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SPECTRAL ANALYSIS FOR THE EXCEPTIONAL X_m-JACOBI EQUATION

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Abstract. We provide the mathematical foundation for the X_m-Jacobi spectral theory. Namely, we present a self-adjoint operator associated to the differential expression with the exceptional X_m-Jacobi orthogonal polynomials as eigenfunctions. This proves that those polynomials are indeed eigenfunctions of the self-adjoint operator (rather than just formal eigenfunctions). Further, we prove the completeness of the exceptional X_m-Jacobi orthogonal polynomials (of degrees $m, m+1, m+2, \dots$) in the Lebesgue-Hilbert space with the appropriate weight. In particular, the self-adjoint operator has no other spectrum.

1. Introduction

The classical orthogonal polynomials of Laguerre, Jacobi, and Hermite are the foundational examples of orthogonal polynomial theory. As shown by Routh in 1884 [21], but most often attributed to Bochner in 1929 [3], these three families of polynomials are, up to affine transformation of x, the only polynomial sequences satisfying the following two conditions: First, they contain an infinite sequence of polynomials $\{p_n\}_{n=0}^{\infty}$, where p_n has degree n, such that for each $n \in \mathbb{N}_0$, $y = p_n$ satisfies a second order eigenvalue equation of the form

$$p(x)y'' + q(x)y' + r(x)y = \lambda y\,,$$

where the polynomials $p(x)$, $q(x)$, and $r(x)$ are determined by the corresponding differential expression (Laguerre, Jacobi or Hermite). Second, each of the eigenpolynomials is orthogonal in a weighted L^2 space where the associated weight has finite moments.

In recent years, there has been interest in the area of *exceptional* orthogonal polynomials, which presents a way to generalize Bochner's classification theorem. The most striking difference between classical orthogonal polynomials and their exceptional counterparts is that the exceptional sequences allow for gaps in the degrees of the polynomials. We denote an exceptional orthogonal polynomial sequence $\{p_{m,n}\}_{n\in\mathbb{N}_0\setminus A}$ by using "X_m", where the subscript $m = |A|$ denotes the number of gaps (or the *codimension* of the sequence). We require that the associated second

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order differential expression preserve the space spanned by the exceptional polynomials, but no space with smaller codimension. Consequently, the coefficients of the second order differential equation are not necessarily polynomial. Remarkably, despite removing any finite number of polynomials, the sequences remain complete in their associated space.

Research in the area of exceptional orthogonal polynomials did not develop from a desire to generalize Bochner's theorem; rather, the exceptional polynomials were discovered in the context of quantum mechanics where researchers were looking for a new approach, outside of the classical Lie algebraic [12, 14, 19] setting, to solving spectral problems for second order linear differential operators with polynomial eigenfunctions. In particular, they were discovered in [6, 8] while developing a direct approach [5] to exact or quasi-exact solvability for spectral problems. The first examples of these exceptional polynomials were introduced in 2009 by Gómez-Ullate, Kamran and Milson [6, 8], who completely characterized all X_1-polynomial sequences. Their result showed that the only polynomial families of codimension one (in particular, having no solution of degree zero) satisfying a second order eigenvalue problem are the X_1-Jacobi and X_1-Laguerre polynomials. Explicit examples of the X_2 families were given by Quesne [19, 20], who used the Darboux transformation and shape invariant potentials to find these new families.

Higher-codimensional families, including the X_m-Laguerre and X_m-Jacobi exceptional polynomial sequences, were first observed by Odake and Sasaki [18]. Further generalizations were observed regarding two distinct types of X_m-Laguerre polynomials by Gómez-Ullate, Kamran and Milson [10, 11]. These X_m-Laguerre polynomial families do not contain polynomials of degree $n \in \mathbb{N}$ for $0 \leq n \leq m-1$. Furthermore, Liaw, Littlejohn, Milson, and Stewart [15] show the existence of a third type of X_m-Laguerre polynomials. The Type III X_m-Laguerre polynomial sequence omits polynomials of degree $n \in \mathbb{N}_0$ for $1 \leq n \leq m$. This new class of polynomials can be derived from the quasi-rational eigenfunctions of the classical Laguerre differential expression by Darboux transform as well as a gauge transformation of the Type I exceptional X_m-Laguerre expression.

Following the discovery of exceptional polynomials, there has been a desire to study the properties of these polynomials more rigorously. The explanation for existence via the Darboux transformation of the higher-codimension X_m-Jacobi and X_m-Laguerre polynomials and a remarkable observation regarding the completeness of the X_m-polynomial families was given by Gómez-Ullate, Kamran and Milson [7]. Gómez-Ullate, Marcellán, and Milson studied the interlacing properties of the zeros for both the exceptional Jacobi and exceptional Type I and Type II Laguerre polynomials along with their asymptotic behavior [11]. The properties of the Type III X_m-Laguerre polynomials is studied [15].

The spectral analysis for the X_1-Jacobi polynomials (for $m = 1$, $A = \{0\}$) may be found in [16] along with an analysis of properties resulting from an extreme parameter choice, and for the X_1-Laguerre polynomials, the spectral analysis was completed in [2]. For all three types of the X_m-Laguerre polynomials, a complete spectral study is completed in [15].

We remark that the exceptional X_m-Jacobi equation is the result of a one-step Darboux transformation. In any one-step process, there is a large gap at the beginning of the degree sequence but all of the rest of the degrees are present in the sequence. It is important to note that there are several multi-step exceptional families in which there are other patterns of gaps in the degree sequence. Along this line, we note that the authors in [4] have classified all multi-step families in the Hermite case. Not much is known, however, at the present time on multi-step families of Jacobi type.

In Section 2 we introduce the X_m-Jacobi differential expression along with some properties. In Section 3 we then apply the Glazman-Krein-Naimark theory to obtain a self-adjoint operator associated to the differential expression, for which the corresponding domain contains the exceptional X_m-Jacobi orthogonal polynomials (see Theorem 3.3). Further, we show the completeness of the exceptional X_m-Jacobi orthogonal polynomials (of degrees $m, m+1, m+2, \dots$) in the Lebesgue–Hilbert space with the appropriate weight (see Theorem 3.4). Summing up, we present the spectral analysis of the X_m-Jacobi differential expression.

2. Some properties of the exceptional X_m-Jacobi expression

We begin by summarizing some properties of the exceptional X_m-Jacobi expression as described in [6, 11]. The parameters α and β are assumed to satisfy

$$\alpha, \beta > -1\,, \quad \alpha+1-m-\beta \notin \{0, 1, \dots, m-1\} \quad \text{and} \quad \operatorname{sgn}(\alpha+1-m) = \operatorname{sgn}\beta \tag{2.1}$$

in accordance with [11, Section 5.2], unless otherwise noted.

The exceptional X_m-Jacobi polynomial of degree $n \ge m$, $P_{m,n}^{(\alpha,\beta)}$ is given in terms of the classical Jacobi polynomials $\{P_k^{(\alpha,\beta)}\}_{k=0}^{\infty}$ by

$$\begin{aligned} &P_{m,n}^{(\alpha,\beta)}(x) \\ &= \frac{(-1)^m}{\alpha+1+n-m}\Big[\frac{1}{2}(\alpha+\beta+n-m+1)(x-1)P_m^{(-\alpha-1,\beta-1)}(x)P_{n-m-1}^{(\alpha+2,\beta)}(x) \\ &\quad + (\alpha-m+1)P_m^{(-\alpha-2,\beta)}(x)P_{n-m}^{(\alpha+1,\beta-1)}(x)\Big]. \end{aligned}$$

The exceptional X_m-Jacobi polynomials satisfy the second-order differential equation

$$T_{\alpha,\beta,m}[y](x) = \lambda_n y(x)$$

for $x \in (-1,1)$, where the exceptional X_m-Jacobi differential expression is given by

$$\begin{aligned} &T_{\alpha,\beta,m}[y](x) \\ &:= (1-x^2)y''(x) + \Big(\beta-\alpha-(\beta+\alpha+2)x - 2(1-x^2) \\ &\quad \times \Big(\log(P_m^{(-\alpha-1,\beta-1)}(x))\Big)'\Big)y'(x) \\ &\quad + \Big((\alpha-\beta-m+1)m - 2\beta(1-x)\Big(\log(P_m^{(-\alpha-1,\beta-1)}(x))\Big)'\Big)y(x) \end{aligned} \tag{2.2}$$

and $\lambda_n = -(n-m)(1+\alpha+\beta+n-m)$.

In Lagrangian symmetric form, the exceptional X_m-Jacobi differential expression (2.2) writes

$$\begin{aligned} &T_{\alpha,\beta,m}[y](x) \\ &= \frac{1}{W_{\alpha,\beta,m}(x)}\Big[\big(W_{\alpha,\beta,m}(x)(1-x^2)y'(x)\big)' \\ &\quad + W_{\alpha,\beta,m}(x)\Big(m(\alpha-\beta-m+1) - 2\beta(1-x)\big(\log(P_m^{(-\alpha-1,\beta-1)}(x))\big)'\Big)y(x)\Big] \end{aligned}$$

for $x \in (-1,1)$, where $W_{\alpha,\beta,m}$ is the exceptional X_m-Jacobi weight function given by

$$W_{\alpha,\beta,m}(x) = \frac{(1-x)^{\alpha}(1+x)^{\beta}}{\left(P_m^{(-\alpha-1,\beta-1)}(x)\right)^2} \quad \text{for } x \in (-1,1)\,.$$

The restrictions on α and β ensure that $W_{\alpha,\beta,m}(x)$ has no singularities for $x \in [-1,1]$ and consequently, all moments are finite.

The exceptional X_m-Jacobi polynomials $\{P_{m,n}^{(\alpha,\beta)}\}_{n=m}^{\infty}$ are orthogonal with respect to the weight function $W_{\alpha,\beta,m}(x)$.

The eigenvalue equation $T_{\alpha,\beta,m}[y] = \lambda y$ does not have any polynomial solutions of degree n for $0 \le n \le m-1$. Despite this fact, it is interesting that the exceptional X_m-Jacobi polynomials $\{P_{m,n}^{(\alpha,\beta)}\}_{n=m}^{\infty}$ form a complete sequence in the Hilbert-Lebesgue space $L^2((-1,1);W_{\alpha,\beta,m})$, defined by

$$\begin{aligned} &L^2((-1,1);W_{\alpha,\beta,m}) \\ &:= \Big\{ f:(-1,1)\to\mathbb{C} : f \text{ is measurable and } \int_{-1}^{1} |f|^2 W_{\alpha,\beta,m} < \infty \Big\}\,. \end{aligned}$$

3. Exceptional X_m-Jacobi spectral analysis

We follow the methods outlined in the classical texts of Akhiezer and Glazman [1], Hellwig [13], and Naimark [17].

The maximal domain associated with $T_{\alpha,\beta,m}[\cdot]$ in $L^2((-1,1),W_{\alpha,\beta,m})$ is:

$$\Delta = \{ f:(-1,1)\to\mathbb{C} \big| f, f' \in AC_{\mathrm{loc}}(-1,1); f, T_{\alpha,\beta,m}[f] \in L^2((-1,1),W_{\alpha,\beta,m}) \}\,. \tag{3.1}$$

The maximal domain Δ is the largest subspace of functions of $L^2((-1,1);W_{\alpha,\beta,m})$ for which $T_{\alpha,\beta,m}$ maps into $L^2((-1,1);W_{\alpha,\beta,m})$. The associated maximal operator is

$$S^1_{\alpha,\beta,m} : \mathcal{D}(S^1_{\alpha,\beta,m}) \subset L^2((-1,1),W_{\alpha,\beta,m}) \to L^2((-1,1),W_{\alpha,\beta,m})$$

where $S^1_{\alpha,\beta,m}$ is defined by

$$\begin{gathered} S^1_{\alpha,\beta,m}[f] := T_{\alpha,\beta,m}[f] \\ f \in \mathcal{D}(S^1_{\alpha,\beta,m}) := \Delta\,. \end{gathered} \tag{3.2}$$

For $f,g \in \Delta$, Green's Formula may be written as

$$\begin{aligned} &\int_{-1}^{1} T_{\alpha,\beta,m}[f](x)\overline{g}(x) W_{\alpha,\beta,m}(x)\,dx \\ &= [f,g](x)\big|_{-1}^{1} + \int_{-1}^{1} f(x) T_{\alpha,\beta,m}[\overline{g}](x) W_{\alpha,\beta,m}(x)\,dx \end{aligned} \tag{3.3}$$

where $[\cdot,\cdot](\cdot)$ is the sesquilinear form defined by

$$\begin{aligned} (x) &= W_{\alpha,\beta,m}(x)(1-x^2)(f'(x)\overline{g}(x) - f(x)\overline{g}'(x)) \\ &= \frac{(1-x)^{\alpha+1}(1+x)^{\beta+1}}{\left(P_m^{(-\alpha-1,\beta-1)}(x)\right)^2}(f'(x)\overline{g}(x) - f(x)\overline{g}'(x)) \quad (x \in (-1,1)) \end{aligned} \tag{3.4}$$

and where

$$[f,g](x)\,\big|_{x=-1}^{x=1} := [f,g](1) - [f,g](-1)\,.$$

By the definition of Δ and the classical Hölder's inequality, notice that the limits

$$[f,g](-1) := \lim_{x\to -1^+}[f,g](x) \quad \text{and} \quad [f,g](1) := \lim_{x\to 1^-}[f,g](x)$$

exist and are finite for each $f,g \in \Delta$.

The adjoint of the maximal operator in $L^2((-1,1);W_{\alpha,\beta,m})$ is the minimal operator,

$$S^0_{\alpha,\beta,m} : \mathcal{D}(S^0_{\alpha,\beta,m}) \subset L^2((-1,1),W_{\alpha,\beta,m}) \to L^2((-1,1),W_{\alpha,\beta,m})$$

where $S^0_{\alpha,\beta,m}$ is defined by

$$\begin{gathered} S^0_{\alpha,\beta,m}[f] := T_{\alpha,\beta,m}[f] \\ f \in \mathcal{D}(S^0_{\alpha,\beta,m}) := \big\{ f \in \Delta \big| [f,g]\big|_{-1}^{1} = 0 \text{ for all } g \in \Delta \big\} \,. \end{gathered}$$

We seek to find a self-adjoint extension $S_{\alpha,\beta,m}$ in $L^2((-1,1);W_{\alpha,\beta,m})$ generated by $T_{\alpha,\beta,m}[\cdot]$, which has the exceptional X_m-Jacobi polynomials $\{P^{(\alpha,\beta)}_{m,n}\}_{n=m}^{\infty}$ as eigenfunctions. To achieve this goal, we need to study the behavior of solutions at the singular endpoints $x=-1$ and $x=1$ so as to determine the deficiency indices and find the appropriate boundary conditions (if any).

First, we obtain the deficiency indices via Frobenius Analysis. They depend on the values of the parameters α and β.

The endpoints $x=-1$ and $x=1$ are, in the sense of Frobenius, regular singular endpoints of the differential expression $T_{\alpha,\beta,m}[\cdot]=0$. We first apply Frobenius analysis to the endpoint $x=1$. By multiplying the exceptional X_m-Jacobi expression $T_{\alpha,\beta,m}[y]$ by $\frac{x-1}{x+1}$, we obtain

$$\big(\frac{x-1}{x+1}\big)\big(T_{\alpha,\beta,m}[y](x) - \lambda_n y(x)\big) = (x-1)^2 y''(x) - (x-1)p(x)y'(x) + q(x)y(x)$$

with

$$p(x) = \frac{\beta-\alpha-(\alpha+\beta+2)x}{x+1} - 2\big(\log(P_m^{(-\alpha-1,\beta-1)})\big)'(x-1)$$

and

$$q(x) = \big(\frac{x-1}{x+1}\big)\Big(-(\alpha-\beta-m+1)m - 2\beta\big(\log(P_m^{(-\alpha-1,\beta-1)})\big)'(x-1)\Big)\,.$$

Evaluating the above equation at $x=1$ yields the indicial equation

$$0 = r(r-1) - rp(1) + q(1) = r(r+\alpha)\,.$$

Therefore, two linearly independent solutions to $T_{\alpha,\beta,m}[y] - \lambda_n y = 0$ behave asymptotically (near $x=1$, e.g. on the interval $(0,1)$) like

$$z_1(x) = 1 \quad \text{and} \quad z_2(x) = (x-1)^{-\alpha}$$

near $x=1$.

For all allowable values of α and β,

$$\int_0^1 |z_1(x)|^2 W_{\alpha,\beta,m}(x)\,dx < \infty\,;$$

while

$$\int_0^1 |z_2(x)|^2 W_{\alpha,\beta,m}(x)\,dx < \infty$$

only for $-1<\alpha<1$.

In a similar way, multiplying the exceptional X_m-Jacobi expression $T_{\alpha,\beta,m}[y] - \lambda_{\alpha,\beta,m}y$ by $(x+1)/(x-1)$, results in an indicial equation

$$r(r+\beta) = 0;$$

and two linearly independent solutions will behave (asymptotically) like

$$y_1(x) = 1 \quad \text{and} \quad y_2(x) = (x+1)^{-\beta}$$

near $x = -1$.

For all allowable values of α and β,

$$\int_{-1}^{0} |y_1(x)|^2 W_{\alpha,\beta,m}(x)\,dx < \infty\,;$$

while

$$\int_{-1}^{0} |y_2(x)|^2 W_{\alpha,\beta,m}(x)\,dx < \infty$$

only for $-1 < \beta < 1$.

As a consequence, we have the following results.

Theorem 3.1. *Let $T_{\alpha,\beta,m}[y] - \lambda_{\alpha,\beta,m}$ be the exceptional X_m-Jacobi differential expression* (2.2) *on the interval* $(-1,1)$.

(1) *$T_{\alpha,\beta,m}[\cdot]$ is in the limit-point case at $x = -1$ for $\beta \geq 1$ and limit-circle for $-1 < \beta < 1$.*
(2) *$T_{\alpha,\beta,m}[\cdot]$ is in the limit-point case at $x = 1$ for $\alpha \geq 1$ and limit-circle for $-1 < \alpha < 1$.*

Corollary 3.2. *The minimal operator $S^0_{\alpha,\beta,m}$ in $L^2((-1,1), W_{\alpha,\beta,m})$ has the following deficiency indices:*

(1) *For $\alpha, \beta \geq 1$, $S^0_{\alpha,\beta,m}$ has deficiency index* $(0,0)$.
(2) *For $\alpha \geq 1$ and $0 < \beta < 1$, $S^0_{\alpha,\beta,m}$ has deficiency index* $(1,1)$.
(3) *Similarly, for $\beta \geq 1$ and $0 < \alpha < 1$, $S^0_{\alpha,\beta,m}$ has deficiency index* $(1,1)$.
(4) *For $-1 < \alpha, \beta < 0$, $S^0_{\alpha,\beta,m}$ has deficiency index* $(2,2)$.

Next we formulate the self-adjoint operators.

Theorem 3.3. *The self-adjoint operator $S_{\alpha,\beta,m}$ in $L^2((-1,1); W_{\alpha,\beta,m})$, generated by the exceptional X_m-Jacobi differential expression $T_{\alpha,\beta,m}$ is given by*

$$S_{\alpha,\beta,m}[f] = T_{\alpha,\beta,m}[f], f \in \mathcal{D}(S_{\alpha,\beta,m}),$$

where

$$\mathcal{D}(S_{\alpha,\beta,m}) \\ = \begin{cases} \Delta \quad \text{if } \alpha \geq 1 \text{ and } \beta \geq 1 \\ \{f \in \Delta : \lim_{x\to -1^+} (1+x)^{\beta+1} f'(x) = 0\} \\ \quad \text{if } \alpha \geq 1 \text{ and } 0 < \beta < 1 \\ \{f \in \Delta : \lim_{x\to 1^-} (1-x)^{\alpha+1} f'(x) = 0\} \\ \quad \text{if } 0 < \alpha < 1 \text{ and } \beta \geq 1 \\ \big\{f \in \Delta : \lim_{x\to 1^-} (1-x)^{\alpha+1} f'(x) = \lim_{x\to -1^+} (1+x)^{\beta+1} f'(x) = 0\big\} \\ \quad \text{for all other choices of parameters that are allowed by (2.1).} \end{cases} \tag{3.5}$$

Proof. If the parameters satisfy $\alpha, \beta \geq 1$, then there is only one self adjoint extension (restriction) of the minimal operator $S^0_{\alpha,\beta,m}$ (maximal operator $S^1_{\alpha,\beta,m}$); that is, the maximal and minimal operator coincide and $S_{\alpha,\beta,m} = S^0_{\alpha,\beta,m} = S^1_{\alpha,\beta,m}$.

Suppose that $0 < \alpha < 1$ and $\beta \geq 1$, then there are infinitely many self-adjoint extensions of the minimal operator $S^0_{\alpha,\beta,m}$. From Corollary 3.2, the deficiency index equals $(1,1)$, which means that $\mathcal{D}(S^0_{\alpha,\beta,m})$ is a subspace of codimension 2 in Δ. We will restrict the maximal domain Δ by imposing a suitable boundary condition which is invoked by the sesquilinear form $[\cdot,\cdot](\cdot)$ defined by (3.4). First note that $h(x) = (1-x)^{-\alpha} \in \Delta$ since

$$T_{\alpha,\beta,m}[(1-x)^{-\alpha}] = \mathcal{O}((1-x)^{-\alpha}) \quad (\text{near } x = 1),$$

which implies $T_{\alpha,\beta,m}[g] \in L^2((-1,1); W_{\alpha,\beta,m})$ because $\alpha < 1$. Further, the constant function satisfies $1 \in \Delta$ and

$$[h,1]\big|_{x=-1}^{x=1} = [h,1](1) = -\frac{\alpha\, 2^{\beta+1}}{\big(P_m^{(-\alpha-1,\beta-1)}(1)\big)^2} = \alpha\, 2^{\beta+1} \neq 0\,, \tag{3.6}$$

where we used standard identities for the Jacobi polynomials and the Gamma function to find

$$\begin{aligned} P_m^{(-\alpha-1,\beta-1)}(1) &= \frac{\Gamma(-\alpha+m)}{m!\,\Gamma(\beta+m-\alpha-1)} \frac{\Gamma(\beta+m-\alpha-1)}{\Gamma(-\alpha)} \\ &= \frac{\Gamma(-\alpha+m)}{m!\,\Gamma(-\alpha)} = \frac{m!\,\Gamma(-\alpha)}{m!\,\Gamma(-\alpha)} = 1. \end{aligned}$$

In particular, we obtain from equation (3.6) that the constant function 1 does not belong to $\mathcal{D}(S^0_{\alpha,\beta,m})$.

For $0 < \beta < 1$ and $\alpha \geq 1$, we can prove the corresponding statement in a similar manner. Lastly, for $\alpha, \beta \leq 1$, we combine the above cases. $\square$

Note that every polynomial, in particular the X_m-Jacobi polynomials, will satisfy all of the boundary conditions given by (3.5).

Next, we adapt ideas introduced in [9] and further developed in [15] (for the case of exceptional Laguerre orthogonal polynomial systems) to prove that the spectrum of the self-adjoint operators from Theorem 3.3 consists exactly of the eigenvalues corresponding to the exceptional X_m-Jacobi polynomials (and nothing more).

Theorem 3.4. *The exceptional X_m-Jacobi polynomials $\{P^{(\alpha,\beta)}_{m,n}\}_{n=m}^\infty$ form a complete set of eigenfunctions of the self-adjoint operator $S_{\alpha,\beta,m}$ in $L^2((-1,1), W_{\alpha,\beta,m})$. Additionally, the spectrum $\sigma(S_{\alpha,\beta,m})$ of $S_{\alpha,\beta,m}$ is pure discrete spectrum consisting of the simple eigenvalues*

$$\sigma(S_{\alpha,\beta,m}) = \sigma_p(S_{\alpha,\beta,m}) = \{-(n-m)(1+\alpha+\beta+n-m) \mid n \geq m\}\,.$$

Proof. The eigenvalue equations follow by the Darboux relations. It remains to prove the completeness of $\{P^{(\alpha,\beta)}_{m,n}\}_{n=m}^\infty$ in $L^2((-1,1), W_{\alpha,\beta,m})$. Fix α, β in the allowed range, pick $f \in \mathcal{H} = L^2((-1,1), W_{\alpha,\beta,m})$ and choose $\varepsilon > 0$.

Define the function

$$\widetilde{f}(x) := \frac{f(x)}{P_m^{(-\alpha-1,\beta-1)}(x)}\,.$$

From the relationship

$$W_{\alpha,\beta,m}(x) = \frac{W_{\alpha,\beta}(x)}{\big(P_m^{(-\alpha-1,\beta-1)}(x)\big)^2}$$

between the exceptional and the classical weight ($W_{\alpha,\beta,m}$ and $W_{\alpha,\beta}$, respectively), it easily follows

$$\|f\|_{\mathcal{H}} = \|\widetilde{f}\|_{L^2((-1,1);W_{\alpha,\beta})}.$$

In particular, we have $\widetilde{f} \in L^2((-1,1);W_{\alpha,\beta})$.

Next we apply Lemma 3.5 with the function

$$\eta(x) = P_m^{(-\alpha-1,\beta-1)}(x)$$

and obtain the existence of $p \in \mathcal{P}$ such that

$$\|\widetilde{f} - P_m^{(-\alpha-1,\beta-1)}(x)p(x)\|_{L^2((-1,1);W_{\alpha,\beta})} < \varepsilon^2.$$

Let N be the degree of p. With this polynomial p we can compute

$$\varepsilon^2 > \|\widetilde{f} - P_m^{(-\alpha-1,\beta-1)}(x)p(x)\|_{L^2((-1,1);W_{\alpha,\beta})} = \|f - \big(P_m^{(-\alpha-1,\beta-1)}(x)\big)^2 p(x)\|_{\mathcal{H}}.$$

Our goal is to show that the approximant $\big(P_m^{(-\alpha-1,\beta-1)}(x)\big)^2 p(x)$ is contained in the (closure of the) vector space spanned by the exceptional Jacobi polynomials. To this end, we consider two $(n+m+1)$-dimensional vector spaces

$$\begin{gathered}
\mathcal{E}_{n+2m} := \{P_{m,j}^{(\alpha,\beta)} : j = m, m+1, \dots, n+2m\}, \\
\mathcal{F}_{n+2m} := \{q \in \mathcal{P}_{n+2m} : (1+x_i)q'(x_i) + \beta q(x_i) = 0\},
\end{gathered}$$

where we let x_i denote the $m-1$ roots of the polynomial $P_m^{(-\alpha-1,\beta-1)}(x)$. The space $\mathcal{F}_{n+2m}$ is motivated by the exceptional term in the exceptional Jacobi differential expression. Clearly, we have $\big(P_m^{(-\alpha-1,\beta-1)}(x)\big)^2 p(x) \in \mathcal{F}_{n+2m}$. Since $\dim \mathcal{F}_{n+2m} = \dim \mathcal{E}_{n+2m}$ we achieve our goal, if we can show that

$$\mathcal{E}_{n+2m} \subset \mathcal{F}_{n+2m}.$$

Take $Q \in \mathcal{E}_{n+2m}$. Since $\mathcal{E}_{n+2m}$ is spanned by a basis of eigenvectors of the exceptional X_m-Jacobi differential expression $T_{\alpha,\beta,m}$ we have $T_{\alpha,\beta,m}[\mathcal{E}_{n+2m}] \subset \mathcal{E}_{n+2m}$. It follows that

$$\begin{aligned}
&T_{\alpha,\beta,m}[Q] \\
&:= (1-x^2)Q'' + (\beta - \alpha - (\beta+\alpha+2)x - 2(1-x^2)\Big(\log(P_m^{(-\alpha-1,\beta-1)})\Big)' Q' \\
&\quad + (\alpha - \beta - m + 1)m - 2\beta(1-x)\Big(\log(P_m^{(-\alpha-1,\beta-1)})\Big)' Q
\end{aligned}$$

is polynomial, and hence the exceptional term (that is, the only term with a denominator):

$$-2(1-x)\frac{\big(P_m^{(-\alpha-1,\beta-1)}(x)\big)'}{P_m^{(-\alpha-1,\beta-1)}(x)}[(1+x)Q'(x) + \beta Q(x)]$$

is a polynomial. Since the roots of the classical orthogonal are simple and 1 is not a root, we have

$$(1+x_i)Q'(x_i) + \beta Q(x_i) = 0.$$

We obtain $Q \in \mathcal{F}_{n+2m}$ as desired. $\square$

Let $\mathcal{P}$ denote the set of all polynomials.

Lemma 3.5. *Given a function η on $[-1,1]$ that satisfies $0 < c < |\eta(x)| < C < \infty$ for all $x \in [-1,1]$. Then the set $\{\eta(x)p(x) : p \in \mathcal{P}\}$ is dense in $L^2((-1,1); W_{\alpha,\beta})$ for the classical range of parameters α, β, and the classical Jacobi weight $W_{\alpha,\beta}$.*

Proof. Fix α, β in the classical parameter range; that is, $\alpha, \beta > -1$. Then, by the theory of classical orthogonal polynomials, the polynomials $\mathcal{P}$ are dense in $\mathcal{H} = L^2((-1,1); W_{\alpha,\beta})$. Therefore, it suffices to show that

$$\mathcal{P} \subset \operatorname{clos}_{\mathcal{H}}(\eta\mathcal{P}).$$

To show this, take $p \in \mathcal{P}$ and fix $\varepsilon > 0$. First observe that

$$\|p/\eta\|_{\mathcal{H}} \leq (1/c)\|p\|_{\mathcal{H}},$$

so that $p/\eta \in \mathcal{H}$. By taking $q \in \mathcal{P}$ such that

$$\varepsilon^2 > C^2\|p/\eta - q\|^2_{\mathcal{H}} \geq \|(p/\eta - q)\eta\|^2_{\mathcal{H}} = \|p - \eta q\|^2_{\mathcal{H}},$$

the lemma is proved. □

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