Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 176, pp. 1–17.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ORBITAL STABILITY OF SOLITARY WAVES FOR A 2D-BOUSSINESQ SYSTEM

ALEX M. MONTES, JOSÉ R. QUINTERO

ABSTRACT. In this article, using a variational approach, we establish the nonlinear orbital stability of ground state solitary waves for a 2D Boussinesq-Benney-Luke system that models the evolution of three dimensional long water waves with small amplitude in the presence of surface tension.

1. Introduction

The focus of the present work is the two-dimensional Boussinesq-Benney-Luke type system

$$\begin{gathered}(I-\frac{\mu}{2}\Delta)\eta_t+\Delta\Phi-\frac{2\mu}{3}\Delta^2\Phi+\epsilon\nabla\cdot(\eta\nabla\Phi)=0,\\ \Big(I-\frac{\mu}{2}\Delta\Big)\Phi_t+\eta-\mu\sigma\Delta\eta+\frac{\epsilon}{2}|\nabla\Phi|^2=0,\end{gathered}\tag{1.1}$$

that arises in the study of the evolution of small amplitude long water waves in the presence of surface tension (see Quintero and Montes [11]). Here μ,ϵ are small positive parameters, σ^{-1} is the Bond number (associated with the surface tension) and the functions $\eta(t,x,y)$ and $\Phi(t,x,y)$ denote the wave elevation and the potential velocity on the bottom $z=0$, respectively. The aspect that we study about the system (1.1) is the orbital stability of solitary wave solutions. It is well know that the study of this kind of states of motion is very important to understand the behavior of many physical systems.

A special feature on the Boussinesq system (1.1) is that the Benney-Luke equation (see [13]) and the Kadomtsev-Petviashvili (KP) equation can be derived up to some order with respect to μ and ϵ from system (1.1). Moreover, for small wave speed and large surface tension, is showed in [11] (see also [6]) that a suitable renormalized family of solitary waves of the Boussinesq system (1.1) converges to a nontrivial solitary wave for the (KP-I) equation. We will use this fact in the stability analysis.

One of the main characteristics behind water wave systems is the existence of a Hamiltonian structure which characterizes travelling waves as critical points of the action functional and also provides relevant information for the stability of travelling

2010 *Mathematics Subject Classification.* 35B35, 76B25, 35Q35.
Key words and phrases. Solitary waves; orbital stability; ground state solutions; Boussinesq system.

Submitted June 2, 2015. Published June 24, 2015.

waves. In our particular Boussinesq system (1.1), the Hamiltonian structure is given by

$$\begin{pmatrix}\eta_t\\ \Phi_t\end{pmatrix} = \mathcal{B}\mathcal{H}'\begin{pmatrix}\eta\\ \Phi\end{pmatrix}, \quad \mathcal{B} = \Big(I - \frac{\mu}{2}\Delta\Big)^{-1}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

where the Hamiltonian $\mathcal{H}$ is defined as

$$\mathcal{H}\begin{pmatrix}\eta\\ \Phi\end{pmatrix} = \frac{1}{2}\int_{\mathbb{R}^2}\Big(|\nabla\Phi|^2 + \eta^2 + \frac{2\mu}{3}|\Delta\Phi|^2 + \mu\sigma|\nabla\eta|^2 + \epsilon\eta|\nabla\Phi|^2\Big)\,dx\,dy.$$

On the other hand, by Noether's Theorem, there is a functional $\mathcal{Q}$ (named the Charge) which is conserved in time for classical solutions defined formally as

$$\mathcal{Q}\begin{pmatrix}\eta\\ \Phi\end{pmatrix} = \frac{1}{2}\langle\mathcal{B}^{-1}\partial_x\begin{pmatrix}\eta\\ \Phi\end{pmatrix}, \begin{pmatrix}\eta\\ \Phi\end{pmatrix}\rangle = -\frac{1}{2}\int_{\mathbb{R}^2}\Big(2\eta\Phi_x + \mu\eta_x\Delta\Phi\Big)\,dx\,dy.$$

We will see that travelling waves of wave speed c for the Boussinesq system (1.1) corresponds to stationary solutions of the modulated system

$$\begin{pmatrix}\eta_t\\ \Phi_t\end{pmatrix} = \mathcal{B}{\mathcal{H}_c}'\begin{pmatrix}\eta\\ \Phi\end{pmatrix},$$

where $\mathcal{H}_c(Y) = \mathcal{H}(Y) + c\mathcal{Q}(Y)$. In other words, solutions of the system

$$\mathcal{H}'(Y) + c\mathcal{Q}'(Y) = 0.$$

We note from the Hamiltonian structure that the well definition of the functionals $\mathcal{H}$ and $\mathcal{H}_c$ require having $\eta, \nabla\Phi \in H^1(\mathbb{R}^2)$. These conditions already characterize the natural space (energy space) to look for travelling waves solutions of the system Bousinesq-Benney-Luke, as shown in the preliminary section. It is important to mention that using the conservation in time of the Hamiltonian, A. Montes *et. al.* (see [7]) established the existence of global solutions for the Cauchy problem associated with the system (1.1) and the initial condition in the energy space. On the other hand, J. Quintero and A. Montes in [11] showed the existence of solitary waves (travelling wave solutions in the energy space) by using a variational approach in which weak solutions correspond to critical points of an energy under a special constrain.

Regarding the stability issue, we need to recall that M. Grillakis, J. Shatah and W. Strauss in [4] gave a general result used to establish orbital stability of solitary waves for a class of abstract Hamiltonian systems. In this case, solitary waves of least energy Y_c are minimums of the action functional $\mathcal{H}_c$ and the stability analysis depends on the positiveness of the symmetric operator $\mathcal{H}_c''(Y_c)$ in a neighborhood of the solitary wave Y_c, except possibly in two directions, and also the strict convexity of the real function

$$d_1(c) = \inf\{\mathcal{H}_c(Y) : Y \in \mathcal{M}_c\},$$

where $\mathcal{M}_c$ is a suitable set. The verification of the positiveness of $\mathcal{H}_c''(Y_c)$ is much simpler for one-dimensional spatial problems since the spectral analysis for the operator $\mathcal{H}_c''(Y_c)$ is reduced to studying the eigenvalues of a ordinary differential equation which at $\pm$ infinity becomes to a ordinary differential equation with constant coefficients (see [1, 8, 12]). The key fact to obtain stability in those cases is that in the one dimensional case solitary waves are unique up to translations, and d_1 can be rescaled allowing to establish the strict convexity $d_1(c)$ in a direct way (see [1, 8, 12]). In the two-dimensional spatial case, we have a harder task to overcome using Grillakis *et al.* approach since the spectral analysis is not straightforward for our problem. In order to avoid making the spectral analysis required in Grillakis *et al.* work, we used a direct approach to prove orbital stability of ground state solitary wave solutions of the system (1.1) in the case of wave speed c near 1^-, using strongly the variational characterization of d_1, as done for other 2D models: see Shatah for nonlinear Klein Gordon equations [14], Quintero for the 2D Benney-Luke equation [9] and also in the case of a 2D Boussinesq-KdV type system [10], Saut for the KP equation [2], Fukuizumi for the nonlinear Schrödinger equation with harmonic potential [3] and Liu for the generalized KP equation [5], among others.

This article is organized as follows. In section 2, we present preliminaries related with the existence of solitons (solitary wave solutions) for the system Boussinesq-Benney-Luke and the link between solitons for the system (1.1) and the (KP) equation. In section 3, we prove the strict convexity of d_1 for $c \in (0,1)$, but near 1. In section 4, we establish the orbital stability result.

2. Preliminaries

To simplify the computation, we rescale the parameters μ and ϵ from the system (1.1) by defining

$$\widetilde{\eta}(t,x,y) = \frac{1}{\epsilon}\eta\Big(\frac{t}{\sqrt{\mu}}, \frac{x}{\sqrt{\mu}}, \frac{y}{\sqrt{\mu}}\Big), \quad \widetilde{\Phi}(t,x,y) = \frac{\sqrt{\mu}}{\epsilon}\Phi\Big(\frac{t}{\sqrt{\mu}}, \frac{x}{\sqrt{\mu}}, \frac{y}{\sqrt{\mu}}\Big).$$

So, by a solitary wave solution for the system (1.1) we mean a solution for the rescaled system of the form

$$\eta(t,x,y) = u\left(x-ct, y\right), \quad \Phi(t,x,y) = v\left(x-ct, y\right),$$

where c denotes the speed of the wave. Then, one sees that the solitary wave profile (u, v) should satisfy the system

$$\begin{gathered} \frac{2}{3}\Delta^2 v - \Delta v + c\Big(I - \frac{1}{2}\Delta\Big)u_x - \nabla\cdot(u\nabla v) = 0, \\ u - \sigma\Delta u - c\Big(I - \frac{1}{2}\Delta\Big)v_x + \frac{1}{2}|\nabla v|^2 = 0. \end{gathered} \tag{2.1}$$

Our stability analysis of the solitary wave solutions will be perform in the following appropriate spaces. Recall that the standard Sobolev space $H^k(\mathbb{R}^2)$, $k \in \mathbb{Z}^+$, is the Hilbert space defined as the closure of $C_0^\infty(\mathbb{R}^2)$ with inner product

$$\langle u, v\rangle_{H^k} = \sum_{|\alpha|\le k}\int_{\mathbb{R}^2} D^\alpha u \cdot D^\alpha v\, dx.$$

We denote $\mathcal{V}$ the closure of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm given by

$$\|v\|_{\mathcal{V}}^2 := \int_{\mathbb{R}^2}\Big(|\nabla v|^2 + |\Delta v|^2\Big)\,dx\,dy = \int_{\mathbb{R}^2}\left(v_x^2 + v_y^2 + v_{xx}^2 + 2v_{xy}^2 + v_{yy}^2\right)\,dx\,dy.$$

Note that $\mathcal{V}$ is a Hilbert space with respect to the inner product

$$\langle v, w\rangle_{\mathcal{V}} = \langle v_x, w_x\rangle_{H^1(\mathbb{R}^2)} + \langle v_y, w_y\rangle_{H^1(\mathbb{R}^2)}.$$

Also, we define the energy space $\mathcal{X} = H^1(\mathbb{R}^2)\times\mathcal{V}$, which is a Hilbert space with respect to the norm

$$\|(u,v)\|_{\mathcal{X}}^2 = \|u\|_{H^1(\mathbb{R}^2)}^2 + \|v\|_{\mathcal{V}}^2 = \int_{\mathbb{R}^2}\left(u^2 + |\nabla u|^2 + |\nabla v|^2 + |\Delta v|^2\right)\,dx\,dy.$$

We can see that solutions (u,v) of system (2.1) are critical points of the functional $J_c = 2\mathcal{H}_c$ given by

$$J_c(u,v) = I_c(u,v) + G(u,v),$$

where the functionals I_c and G are defined on the space $\mathcal{X}$ by

$$\begin{aligned} I_c(u,v) &= I_1(u,v) + I_{2,c}(u,v),\\ I_1(u,v) &= \int_{\mathbb{R}^2} \left(u^2 + \sigma|\nabla u|^2 + |\nabla v|^2 + \tfrac{2}{3}(\Delta v)^2\right)\,dx\,dy,\\ I_{2,c}(u,v) &= -c\int_{\mathbb{R}^2} (2uv_x + u_x\Delta v)\,dx\,dy,\\ G(u,v) &= \int_{\mathbb{R}^2} u|\nabla v|^2\,dx\,dy. \end{aligned}$$

In fact, note that $I_c, G \in C^1(\mathcal{X},\mathbb{R})$ and its derivatives in (u,v) in the direction of (U,V) are given by

$$\begin{aligned} \langle I_c'(u,v),(U,V)\rangle &= 2\int_{\mathbb{R}^2} \left(uU + \sigma\nabla u\cdot\nabla U + \nabla v\cdot\nabla V + \tfrac{2}{3}\Delta v\Delta V\right)\,dx\,dy\\ &\quad - c\int_{\mathbb{R}^2} (2uV_x + 2v_xU + u_x\Delta V + \Delta v U_x)\,dx\,dy,\\ \langle G'(u,v),(U,V)\rangle &= \int_{\mathbb{R}^2} \left(|\nabla v|^2 U + 2u\nabla v\cdot\nabla V\right)\,dx\,dy. \end{aligned}$$

Then we see that

$$J_c'(u,v) = 2\begin{pmatrix} u - \sigma\Delta u - c(I - \frac{1}{2}\Delta)v_x + \frac{1}{2}|\nabla v|^2 \\ \frac{2}{3}\Delta^2 v - \Delta v + c(I - \frac{1}{2}\Delta)u_x - \nabla\cdot(u\nabla v)\end{pmatrix},$$

meaning that critical points of the functional J_c satisfy the solitary wave system (2.1).

2.1. **Existence of solitary waves.** Quintero and Montes [11] established the existence of solitary wave solutions for the Boussinesq-Benney-Luke system (1.1) for $\sigma > 0$ and $0 < c < \min\{1, \frac{8\sigma}{3}\}$, by using the Concentration-Compactness principle and the existence of a local compact embedding result. The strategy was to consider the following minimization problem

$$\mathcal{I}_c := \inf\{I_c(u,v) : (u,v)\in\mathcal{X} \text{ with } G(u,v) = 1\}. \tag{2.2}$$

The existence of solitary waves is consequence of the following results [11], which we will use throughout this work. Next, we assume that $\sigma > 0$ and $0 < c < \min\{1, \frac{8\sigma}{3}\}$.

Lemma 2.1. *The functional I_c is nonnegative and there are positive constants $C_1(\sigma,c) < C_2(\sigma,c)$ defined as*

$$C_1(\sigma,c) = \min\big\{1-c, \sigma(1-c), \frac{2}{3} - \frac{c}{4\sigma}\big\}, \quad C_2(\sigma,c) = \max\big\{1+c, \frac{2}{3} + \frac{c}{2}, \sigma + \frac{c}{2}\big\}$$

such that

$$C_1(\sigma,c)I_c(u,v) \le \|(u,v)\|_{\mathcal{X}}^2 \le C_2(\sigma,c)I_c(u,v). \tag{2.3}$$

Furthermore, $\mathcal{I}_c$ is finite and positive.

Theorem 2.2. *If (u_0,v_0) is a minimizer for problem (2.2), then $(u,v) = -k(u_0,v_0)$ is a nontrivial solution of (2.1) for $k = \frac{2}{3}\mathcal{I}_c$.*

Theorem 2.3. *If $\{(u_m, v_m)\}$ is a minimizing sequence for (2.2), then there is a subsequence (which we denote the same), a sequence of points $(x_m, y_m) \in \mathbb{R}^2$, and a minimizer $(u_0, v_0) \in \mathcal{X}$ of (2.2), such that the translated functions*

$$(\tilde{u}_m, \tilde{v}_m) = (u_m(\cdot + x_m, \cdot + y_m), v_m(\cdot + x_m, \cdot + y_m))$$

converge to (u_0, v_0) strongly in $\mathcal{X}$.

2.2. **Link between solitary waves for** (1.1) **and the KP equation.** Assuming $\sigma > 3/8$, c is close to 1^-, and balancing the effects of nonlinearity and dispersion, Quintero and Montes [11] established that a renormalized family of solitons of the Boussinesq-Benney-Luke system converges to a nontrivial soliton for a KP-I type equation. More precisely, set $\sigma > 0$, $\epsilon > 0$, $\mu = \epsilon$, $c^2 = 1 - \epsilon$ and for a given couple $(u, v) \in \mathcal{X}$ define the functions z and w by

$$u(x, y) = \epsilon^{1/2} z(X, Y), \quad v(x, y) = w(X, Y), \quad X = \epsilon^{1/2} x, \quad Y = \epsilon y. \tag{2.4}$$

Then a simple calculation shows that

$$I_1(u, v) = \epsilon^{1/2} I^{1,\epsilon}(z, w), \quad I_{2,c}(u, v) = \epsilon^{1/2} I^{2,\epsilon}(z, w),$$
$$I_{c(\epsilon)}(u, v) = \epsilon^{1/2} I^{\epsilon}(z, w), \quad G(u, v) = G^{\epsilon}(z, w),$$

where I^1, $I^{2,\epsilon}$, I^{ϵ} and G^{ϵ} are given by

$$\begin{aligned} I^{\epsilon}(z, w) &= I^{1,\epsilon}(z, w) + I^{2,\epsilon}(z, w), \\ I^{1,\epsilon}(z, w) &= \int_{\mathbb{R}^2} \left(\epsilon^{-1} z^2 + \sigma(z_x^2 + \epsilon z_y^2) + \epsilon^{-1} w_x^2 + w_y^2\right) dx\, dy \\ &\quad + \frac{2}{3} \int_{\mathbb{R}^2} \left(w_{xx}^2 + 2\epsilon w_{xy}^2 + \epsilon^2 w_{yy}^2\right) dx\, dy, \\ I^{2,\epsilon}(z, w) &= -c \int_{\mathbb{R}^2} \left(2\epsilon^{-1} z w_x + z_x(w_{xx} + \epsilon w_{yy})\right) dx\, dy, \\ G^{\epsilon}(z, w) &= \int_{\mathbb{R}^2} z \left(w_x^2 + \epsilon w_y^2\right) dx\, dy. \end{aligned}$$

Note that if $\sigma > 3/8$ then there is a family $\{(u_c, v_c)\}_c$ such that

$$I_c(u_c, v_c) = \mathcal{I}_c, \quad G(u_c, v_c) = 1, \quad 0 < c < 1.$$

Then, if we denote

$$\mathcal{I}^{\epsilon} := \inf\{I^{\epsilon}(z, w) : (z, w) \in \mathcal{X} \text{ with } G^{\epsilon}(z, w) = 1\},$$

there is a correspondent family $\{(z^{\epsilon}, w^{\epsilon})\}_{\epsilon}$ such that

$$\mathcal{I}^{\epsilon} = I^{\epsilon}(z^{\epsilon}, w^{\epsilon}), \quad G^{\epsilon}(z^{\epsilon}, w^{\epsilon}) = 1, \quad \mathcal{I}_c = \epsilon^{1/2} \mathcal{I}^{\epsilon}. \tag{2.5}$$

We have the following results (see [11]).

Lemma 2.4. *Let $\sigma > 3/8$. Then we have*

$$\lim_{\epsilon \to 0^+} \mathcal{I}^{\epsilon} = \lim_{\epsilon \to 0^+} I^{\epsilon}(z^{\epsilon}, w^{\epsilon}) = \mathcal{J}^0 > 0, \tag{2.6}$$

where

$$\mathcal{J}^0 = \inf\{J^0(w) : w \in \mathcal{V},\, G^0(w) = 1\},$$
$$J^0(w) = \int_{\mathbb{R}^2} \left(w_x^2 + w_y^2 + \left(\sigma - \tfrac{1}{3}\right) w_{xx}^2\right) dx\, dy,$$

$$G^0(w) = \int_{\mathbb{R}^2} w_x^3 \, dx \, dy.$$

Lemma 2.5. *Let $\sigma > 3/8$. Then we have*

$$\lim_{\epsilon \to 0^+} (z^\epsilon - \partial_x w^\epsilon) = 0 \quad \text{in } L^2(\mathbb{R}^2).$$

Moreover, there is a nontrivial distribution $w_0 \in \mathcal{V}$ such that

$$\lim_{\epsilon \to 0^+} \partial_x w^\epsilon = \partial_x w_0 \quad \text{in } L^2(\mathbb{R}^2).$$

Furthermore,

$$\|z^\epsilon\|_{L^2(\mathbb{R}^2)} + \|\partial_x z^\epsilon\|_{L^2(\mathbb{R}^2)} = O(1), \quad \|\partial_{yy} w^\epsilon\|_{L^2(\mathbb{R}^2)} = O(\epsilon^{-1}),$$
$$\|\partial_x w^\epsilon\|_{L^2(\mathbb{R}^2)} + \|\partial_{xx} w^\epsilon\|_{L^2(\mathbb{R}^2)} = O(1).$$

Using the previous lemmas, Quintero *et al.* showed that there are nontrivial distributions $w_0 \in \mathcal{V}$, $z_0 \in H^1(\mathbb{R}^2)$ such that as $\epsilon \to 0^+$,

$$w^\epsilon \to w_0 \quad \text{in } \mathcal{V}, \quad z^\epsilon \to z_0 \quad \text{in } H^1(\mathbb{R}^2),$$

and $\partial_x w_0$ being a solution of the solitary wave equation for the (KP-I) type equation

$$\left(u_x - (\sigma - \tfrac{1}{3}) u_{xxx} + 3 u u_x\right)_x + u_{yy} = 0.$$

We shall use Lemmas 2.4 and 2.5 in our proof of stability.

3. Variational approach for stability

Recall that the solitary waves for the Boussinesq-Benney-Luke system (1.1) are characterized as critical points of the functional defined on $\mathcal{X}$ by

$$J_c(u, v) = I_c(u, v) + G(u, v).$$

In particular, if

$$K_c(u, v) = \langle J_c'(u, v), (u, v) \rangle$$

we have

$$K_c(u, v) = 2 I_c(u, v) + 3 G(u, v) = 2 J_c(u, v) + G(u, v).$$

Thus, on any critical point (u, v) of the functional J_c we have that

$$J_c(u, v) = \frac{1}{3} I_c(u, v), \tag{3.1}$$

$$J_c(u, v) = -\frac{1}{2} G(u, v), \tag{3.2}$$

$$I_c(u, v) = -\frac{3}{2} G_(u, v). \tag{3.3}$$

Now, define the set

$$\mathcal{M}_c = \{(u, v) \in \mathcal{X} : K_c(u, v) = 0, \, (u, v) \neq 0\}.$$

Note that $\mathcal{M}_c$ is just the "artificial constrain" for minimizing the functional J_c on $\mathcal{X}$. We will see that the analysis of the orbital stability of ground states solutions depends upon some properties of the function d defined by

$$d(c) = \inf\{J_c(u, v) : (u, v) \in \mathcal{M}_c\}.$$

A ground state solution is a solitary wave which minimizes the action functional J_c among all the nonzero solutions of (2.1). Moreover, the set of ground state solutions

$$\mathcal{G}_c = \{(u,v) \in \mathcal{M}_c : d(c) = J_c(u,v)\}$$

can be characterized as

$$\mathcal{G}_c = \{(u,v) \in \mathcal{X} \setminus \{0\} : d(c) = \frac{1}{3} I_c(u,v) = -\frac{1}{2} G(u,v)\} \subset \mathcal{M}_c.$$

We note that there is a simple relationship between d_1 and d, and so regarding the convexity of them. In fact,

$$\begin{aligned} d(c) &= \inf\{J_c(u,v) : (u,v) \in \mathcal{M}_c\} \\ &= 2\inf\{\mathcal{H}_c(u,v) : (u,v) \in \mathcal{M}_c\} = 2d_1(c). \end{aligned}$$

In the next lemmas we present important variational properties of $d(c)$.

Lemma 3.1. *Let $0 < c < 1$ and $\sigma > 3/8$. Then*

(1) *$d(c)$ exist and is positive.*
(2) *$d(c) = \inf\{\frac{1}{3} I_c(u,v) : K_c(u,v) \le 0,\ (u,v) \neq 0\}$.*

Proof. (1) Let $(u,v) \in \mathcal{M}_c$, then we have that

$$J_c(u,v) = \frac{1}{3} I_c(u,v) \ge 0.$$

This implies that $d(c)$ exists. Now, Using the Young inequality and that the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ is continuous for $q \ge 2$, we see that there is a constant $C > 0$ such that

$$|G(u,v)| \le C\Big(\|u\|^3_{H^1(\mathbb{R}^2)} + \|\nabla v\|^3_{H^1(\mathbb{R}^2)}\Big). \tag{3.4}$$

Thus, using (2.3) we see that

$$J_c(u,v) = \frac{1}{3} I_c(u,v) = -\frac{1}{2} G(u,v) \le C\|(u,v)\|^3_{\mathcal{X}} \le C\left(I_c(u,v)\right)^{3/2}.$$

Then follows that $\frac{1}{3} I_c(u,v) \ge C$, and this implies that $d(c) \ge C > 0$.

(2) For $(u,v) \in \mathcal{X}$ such that $K_c(u,v) \le 0$ we have that $G(u,v) < 0$. Define $\alpha \in [0,1)$ by

$$\alpha = -\frac{2I_c(u,v)}{3G(u,v)}.$$

Then a direct computation shows that $K_c(\alpha(u,v)) = 0$. In other words, $\alpha(u,v) \in \mathcal{M}_c$. So that,

$$d(c) \le J_c(\alpha(u,v)) = \frac{\alpha^2}{3} I_c(u,v) \le \frac{1}{3} I_c(u,v).$$

Hence, we obtain

$$d(c) \le \inf\Big\{\frac{1}{3} I_c(u,v) : K_c(u,v) \le 0\Big\}.$$

If $(u,v) \in \mathcal{M}_c$, we see that $J_c(u,v) = \frac{1}{3} I_c(u,v)$ and

$$\inf\big\{\tfrac{1}{3} I_c(u,v) : K_c(u,v) \le 0,\ (u,v) \neq 0\big\} \le \inf\big\{J_c(u,v) : (u,v) \in \mathcal{M}_c\big\} = d(c).$$

Then the statement 2 of the lemma follows. □

Lemma 3.2. *Let $0 < c < 1$ and $\sigma > 3/8$. Then*

(1) If $\{(u_m, v_m)\}$ is a minimizing sequence of $d(c)$, then there is a subsequence, which we denote the same, a sequence of points $(x_m, y_m) \in \mathbb{R}^2$, and $(u^c, v^c) \in \mathcal{X}\setminus\{0\}$ such that the translated functions

$$(u_m(\cdot + x_m, \cdot + y_m), v_m(\cdot + x_m, \cdot + y_m))$$

converge to (u^c, v^c) strongly in $\mathcal{X}$, $(u^c, v^c) \in \mathcal{M}_c$, $d(c) = J_c(u^c, v^c)$ and (u^c, v^c) is a solution of (2.1). Moreover,

$$d(c) = \frac{4}{27}\mathcal{I}_c^3, \tag{3.5}$$

where $\mathcal{I}_c = \inf\{I_c(u, v) : G(u, v) = 1,\ (u, v) \in \mathcal{X}\}$.

(2) Let $\{(u_m, v_m)\}$ be a sequence in $\mathcal{X}$ such that

$$\frac{1}{3}I_c(u_m, v_m) \to d(c) \quad \text{and} \quad J_c(u_m, v_m) \to \tilde{d} \le d(c).$$

Then there exist a subsequence of $\{(u_m, v_m)\}$ which denote the same, a sequence $(x_m, y_m) \in \mathbb{R}^2$ and $(u^c, v^c) \in \mathcal{M}_c$ such that the translated functions

$$(u_m(\cdot + x_m, \cdot + y_m), v_m(\cdot + x_k, \cdot + y_k))$$

converge to (u^c, v^c) strongly in $\mathcal{X}$ and $\tilde{d} = d(c) = \frac{1}{3}I_c(u^c, v^c)$.

Proof. The first part of this result is consequence of the Theorem 2.2, Theorem 2.3 and the following argument. Let $(u, v) \in \mathcal{X} \setminus \{0\}$ be such that $K_c(u, v) = 0$, then

$$I_c(u, v) = -\frac{3}{2}G(u, v) = \frac{3}{2}|G(u, v)| = 3J_c(u, v).$$

Consider the couple

$$(z, w) = \frac{1}{G^{1/3}(u, v)}(u, v).$$

Then $G(z, w) = 1$. Thus,

$$\mathcal{I}_c \le I_c(z, w) = \frac{1}{G^{\frac{2}{3}}(u, v)}I_c(u, v) = \left(\frac{3}{2}\right)^{2/3} I_c^{1/3}(u, v) = \left(\frac{3}{2}\right)^{2/3} \Big(3J_c(u, v)\Big)^{1/3}.$$

So that, we concluded

$$\frac{4}{27}\mathcal{I}_c^3 \le d(c).$$

Now, suppose that $(u, v) \ne 0$ such that $G(u, v) = 1$. Take t such that $K_c(tu, tv) = 0$. In this case, $2I_c(u, v) + 3t = 0$. Therefore

$$t^2 = \frac{4}{9}I_c^2(u, v).$$

Then we obtain,

$$d(c) \le J_c(tu, tv) = t^2\left(I_c(u, v) + t\right) = \frac{4}{27}I_c^3(u, v).$$

Thus, we have shown that

$$d(c) \le \frac{4}{27}\left(\mathcal{I}_c\right)^3.$$

This proves (3.5). Now, we show the second part. Since $K_c = 2I_c + 3G$ then we see that

$$J_c(u_m, v_m) = \frac{1}{3}\left(I_c(u_m, v_m) + K_c(u_m, v_m)\right) \to \tilde{d} \le d(c).$$

Then for m large enough we have that $K_c(u_m, v_m) \le 0$. This fact implies that the sequence $\{(u_m, v_m)\}$ is a minimizing sequence for $d(c)$. Then using the part 1 we have that there exist a subsequence of $\{(u_m, v_m)\}$, which denote the same, a sequence $(x_m, y_m) \in \mathbb{R}^2$ and $(u^c, v^c) \in \mathcal{M}_c$ such that

$$(u_m(\cdot + x_m, \cdot + y_m), v_m(\cdot + x_k, \cdot + y_k)) \to (u^c, v^c)$$

in $\mathcal{X}$. In particular $K_c(u^c, v^c) = 0$ and $\tilde{d} = d(c) = \frac{1}{3} I_c(u^c, v^c)$. □

Lemma 3.3. *Let $0 < c < 1$ and $\sigma > 3/8$. Then*

(1) If $0 < c_1 < c_2 < 1$ and $(u, v) \in \mathcal{G}_c$, then we have that $d(c)$ and $I_{2,c}(u, v)$ are uniformly bounded functions on $[c_1, c_2]$.

(2) If $c_1 < c_2$ and $(u^{c_i}, v^{c_i}) \in \mathcal{G}_{c_i}$, we have the following inequalities

$$d(c_1) \le d(c_2) - \Big(\frac{c_2 - c_1}{c_2}\Big) I_{2,c_2}(u^{c_2}, v^{c_2}) + o(c_2 - c_1),$$
$$d(c_2) \le d(c_1) + \Big(\frac{c_2 - c_1}{c_1}\Big) I_{2,c_1}(u^{c_1}, v^{c_1}) + o(c_2 - c_1).$$

(3) If $0 < c_1 < c_2 < 1$, $(u^{c_1}, v^{c_1}) \in \mathcal{G}_{c_1}$ and $I_{2,c_1}(u^{c_1}, v^{c_1}) \le 0$, then

$$d(c_2) \le d(c_1) + \Big(\frac{2(c_2 - c_1)}{3c_1}\Big) I_{2,c_1}(u^{c_1}, v^{c_1}).$$

In particular, d is a strictly decreasing function on $(c_1, 1)$.

Proof. (1) Let c_1, c_2 be such that $0 < c_1 < c_2 < 1$ and let $(u, v) \in \mathcal{X}$ be such that $G(u, v) \neq 0$. Define t_c by

$$t_c = -\frac{2}{3} \frac{I_c(u, v)}{G(u, v)}.$$

Then we have that $K_c(t_c(u, v)) = 0$ and $J_c(t_c(u, v)) = \frac{t_c^2}{3} I_c(u, v)$. Using (2.3) we see that there exist $C > 0$ that depends only on σ such that for all $c \in [c_1, c_2]$,

$$d(c) \le J_c(t_c(u, v)) = \frac{4}{27} \frac{I_c^3(u, v)}{G^2(u, v)} \le C \frac{\|(u, v)\|_{\mathcal{X}}^6}{G^2(u, v)}.$$

Now, let $(z, w) \in \mathcal{G}_c$, then we have that $2I_c(z, w) + 3G(z, w) = 0$. Moreover,

$$C_1(\sigma, c_1, c_2)\|(z, w)\|_{\mathcal{X}}^2 \le 2I_c(z, w) = 3|G(z, w)| \le C\|(z, w)\|_{\mathcal{X}}^3.$$

Then we conclude that

$$C_1(\sigma, c_1, c_2) \le \|(z, w)\|_{\mathcal{X}} \le C_2(\sigma, c_1, c_2)\Big(\frac{1}{3} I_c(z, w)\Big)^{1/2}.$$

Thus, we have shown that

$$d(c) \ge \Big(\frac{C_1(\sigma, c_1, c_2)}{C_2(\sigma, c_1, c_2)}\Big)^2.$$

Hence, if $(u, v) \in \mathcal{G}_c$ we see that $I_c(u, v)$ and $G(u, v)$ are uniformly bounded on $[c_1, c_2]$ since

$$d(c) = \frac{1}{3} I_c(u, v) = -\frac{1}{2} G(u, v),$$

which implies that $I_{2,c}(u, v)$ is also uniformly bounded because $K_c(u, v) = 0$ and

$$I_1(u, v) \cong \|(u, v)\|_{\mathcal{X}}^2.$$

(2) Let (z,w) be defined by $(z,w)=t(u^{c_2},v^{c_2})$. We want t such that $K_{c_1}(z,w)=0$. Note that

$$\begin{aligned}
K_{c_1}(z,w) &= 2t^2 I_{c_1}(u^{c_2},v^{c_2}) + 3t^3 G(u^{c_2},v^{c_2})\\
&= t^2\Big(2I_{c_2}(u^{c_2},v^{c_2}) - \frac{2(c_2-c_1)}{c_2} I_{2,c_2}(u^{c_2},v^{c_2})\Big) + 3t^3 G(u^{c_2},v^{c_2})\\
&= t^2\Big(3tG(u^{c_2},v^{c_2}) - 3G(u^{c_2},v^{c_2}) - \frac{2(c_2-c_1)}{c_2} I_{2,c_2}(u^{c_2},v^{c_2}\Big).
\end{aligned}$$

Thus, t has to be such that

$$tG(u^{c_2},v^{c_2}) = G(u^{c_2},v^{c_2}) + \frac{2(c_2-c_1)}{3c_2} I_{2,c_2}(u^{c_2},v^{c_2})$$

or equivalently

$$t = 1 + \frac{2(c_2-c_1)}{3c_2}\Big(\frac{I_{2,c_2}(u^{c_2},v^{c_2})}{G(u^{c_2},v^{c_2})}\Big) = 1 - \frac{(c_2-c_1)}{3c_2}\Big(\frac{I_{2,c_2}(u^{c_2},v^{c_2})}{d(c_2)}\Big).$$

Then for this t, we conclude that $K_{c_1}(z,w)=0$. Now,

$$\begin{aligned}
d(c_1) \le J_{c_1}(w,z) &= t^2\Big(I_{c_1}(u^{c_2},v^{c_2}) + tG(u^{c_2},v^{c_2})\Big)\\
&= t^2\Big(I_{c_2}(u^{c_2},v^{c_2}) + \frac{c_1-c_2}{c_2} I_{2,c_2}(u^{c_2},v^{c_2}) + tG(u^{c_2},v^{c_2})\Big)\\
&= t^2\Big(d(c_2) - \frac{c_2-c_1}{3c_2} I_{2,c_2}(u^{c_2},v^{c_2})\Big).
\end{aligned}$$

But we have that

$$\begin{aligned}
t^2 &= \Big(1 - \frac{(c_2-c_1)}{3c_2}\Big(\frac{I_{2,c_2}(u^{c_2},v^{c_2})}{d(c_2)}\Big)\Big)^2\\
&= 1 - \frac{2(c_2-c_1)}{3c_2}\Big(\frac{I_{2,c_2}(u^{c_2},v^{c_2})}{d(c_2)}\Big) + O\left((c_2-c_1)^2\right).
\end{aligned}$$

Then we see that

$$\begin{aligned}
&t^2\Big(d(c_2) - \frac{(c_2-c_1)}{3c_2} I_{2,c_2}(u^{c_2},v^{c_2})\Big)\\
&= d(c_2) - \frac{(c_2-c_1)}{c_2} I_{2,c_2}(u^{c_2},v^{c_2}) + O\left((c_2-c_1)^2\right),
\end{aligned}$$

which implies the desired result,

$$d(c_1) \le d(c_2) - \Big(\frac{c_2-c_1}{c_2}\Big) I_{2,c_2}(u^{c_2},v^{c_2}) + o(c_2-c_1).$$

Now, let (z,w) be defined by $(z,w)=t(u^{c_1},v^{c_1})$. As before, we want t such that $K_{c_2}(z,w)=0$. In this case,

$$t = 1 - \frac{2(c_2-c_1)}{3c_1}\Big(\frac{I_{2,c_1}(u^{c_1},v^{c_1})}{G(u^{c_1},v^{c_1})}\Big) = 1 + \frac{(c_2-c_1)}{3c_1}\Big(\frac{I_{2,c_1}(u^{c_1},v^{c_1})}{d(c_1)}\Big).$$

Since $K_{c_1}(z,w)=0$, we see that

$$d(c_2) \le J_{c_2}(z,w) = t^2\Big(d(c_1) + \frac{c_2-c_1}{3c_1} I_{2,c_1}(u^{c_1},v^{c_1})\Big).$$

Then, as above, we have that

$$t^2 = 1 + \frac{2(c_2-c_1)}{3c_1}\Big(\frac{I_{2,c_1}(u^{c_1},v^{c_1})}{d(c_1)}\Big) + O\left((c_2-c_1)^2\right).$$

Using this we conclude that

$$\begin{aligned} & t^2\Big(d(c_1) + \frac{(c_2-c_1)}{3c_1} I_{2,c_1}(u^{c_1}, v^{c_1})\Big) \\ & = d(c_1) + \frac{(c_2-c_1)}{c_1} I_{2,c_1}(u^{c_1}, v^{c_1}) + O\left((c_2-c_1)^2\right), \end{aligned}$$

which implies the other inequality.

(3) Assume that $K_{c_1}(u^{c_1}, v^{c_1}) = 0$. Hence we see that $G(u^{c_1}, v^{c_1}) \le 0$. Now, if $I_{2,c_1}(u^{c_1}, v^{c_1}) \le 0$ then for $c_1 < c_2$ we have that

$$K_{c_2}(u^{c_1}, v^{c_1}) = K_{c_1}(u^{c_1}, v^{c_1}) + \frac{2(c_2-c_1)}{c_1} I_{2,c_1}(u^{c_1}, v^{c_1}) \le 0.$$

Thus, we obtain

$$\begin{aligned} d(c_2) &\le \frac{1}{3} I_{c_2}(u^{c_1}, v^{c_1}) \\ &= \frac{1}{3}\Big(I_{c_1}(u^{c_1}, v^{c_1}) + \frac{c_2-c_1}{c_1} I_{2,c_1}(u^{c_1}, v^{c_1})\Big) \\ &\le d(c_1) + \frac{c_2-c_1}{3c_1} I_{2,c_1}(u^{c_1}, v^{c_1}). \end{aligned}$$

This also implies that $d(c_2) < d(c_2)$, provided that $0 < c_1 < c_2 < 1$. □

Convexity of d**.** Now, we prove that the function d is strictly convex on $(c_0, 1)$ with $c_0 > 0$ near 1. To do this, we compute d' and analyze the behavior of d and d' near 1^-. We have the following results.

Lemma 3.4. *If $(u^c, v^c) \in \mathcal{G}_c$, then we have that*

$$d'(c) = \frac{I_{2,c}(u^c, v^c)}{c}. \tag{3.6}$$

Proof. Note that d' can be computed by taking approptiate limits in part 2 of Lemma 3.3 □

Theorem 3.5. *Let $\sigma > 3/8$ and $(u^c, v^c) \in \mathcal{G}_c$. Then we have that*

$$\lim_{c\to 1^-} d(c) = 0 \quad \textit{and} \quad I_{2,c}(u^c, v^c) < 0 \quad \textit{for cnear } 1^-.$$

Proof. From Equations (2.4)-(2.6) and (3.5) we obtain the first part. Now, using the same notation as Section 2.2 we have

$$\begin{aligned} \epsilon I^{2,\epsilon}(z^\epsilon, w^\epsilon) &= -c\int_{\mathbb{R}^2} \Big(2z^\epsilon \partial_x w^\epsilon + \epsilon \partial_x z^\epsilon \left(\partial_{xx} w^\epsilon + \epsilon \partial_{yy} w^\epsilon\right)\Big)\, dx\, dy \\ &= -2c\int_{\mathbb{R}^2} \left(z^\epsilon - \partial_x w^\epsilon\right) \partial_x w^\epsilon \, dx\, dy \\ &\quad - c\epsilon \int_{\mathbb{R}^2} \partial_x z^\epsilon \left(\partial_{xx} w^\epsilon + \epsilon z^\epsilon \partial_{yy} w^\epsilon\right)\, dx\, dy - 2c\int_{\mathbb{R}^2} \left(\partial_x w^\epsilon\right)^2\, dx\, dy. \end{aligned}$$

Then using Lemma 2.5 we see that

$$\lim_{\epsilon\to 0^+} \epsilon I^{2,\epsilon}(z^\epsilon, w^\epsilon) < 0,$$

meaning that for ϵ near 0^+ we have $I^{2,\epsilon}(z^\epsilon, w^\epsilon) < 0$, which implies that for c near 1^-, we ahve $I_{2,c}(u_c, v_c) < 0$. □

Theorem 3.6. *Let $\sigma > 3/8$. Then there exist $0 < c_0 < 1$ enough near 1 such that d is a decreasing function on $(c_0, 1)$. Furthermore, $\lim_{c\to 1^-} d'(c) = 0$.*

Proof. Using (3.6) and Theorem 3.5 we have that d is a decreasing function for c near 1^- and we also have that $\lim_{c\to 1^-} \|(u^c, v^c)\|_{\mathcal{X}} = 0$ for any $(u^c, v^c) \in \mathcal{X}$ such that $d(c) = \frac{1}{3} I_c(u^c, v^c)$, since from (2.3) we see that

$$\|(u^c, v^c)\|_{\mathcal{X}}^2 \le C(\sigma) I_c(u^c, v^c) = C(\sigma) d(c).$$

Thus, from (3.6) and definition of $I_{2,c}$ we conclude that

$$|d'(c)| \le 2\|u^c\|_{L^2(\mathbb{R}^2)} \|v_x^c\|_{L^2(\mathbb{R}^2)} + \|u_x^c\|_{L^2(\mathbb{R}^2)} \|\Delta v^c\|_{L^2(\mathbb{R}^2)} \le 3\|(u^c, v^c)\|_{\mathcal{X}}^2.$$

Therefore, $\lim_{c\to 1^-} d'(c) = 0$. □

From the previous results we have the following lemma.

Lemma 3.7. *Let $\sigma > 3/8$, then d and d_1 are strictly convex for c near 1^-.*

We will use the following result by Shatah [14].

Lemma 3.8. *Suppose that h is a strictly convex function in a neighborhood of c_0. Then given $\varepsilon > 0$, there exist $N(\varepsilon) > 0$ such that for $|c_\varepsilon - c_0| = \varepsilon$,*

(1) *If $c_\varepsilon < c_0 < c$ and $|c - c_0| < \varepsilon/2$,*
$$\frac{h(c_\varepsilon) - h(c)}{c_\varepsilon - c} \le \frac{h(c_0) - h(c)}{c_0 - c} - \frac{1}{N(\varepsilon)}.$$
(2) *If $c < c_0 < c_\varepsilon$ and $|c - c_0| < \varepsilon/2$,*
$$\frac{h(c_\varepsilon) - h(c)}{c_\varepsilon - c} \ge \frac{h(c_0) - h(c)}{c_0 - c} + \frac{1}{N(\varepsilon)}.$$

Theorem 3.9. *Let $\sigma > 3/8$. If $0 < c_0 < 1$ with c_0 near 1 and $(u^{c_0}, v^{c_0}) \in \mathcal{G}_{c_0}$, then for c close to c_0, there exist $\rho(c) > 0$ such that $\rho(c_0) = 0$ and*

$$d(c) - d(c_0) \ge \Big(\frac{c - c_0}{c_0}\Big) I_{2,c_0}(u^{c_0}, v^{c_0}) + \rho(c).$$

Proof. Let $c < c_0$, c close to c_0. Then by Lemma 3.8, for $c < c_0 < c_1$ we see that

$$\frac{d(c) - d(c_1)}{c - c_1} \le \frac{d(c_0) - d(c_1)}{c_0 - c_1} - \frac{1}{N(c)}.$$

From Lemma 3.3 we have

$$d(c_1) \le d(c_0) + \Big(\frac{c_1 - c_0}{c_0}\Big) I_{2,c_0}(u^{c_0}, v^{c_0}) + o(c_1 - c_0).$$

Then we obtain

$$\frac{d(c) - d(c_1)}{c - c_1} \le \frac{d(c_1) - d(c_0)}{c_1 - c_0} - \frac{1}{N(c)} \le \frac{I_{2,c_0}(u^{c_0}, v^{c_0})}{c_0} + \frac{o(c_1 - c_0)}{c_1 - c_0} - \frac{1}{N(c)}.$$

Using the continuity of d, we have as $c_1 \to c_0$ that

$$\frac{d(c) - d(c_0)}{c - c_0} \le \frac{I_{2,c_0}(u^{c_0}, v^{c_0})}{c_0} - \frac{1}{N(c)}.$$

As a consequence of this inequality follows that

$$d(c) - d(c_0) \ge \Big(\frac{c - c_0}{c_0}\Big) I_{2,c_0}(u^{c_0}, v^{c_0}) + \frac{c_0 - c}{N(c)}.$$

Now, let $c_0 < c$ be c close to c_0. If $c_1 < c_0 < c$, then by using Lemma 3.8,

$$\frac{d(c)-d(c_1)}{c-c_1} \geq \frac{d(c_0)-d(c_1)}{c_0-c_1} + \frac{1}{N(c)}.$$

Then from Lemma 3.3,

$$d(c_1) \leq d(c_0) - \Big(\frac{c_0-c_1}{c_0}\Big) I_{2,c_0}(u^{c_0}, v^{c_0}) + o(c_1-c_0).$$

Thus, we obtain

$$\frac{d(c)-d(c_1)}{c-c_1} \geq \frac{d(c_1)-d(c_0)}{c_1-c_0} + \frac{1}{N(c)} \geq \frac{I_{2,c_0}(u^{c_0}, v^{c_0})}{c_0} + \frac{o(c_1-c_0)}{c_1-c_0} + \frac{1}{N(c)}.$$

Again, using the continuity of d, we have as $c_1 \to c_0$ that

$$\frac{d(c)-d(c_0)}{c-c_0} \geq \frac{I_{2,c_0}(u^{c_0}, v^{c_0})}{c_0} + \frac{1}{N(c)}.$$

As a consequence of this inequality holds

$$d(c) - d(c_0) \geq \Big(\frac{c-c_0}{c_0}\Big) I_{2,c_0}(u^{c_0}, v^{c_0}) + \frac{c-c_0}{N(c)},$$

and the result follows. □

4. Orbital stability of the solitary waves

We first consider the modulated system associated with the system (2.1) on $\mathcal{X}$. In other words, we assume that the solution $(\eta(t), \Phi(t))$ of the system (1.1) has the form

$$\eta(t,x,y) = z(t, x-ct, y), \quad \Phi(t,x,y) = w(t, x-ct, y)$$

Then we see that $(z(t), w(t))$ satisfies the modulated system

$$\begin{gathered} \Big(I - \frac{1}{2}\Delta\Big) z_t - c\Big(I - \frac{1}{2}\Delta\Big) z_x - \frac{2}{3}\Delta^2 w + \Delta w + \nabla\cdot(z\nabla w) = 0, \\ \Big(I - \frac{1}{2}\Delta\Big) w_t - c\Big(I - \frac{1}{2}\Delta\Big) w_x + z - \sigma\Delta z + \frac{1}{2}|\nabla w|^2 = 0. \end{gathered} \tag{4.1}$$

Observe that the modulated Hamiltonian for this system has the form

$$\mathcal{H}_c(z,w) = \frac{1}{2} J_c(z,w) = \mathcal{H}(z,w) + \frac{1}{2} I_{2,c}(z,w),$$

We also observe that $\mathcal{H}_c$ is conserved in time on solutions since

$$\begin{gathered} \Big(I - \frac{1}{2}\Delta\Big) z_t = \partial_w \mathcal{H}_c(z,w) = c\Big(I - \frac{1}{2}\Delta\Big) z_x + \frac{2}{3}\Delta^2 w - \Delta w - \nabla\cdot(z\nabla w), \\ -\Big(I - \frac{1}{2}\Delta\Big) w_t = \partial_z \mathcal{H}_c(z,w) = -c\Big(I - \frac{1}{2}\Delta\Big) w_x + z - \sigma\Delta z + \frac{1}{2}|\nabla w|^2. \end{gathered}$$

Now we introduce the regions $\mathcal{R}_c^i, i = 1, 2$, in the energy space $\mathcal{X}$ by

$$\begin{gathered} \mathcal{R}_c^1 = \{(z,w) \in \mathcal{X} : \mathcal{H}_c(z,w) < \frac{1}{2} d(c),\ \frac{1}{3} I_c(z,w) < d(c)\} \\ \mathcal{R}_c^2 = \{(z,w) \in \mathcal{X} : \mathcal{H}_c(z,w) < \frac{1}{2} d(c),\ \frac{1}{3} I_c(z,w) > d(c)\}, \end{gathered}$$

and have the following result.

Lemma 4.1. *$\mathcal{R}_c^1, \mathcal{R}_c^2$ are invariant regions under the flow for the modulated system* (4.1).

Proof. Let $(u_0, v_0) \in \mathcal{R}_c^1$. Suppose that $(z(t), w(t))$ satisfies the modulated system (4.1) with initial condition

$$z(0) = u_0, \quad w(0) = v_0.$$

By characterization of $d(c)$ and definition of $\mathcal{R}_c^1$, we must have that

$$K_c(u_0, v_0) > 0.$$

In fact, suppose that $K_c(u_0, v_0) \le 0$. Then we see that $d(c) \le \frac{1}{3} I_c(u_0, v_0)$. Moreover, if $(z(t), w(t)) \in \mathcal{R}_c^1$ for some $t > 0$, we have that $K_c(z(t), w(t)) > 0$. Now, suppose that there exists a minimum t_0 such that $K_c(z(t), w(t)) > 0$ for $t \in [0, t_0)$ and $K_c(z(t_0), w(t_0)) = 0$. Observe that

$$\begin{aligned}
d(c) &\le \frac{1}{3} I_c(z(t_0), w(t_0)) \\
&\le \liminf_{t \to t_0^-} \Big(\frac{1}{3} I_c(z(t), w(t)) + \frac{1}{3} K_c(z(t), w(t)) \Big) \\
&\le \liminf_{t \to t_0^-} J_c(z(t), w(t)) \\
&\le 2 \liminf_{t \to t_0^-} \mathcal{H}_c(z(t), w(t)) \\
&\le 2\mathcal{H}_c(z_0, w_0) < d(c).
\end{aligned}$$

This is a contradiction, which shows that $\mathcal{R}_c^1$ is invariant under the flow for the modulated system (4.1). In a similar fashion we have that $\mathcal{R}_c^2$ is also invariant under the flow for the modulated system (4.1). □

The following lemma will be used to obtain stability with respect to the ground state solutions.

Lemma 4.2. *Let $\sigma > 3/8$ and $0 < c_0 < 1$ be near 1. If $U(t) = (\eta(t), \Phi(t))$ is a global solution of the Boussinesq-Benney-Luke system (1.1) with initial condition $U(0) = U_0 \in \mathcal{X}$, then for every M, there is $\delta(M)$ such that if*

$$\|U_0 - U^{c_0}\|_{\mathcal{X}} < \delta(M).$$

Then we have

$$d\Big(c_0 + \frac{1}{M}\Big) \le \frac{1}{3} I_{c_0}(U(t)) \le d\Big(c_0 - \frac{1}{M}\Big), \quad \text{for all } t \in \mathbb{R}.$$

Proof. Let M be fixed and define $c_1 = c_0 - \frac{1}{M}$ and $c_2 = c_0 + \frac{1}{M}$. Now, let $(z^i(t), w^i(t))$ be defined by the formulas

$$\eta(t, x, y) = z^i(t, x - c_i t, y), \quad \Phi(t, x, y) = w^i(t, x - c_i t, y), \quad i = 1, 2.$$

Then the couple $(z^i(t), w^i(t))$ satisfies the modulated system (4.1) with initial condition

$$(z^i(0), w^i(0)) = U(0).$$

For this solution we have that the modulated Hamiltonian is conserved in time, in other words

$$\mathcal{H}_{c_i}(U(t)) = \mathcal{H}_{c_i}(U(0)).$$

Now, using hypothesis and inequality (2.3) we conclude for small δ that

$$I_{c_i}(U^{c_0}) = I_{c_i}(U(0)) + O(\delta).$$

Since d is a strictly decreasing function such that $d(c_0) = \frac{1}{3} I_{c_0}(U^{c_0})$, we can choose δ small enough in such a way that

$$d(c_2) < \frac{1}{3} I_{c_0}(U(0)) < d(c_1).$$

We also have that

$$\begin{aligned} J_{c_i}(U(0)) &= J_{c_i}(U^{c_0}) + O(\delta) \\ &= J_{c_0}(U^{c_0}) + \frac{c_i - c_0}{c_0} I_{2,c_0}(U^{c_0}) + O(\delta) \\ &= d(c_0) + \frac{c_i - c_0}{c_0} I_{2,c_0}(U^{c_0}) + O(\delta) \\ &\le d(c_i) - \rho(c_i) + O(\delta), \end{aligned}$$

where we have make used of Theorem 3.9. Next, let δ be small enough such that

$$2\delta < \min\{\rho\big(c_0 - \frac{1}{M}\big), \rho\big(c_0 + \frac{1}{M}\big)\}.$$

This implies

$$2\mathcal{H}_{c_i}(U(0)) = J_{c_i}(U(0)) < d(c_i). \tag{4.2}$$

Then, using Lemma 4.1, we have for all $t \in \mathbb{R}$ that

$$\mathcal{H}_{c_i}(U(t)) < \frac{1}{2} d(c_i), \quad d\Big(c_0 + \frac{1}{M}\Big) \le \frac{1}{3} I_{c_0}(U(t)) \le d\Big(c_0 - \frac{1}{M}\Big).$$

□

Finally we establish the main result in this work.

Theorem 4.3 (Orbital stability)**.** *Let* $\sigma > 3/8$ *and* $0 < c_0 < 1$ *be near 1. Then the ground state solitary wave solutions of the Boussinesq-Benney-Luke system* (1.1) *are stable in the following sense: Given* $\varepsilon > 0$*, there exist* $\delta(\varepsilon) > 0$ *such that if* $U_0 \in \mathcal{X}$ *satisfies*

$$\|U_0 - U^{c_0}\|_{\mathcal{X}} < \delta(\varepsilon),$$

then there exist a unique solution $U(t)$ *of the Boussinesq-Benney-Luke system* (1.1) *with initial condition* U_0 *such that*

$$\inf_{V \in \mathcal{G}_{c_0}} \|U(t) - V\|_{\mathcal{X}} < \varepsilon, \quad \text{for all } t \in \mathbb{R}.$$

Proof. We will argue by contradiction. Suppose that there exist a positive number ε_0, and sequences $\{t_k\} \subset \mathbb{R}$ and $\{U_0^k\} \subset \mathcal{X}$, such that

$$\lim_{k\to\infty} \|U_0^k - U^{c_0}\|_{\mathcal{X}} = 0, \quad \inf_{V \in \mathcal{G}_{c_0}} \|U^k(t_k) - V\|_{\mathcal{X}} > \varepsilon_0,$$

where U^k denotes the unique solution of system (1.1) with initial condition $U^k(0) = U_0^k$. Now, from the Lemma 4.2 and the assumption, given $m > 0$ we have the existence of $\delta(m)$ and a subsequence k_m such that

$$\|U_0^{k_m} - U^{c_0}\|_{\mathcal{X}} < \delta(m)$$

and

$$d\Big(c_0 + \frac{1}{k_m}\Big) \le \frac{1}{3} I_{c_0}\left(U^{k_m}(t_{k_m})\right) \le d\Big(c_0 - \frac{1}{k_m}\Big),$$

meaning that there exist a subsequence of $\{U^k(t_k)\}$, which we denote the same, such that

$$d\Big(c_0+\frac{1}{k}\Big) \le \frac{1}{3} I_{c_0}\left(U^k(t_k)\right) \le d\Big(c_0-\frac{1}{k}\Big).$$

In particular, we have that

$$\frac{1}{3} I_{c_0}\left(U^k(t_k)\right) \to d(c_0) \quad \text{as} \quad k\to\infty.$$

Now, we consider $c_2 = c_0 + \frac{1}{k}$ and $V^{k,2}(t)$ defined by

$$U^k(t,x,y) = V^{k,2}(t, x-c_2t, y).$$

Then as in proof of previous lemma (see (4.2)), we obtain that

$$2\mathcal{H}_{c_2}\left(U^k(t_k)\right) = J_{c_2}\left(U^k(t_k)\right) < d(c_2) < d(c_0) < d\Big(c_0-\frac{1}{k}\Big).$$

On the other hand,

$$\begin{aligned} J_{c_2}\left(U^k(t_k)\right) &= J_{c_0}\left(U^k(t_k)\right) + \Big(\frac{c_2-c_0}{c_0}\Big) I_{2,c_0}\left(U^k(t_k)\right) \\ &= J_{c_0}\left(U^k(t_k)\right) + \Big(\frac{1}{kc_0}\Big) I_{2,c_0}\left(U^k(t_k)\right). \end{aligned}$$

But note that

$$\lim_{k\to\infty}\Big(\frac{1}{kc_0}\Big)\left|I_{2,c_0}\left(U^k(t_k)\right)\right| \le \lim_{k\to\infty}\frac{1}{k}\|U^k(t_k)\|^2_{\mathcal{X}} \le \lim_{k\to\infty}\Big(\frac{1}{k}C\Big) = 0,$$

since we have that

$$\|U^k(t_k)\|^2_{\mathcal{X}} \cong \frac{1}{3} I_{2,c_0}\left(U^k(t_k)\right) \to d(c_0).$$

Using these facts, we conclude that

$$J_{c_0}\left(U^k(t_k)\right) \to \tilde{d} \le d(c_0).$$

Then by Corollary 3.2, there exist $U_{c_0} \in \mathcal{G}_{c_0}$ such that as $k\to\infty$,

$$U^k(t_k) \to U_{c_0} \quad \text{in } \mathcal{X}, \quad \frac{1}{3} I_{c_0}\left(U^k(t_k)\right) \to d(c_0) = \tilde{d},$$

also $J_{c_0}\left(U^k(t_k)\right) \to d(c_0)$. But this contradicts the assumption of instability

$$\inf_{V\in\mathcal{G}_{c_0}} \|U^k(t_k) - V\|_{\mathcal{X}} > \varepsilon_0.$$

□

Acknowledgments. A. M. Montes was supported by the Universidad del Cauca (Colombia) under the project I.D. 3982. J. R. Quintero was supported by the Mathematics Department at Universidad del Valle (Colombia) under the project CI 7001. A. M. and J. Q. are supported by Colciencias grant No 42878.

References

[1] J. Bona, P. Souganidis, W. Strauss; *Stability and instability of solitary waves of Korteweg de Vries type equation*. Proc. Roy. Soc. London. Ser A 411 (1987), 395-412.

[2] A. de Bouard, J. C. Saut; *Remarks on the stability of generalized KP solitary waves*. Comtemp. Math. 200 (1996), 75-84.

[3] R. Fukuizumi; *Stability and instability of standing waves for the nonlinear Schrödinger equation with harmonic potential*. Discrete contin. Dynam. Syst. 7 (2001), 525-544.

[4] M. Grillakis, J. Shatah, W. Strauss; *Stability Theory of Solitary Waves in Presence of Symmetry, I.* Functional Anal. 74 (1987), 160-197.

[5] Y. Liu, X-P. Wang; *Nonlinear stablitity of solitary waves of a generalized Kadomtsev-Petviashvili equation*. Comm. Math. Phys. 183 (1997), 253-266.

[6] A. M. Montes; *Boussinesq-Benney-Luke type systems related with water wave models*. Doctoral Thesis, Universidad del Valle (Colombia), 2013.

[7] A. M. Montes, J. R. Quintero; *On the nonlinear scattering and well-posedness for a 2D-Boussinesq-Benney-Luke type system*. Submmited to Journal Diff. Eq. (2013).

[8] J. R. Quintero; *Nonlinear stability of a one-dimensional Boussinesq equation*. Dynamic of PDE 15 (2003), 125-142.

[9] J. R. Quintero; *Nonlinear stability of solitary waves for a 2-D Benney-Luke equation*. Discrete Contin. Dynam. Systems 13 (2005), 203-218.

[10] J. R. Quintero; *The Cauchy problem and stability of solitary waves for a 2D Boussinesq-KdV type system*. Diff. and Int. Eq. 23 (2010), 325-360.

[11] J. R. Quintero, A. M. Montes; *Existence, physical sense and analyticity of solitons for a 2D Boussinesq-Benney-Luke System*. Dynamics of PDE 10 (2013), 313-342.

[12] J. R. Quintero, J. C. Muñoz; *Solitons and nonlinear stability/instability for a 1D Benney-Luke-Paumond Equation*. In preparation (2014).

[13] J. R. Quintero, R. L. Pego; *Two-dimensional solitary waves for a Benney-Luke equation*. Physica D 45 (1999), 476-496.

[14] J. Shatah; *Stable standing waves of nonlinear Klein-Gordon equations*. Comm. Math. Phys. A. 91 (1983), 313-327.

Alex M. Montes

Universidad del Cauca, Carrera 3 3N-100, Popayán, Colombia

E-mail address: amontes@unicauca.edu.co

José R. Quintero

Universidad del Valle, A. A. 25360, Cali, Colombia

E-mail address: jose.quintero@correounivalle.edu.co