_*)} u < c, \qquad \sup_{(x_3,a)\times(0,T_*)} u < c. \tag{3.2}$$

Set

$$v(x,t) = \frac{\partial u}{\partial t}(x,t), \quad (x,t) \in [0,a]\times[0,T_*),$$

which solves

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + b(x)\frac{\partial v}{\partial x} = f'(u)v, \quad (x,t) \in (0,a)\times(0,T). \tag{3.3}$$

Proposition 2.1 gives

$$v(x,t) > 0, \quad (x,t) \in (0,a)\times(0,T_*). \tag{3.4}$$

Let z be the solution to the linear problem

$$\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} + b(x)\frac{\partial z}{\partial x} = 0, \quad (x,t) \in (x_1,x_4)\times(T_*/2,T_*), \tag{3.5}$$

$$z(x_1,t) = z(x_4,t) = 0, \quad t \in (T_*/2,T_*), \tag{3.6}$$

$$z(x,T_*/2) = \delta \sin\Big(\frac{\pi(x-x_1)}{x_4-x_1}\Big), \quad x \in (x_1,x_4) \tag{3.7}$$

with $\delta = \min_{(x_1,x_4)} v(\cdot,T_*/2)$. Owing to (3.3) and (3.4), v is a supersolution to (3.5)–(3.7). The comparison principle and the maximum principle give

$$v(x,t) \geq z(x,t) \geq \gamma, \quad (x,t) \in (x_1,x_4)\times(T_*/2,T_*) \tag{3.8}$$

with some $\gamma > 0$. Set

$$w(x,t) = v(x,t) - \kappa f(u(x,t)), \quad (x,t) \in [x_2,x_3]\times[T_*/2,T_*).$$

By (3.2) and (3.8), there exists $\kappa > 0$ such that

$$w(x,t) \geq 0, \quad (x,t) \in \{x_2,x_3\}\times[T_*/2,T_*) \cup [x_1,x_2]\times\{T_*/2\}. \tag{3.9}$$

Thanks to (1.1) and (3.3), v solves

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} + b(x)\frac{\partial w}{\partial x} - f'(u)w = \kappa f''(u)\Big(\frac{\partial w}{\partial x}\Big)^2 \geq 0,$$
$$(x,t) \in (x_2,x_3)\times(T_*/2,T_*).$$

Then, it follows from the maximal principle with (3.9) that $w \geq 0$ in $(x_2,x_3)\times[T_*/2,T_*)$, which, together with (3.2), yields $\lim_{t\to T_*^-} \sup_{(x_2,x_3)} v(\cdot,t) = +\infty$. □

Remark 3.4. As in Remark 2.8, we note that Theorems 3.2 and 3.3 remain valid if $b \in L^\infty([0,+\infty))$.

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