Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 160, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

CONTINUITY OF THE FREE BOUNDARY IN ELLIPTIC PROBLEMS WITH NEUMAN BOUNDARY CONDITION

ABDERACHID SAADI

Abstract. We show the continuity of the free boundary in a class of two dimensional free boundary problems with Neuman boundary condition, which includes the aluminium electrolysis problem and the heterogeneous dam problem with leaky boundary condition.

1. Statement of the problem and preliminary results

Let Ω be the open bounded domain of $\mathbb{R}^2$ defined by

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in (a_0, b_0),\ d_0 < x_2 < \gamma(x_1)\}$$

where a_0, b_0, d_0 are real numbers and γ is a real-valued Lipschitz continuous function on (a_0, b_0). Let $a(x) = (a_{ij}(x))$ be a two-by-two matrix and h a function defined in Ω with

$$a_{ij} \in L^\infty(\Omega), \quad |a(x)| \le \Lambda, \quad \text{for a.e. } x \in \Omega, \tag{1.1}$$

$$a(x)\xi \cdot \xi \ge \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^2, \quad \text{for a.e. } x \in \Omega, \tag{1.2}$$

$$\underline{h} \le h(x) \le \bar{h} \quad \text{for a.e. } x \in \Omega \tag{1.3}$$

$$h_{x_2} \in L^p_{\rm loc}(\Omega) \tag{1.4}$$

$$h_{x_2}(x) \ge 0 \quad \text{for a.e. } x \in \Omega. \tag{1.5}$$

where λ, Λ, $\bar{h}, \underline{h}$ and p are positive constants such that $\bar{h} \ge \underline{h}$ and $p > 2$.

Let $\Gamma = \{(x_1, \gamma(x_1)) : x_1 \in (a_0, b_0)\}$ and let $\beta(x, u)$ be a function defined on $\Gamma \times \mathbb{R}$ satisfying

$$\beta(x, .) \text{ is Lipschitz continuous for a.e. } x \in \Gamma \tag{1.6}$$

$$\beta(x, .) \text{ is non-decreasing for a.e. } x \in \Gamma. \tag{1.7}$$

Let φ be a Lipschitz continuous function on Γ, e the vector $(0, 1)$, and $\Upsilon = \partial\Omega \setminus \Gamma$. Then we consider the following problem

Problem (P1) Find $(u, \chi) \in H^1(\Omega) \times L^\infty(\Omega)$ such that

(i) $u \ge 0$, $0 \le \chi \le 1$, $u(1 - \chi) = 0$ a.e. in Ω,

2010 *Mathematics Subject Classification.* 35J15, 35R35.
Key words and phrases. Continuity; free boundary; Neuman boundary condition.

Submitted May 13, 2015. Published June 16, 2015.

(ii)
$$\int_\Omega \big(a(x)\nabla u + \chi h(x)e\big)\cdot\nabla\xi dx \le \int_\Gamma \beta(x,\varphi-u)\xi d\sigma(x)$$
for all $\xi \in H^1(\Omega)$ and $\xi \ge 0$ on Υ.

This problem describes many free boundary problems including the aluminium electrolysis problem [5], the heterogeneous dam problem with leaky boundary condition [7, 9, 10, 18, 19, 20, 23]. For the problem with Dirichlet condition on Γ, we refer for example to [1] and [21] in the case of the heterogeneous dam problem, to [3] and [4] in the case of the lubrication problem, and to [11, 13, 14] for a more general framework. Regarding the existence of a solution under suitable boundary conditions, we refer for example to [8, 6, 9, 10, 18, 23].

In this paper, we shall be interested in studying the free boundary $\Gamma_f = \partial\{u > 0\}\cap\Omega$ separating two different regions, which in the case of the dam and lubrication problems, separates the region that contains the fluid from the rest of the domain. In the case of the aluminium electrolysis problem, the free boundary separates the regions containing liquid and solid aluminium.

The regularity of Γ_f has been addressed in the case of Dirichlet boundary condition in [11] and [13], where the authors have established that Γ_f is a continuous curve $x_2 = \Phi(x_1)$. This result was later on extended in [14] to a more general framework and also in [15] in the case of the $p-$Laplacian.

2. Preliminary results

Remark 2.1. By Harnack's inequality [17], we know that u is locally bounded. Due to the local character of this study, we shall assume that there exists a positive constant M such that
$$0 \leqslant u \leqslant M \quad \text{a.e. in } \Omega. \tag{2.1}$$

Remark 2.2. We have (see [13, Remark 2.1])

- (i) $u \in C^{0,\alpha}_{\rm loc}(\Omega)$ for some $\alpha \in (0,1)$. As a consequence the set $\{u > 0\}$ is open.
- (ii) If $a \in C^{0,\alpha}_{\rm loc}(\Omega)$ $(0 < \alpha < 1)$, then we have $u \in C^{1,\alpha}_{\rm loc}(\{u > 0\})$.

The following three propositions were established in [11] where the Dirichlet condition $u = 0$ was imposed on Γ instead of the Neuman boundary condition that we are considering in this work. The proofs are the same and will be omitted.

Proposition 2.3. *Let (u,χ) be a solution of* (P1)*. We have*
$$\chi_{x_2} \le 0 \quad in \quad \mathcal{D}'(\Omega). \tag{2.2}$$

Proposition 2.4. *Let (u,χ) be a solution of* (P1) *and $x_0 = (x_{01}, x_{02}) \in \Omega$.*

- (i) *If $u(x_0) > 0$, then there exists $\varepsilon > 0$ such that $u(x_1,x_2) > 0$ for all $(x_1,x_2) \in C_\varepsilon(x_0) = B_\varepsilon(x_0)\cup\{(x_1,x_2)\in\Omega : |x_2 - x_{02}| < \varepsilon,\, x_2 < x_{02}\}$, where $B_\varepsilon(x_0)$ is the open ball of center x_0 and radius r.*
- (ii) *If $u(x_0) = 0$, then $u(x_{01}, x_2) = 0$ for all $x_2 \ge x_{02}$.*

We then define the function Φ by
$$\Phi(x_1) = \begin{cases} d_0 & \text{if } \{x_2 : (x_1,x_2)\in\Omega,\, u(x_1,x_2) > 0\} = \emptyset \\ \sup\{x_2 : (x_1,x_2)\in\Omega,\, u(x_1,x_2) > 0\} & \text{otherwise.}\end{cases} \tag{2.3}$$
Then the function Φ is well defined.

Proposition 2.5. *Φ is lower semi-continuous on (a_0, b_0) and*

$$\{u > 0\} = \{x_2 < \Phi(x_1)\}.$$

The following lemma is an extension of [11, Lemma 3.4].

Lemma 2.6. *Let (u, χ) be a solution of (P1). Let $(x_{11}, \underline{x}_2), (x_{12}, \underline{x}_2) \in \Omega$ with $x_{11} < x_{12}$ and $u(x_{1i}, \underline{x}_2) = 0$ for $i = 1, 2$. Let $D = \big((x_{12}, x_{22}) \times (\underline{x}_2, +\infty)\big) \cap \Omega$. Then we have*

$$\int_D \big(a(x)\nabla u + \chi h(x)e\big) \cdot \nabla\zeta dx \le \int_\Gamma \beta(x, \varphi - u)\zeta d\sigma(x)$$
$$\forall \zeta \in H^1(D) \cap L^\infty(D),\ \zeta \ge 0,\ \zeta(x_1, \underline{x}_2) = 0 \ a.e.\ x_1 \in (x_{11}, x_{12}).$$

Proof. For $\epsilon > 0$ small enough, one sets:

$$\alpha_\epsilon(x_1) = \min\Big(1, \frac{(x_1 - x_{11})^+}{\epsilon}, \frac{(x_{12} - x_1)^+}{\epsilon}\Big).$$

Note that

$$\alpha_\epsilon(x_1) = \begin{cases} \frac{x_1 - x_{11}}{\epsilon} & \text{for } x_1 \in (x_{11}, x_{11} + \epsilon) \\ 1 & \text{for } x_1 \in (x_{11} + \epsilon, x_{12} - \epsilon) \\ \frac{x_{12} - x_1}{\epsilon} & \text{for } x_1 \in (x_{12} - \epsilon, x_{12}) \end{cases} \tag{2.4}$$

Then $\chi(D)\alpha_\epsilon\zeta$ is a test function for (P), and we have:

$$\int_D \big(a(x)\nabla u + \chi h(x)e\big) \cdot \nabla(\alpha_\epsilon\zeta) dx \le \int_{\partial D \cap \Gamma} \beta(x, \varphi - u)\alpha_\epsilon\zeta d\sigma(x) \tag{2.5}$$

We set $\xi_\epsilon = (1 - \alpha_\epsilon)\zeta$, and for $\delta > 0$, we denote by H_δ the following approximation of the Heaviside function i.e. the function defined by

$$H_\delta(s) = \min\Big(1, \frac{s^+}{\delta}\Big) = \begin{cases} 1 & \text{for} s \ge \delta \\ s/\delta & \text{for } 0 \le s \le \delta \\ 0 & \text{for } s \le 0 \end{cases} \tag{2.6}$$

Then $\chi(D)H_\delta(u)\xi_\epsilon$ is a test function for (P1), and we have

$$\int_D \big(a(x)\nabla u + \chi h(x)e\big) \cdot \nabla(H_\delta(u)\xi_\epsilon) dx \le \int_{\partial D \cap \Gamma} \beta(x, \varphi - u)H_\delta(u)\xi_\epsilon d\sigma(x)$$

or

$$\begin{aligned} &\int_D \big(H_\delta(u)a(x)\nabla u\nabla\xi_\epsilon + H'_\delta(u)a(x)\nabla u\nabla u + \chi h(x)(H_\delta(u)\xi_\epsilon)_{x_2}\big) dx \\ &\le \int_{\partial D \cap \Gamma} \beta(x, \varphi - u)H_\delta(u)\xi_\epsilon d\sigma(x). \end{aligned}$$

which leads by (1.2) and the monotonicity of H_δ to

$$\int_D \big(H_\delta(u)a(x)\nabla u\nabla\xi_\epsilon + \chi h(x)(H_\delta(u)\xi_\epsilon)_{x_2}\big) dx \le \int_{\partial D \cap \Gamma} \beta(x, \varphi - u)H_\delta(u)\xi_\epsilon d\sigma(x)$$

Hence we have

$$\begin{aligned} &\int_D H_\delta(u)a(x)\nabla u\nabla\xi_\epsilon dx + \int_{D \cap \{u > 0\}} \big((h.H_\delta(u)\xi_\epsilon)_{x_2} - h_{x_2}H_\delta(u)\xi_\epsilon\big) dx \\ &\le \int_{\partial D \cap \Gamma} \beta(x, \varphi - u)H_\delta(u)\xi_\epsilon d\sigma(x). \end{aligned} \tag{2.7}$$

Since

$$\int_{D\cap\{u>0\}}(h.H_\delta(u)\xi_\epsilon)_{x_2}dx=\int_{\partial D\cap\Gamma}\big(h.H_\delta(u)\xi_\epsilon\big)\nu_2 d\sigma(x)\geq 0,$$

it follows from (2.7) that

$$\int_D H_\delta(u)a(x)\nabla u\nabla\xi_\epsilon dx-\int_{D\cap\{u>0\}}h_{x_2}H_\delta(u)\xi_\epsilon dx\leq\int_{\partial D\cap\Gamma}\beta(x,\varphi-u)H_\delta(u)\xi_\epsilon d\sigma(x). \tag{2.8}$$

Letting δ go to 0 in (2.8), we obtain

$$\begin{aligned}&\int_D a(x)\nabla u\nabla((1-\alpha_\epsilon)\zeta)dx\\&\leq\int_{D\cap\{u>0\}}h_{x_2}(1-\alpha_\epsilon)\zeta dx+\int_{\partial D\cap\Gamma}\beta(x,\varphi-u)(1-\alpha_\epsilon)\zeta d\sigma(x).\end{aligned}\tag{2.9}$$

Now, from (2.5) and (2.9), we deduce that

$$\begin{aligned}&\int_D\big(a(x)\nabla u+\chi h(x)e\big)\cdot\nabla\zeta dx\\&=\int_D\big(a(x)\nabla u+\chi h(x)e\big)\cdot\nabla(\alpha_\epsilon\zeta)dx\\&\quad+\int_D\big(a(x)\nabla u+\chi h(x)e\big)\nabla((1-\alpha_\epsilon)\zeta)dx\\&\leq\int_{\partial D\cap\Gamma}\beta(x,\varphi-u)(\alpha_\epsilon\zeta)d\sigma(x)+\int_D\chi h(x)(1-\alpha_\epsilon)\zeta_{x_2}dx\\&\quad+\int_{D\cap\{u>0\}}h_{x_2}(1-\alpha_\epsilon)\zeta dx+\int_{\partial D\cap\Gamma}\beta(x,\varphi-u)(1-\alpha_\epsilon)\zeta d\sigma(x)\\&\leq\int_D\chi h(x)(1-\alpha_\epsilon)\zeta_{x_2}dx+\int_{D\cap\{u>0\}}h_{x_2}(1-\alpha_\epsilon)\zeta dx\\&\quad+\int_{\partial D\cap\Gamma}\beta(x,\varphi-u)\zeta d\sigma(x).\end{aligned}$$

Taking into account (2.4), the result follows by letting ϵ approach 0. □

Proposition 2.7. *Let (u,χ) be a solution of* (P1) *and $B_r(x_0)\subset\Omega$. If $u=0$ in $B_r(x_0)$, then we have*

$$\chi(x_1,x_2)=\frac{\beta((x_1,\gamma(x_1)),\varphi(x_1,\gamma(x_1)))}{h(x)\nu_2(x_1,\gamma(x_1))}\quad\textit{for a.e. } (x_1,x_2)\in C_r(x_0).$$

Proof. Since $u(x)=0$ in $B_r(x_0)$, we obtain by Proposition 2.4

$$u=0\quad\text{in } C_r(x_0).$$

Moreover, since we have in the distributional sense

$$\operatorname{div}\big(a(x)\nabla u+\chi h(x)e\big)=0\quad\text{in } C_r(x_0),$$

we obtain in particular

$$\big(\chi h(x)\big)_{x_2}=0\quad\text{in } \mathcal{D}'(C_r(x_0)). \tag{2.10}$$

Let $\xi \in H^1(C_r(x_0))$ such that $\xi = 0$ on $\partial C_r(x_0) \cap \Omega$. Then $\pm\chi(C_r(x_0))\xi$ are test functions for (P1), and we have

$$\int_{C_r(x_0)} \chi h \xi_{x_2} dx = \int_{\partial C_r(x_0)\cap\Gamma} \beta(x,\varphi)\xi d\sigma(x). \tag{2.11}$$

Integrating by parts and using (2.10), we obtain

$$\int_{C_r(x_0)} \chi h \xi_{x_2} dx = \int_{\partial C_r(x_0)\cap\Gamma} \chi h \nu_2 \xi d\sigma(x). \tag{2.12}$$

We deduce then from (2.11) and (2.12) that

$$\int_{\partial C_r(x_0)\cap\Gamma} \chi h \nu_2 \xi d\sigma(x) = \int_{\partial C_r(x_0)\cap\Gamma} \beta(x,\varphi)\xi d\sigma(x),$$

for all $\xi \in H^1(C_r(x_0))$, $\xi = 0$ on $\partial C_r(x_0) \cap \Omega$, which leads to $\chi h \nu_2 = \beta(x,\varphi)$, or

$$\chi(x) = \frac{\beta((x_1,\gamma(x_1)),\varphi(x_1,\gamma(x_1)))}{h(x)\nu_2(x_1,\gamma(x_1))} \quad \text{a.e. in } C_r(x_0).$$

□

Proposition 2.8. *Let* (u,χ) *be a solution of* (P1). *If the Lebesgue measure of the free boundary is zero, then we have*

$$\chi = \chi_{\{u>0\}} + \frac{\beta(x,\varphi)}{h\nu_2}\chi_{\{u=0\}} \quad \textit{for a.e. } x \in \Omega.$$

Proof. From (P1)(i), we know that

$$\chi = 1 \quad \text{a.e. in } \{u > 0\}. \tag{2.13}$$

Let $x \in Int(\{u = 0\})$. Then there exists a ball $B_r(x)$ of center x and radius r such that $B_r(x) \subset \{u = 0\}$, i.e. $u = 0$ in $B_r(x)$. By Proposition 2.7, we have $\chi = \frac{\beta(x,\varphi)}{h\nu_2}$ a.e. in $B_r(x)$. Therefore

$$\chi = \frac{\beta(x,\varphi)}{h\nu_2} \quad \text{a.e. in } \operatorname{Int}(\{u = 0\}). \tag{2.14}$$

Since $\partial\{u > 0\} \cap \Omega$ is of measure zero, from (2.13)-(2.14) we obtain

$$\chi = \chi_{\{u>0\}} + \frac{\beta(x,\varphi)}{h\cdot\nu_2}\chi_{\{u=0\}} \quad \text{a.e. in } \Omega.$$

□

Proposition 2.9. *Let* (u,χ) *be a solution of* (P1). *If* $x_0 = (x_{01}, x_{02}) \in \Omega$ *and* $r > 0$ *are such that* $B_r(x_0) \subset \Omega$*, then we cannot have the following situations in* $B_r(x_0)$*:*

(i) $u(x) = 0$ *for* $x_1 = x_{01}$ *and* $u(x) > 0$ *for* $x_1 \neq x_{01}$.
(ii) $u(x) = 0$ *for* $x_1 \geq x_{01}$ *and* $u(x) > 0$ *for* $x_1 < x_{01}$.
(iii) $u(x) > 0$ *for* $x_1 > x_{01}$ *and* $u(x) = 0$ *for* $x_1 \leq x_{01}$.

Proof. Let $\xi \in \mathcal{D}(B_r), \xi \geq 0$. Since $\pm\xi$ are test functions for (P1), we have

$$\int_{B_r(x_0)} \big(a(x)\nabla u + \chi h(x)e\big)\nabla\xi dx = 0. \tag{2.15}$$

(i) Under assumption (i), we have $\chi = 1$ a.e. in $B_r(x_0)$. Taking into account (1.5), from (2.15) we obtain

$$\int_{B_r(x_0)} a(x)\nabla u\nabla\xi dx = -\int_{B_r(x_0)} h(x)\xi_{x_2} \geq 0$$

Using the maximum principle, we deduce that either $u > 0$ in $B_r(x_0)$ or $u = 0$ in $B_r(x_0)$, which both contradict (i).

(ii) From (P1)(i) and Proposition 2.8, we have under the assumption (ii), $\chi = 1$ a.e. in $B_r^- = B_r(x_0)\cap\{x_1 < x_{01}\}$ and $\chi = \frac{\beta(x,\varphi)}{h\nu_2}$ a.e. in $B_r^+ = B_r(x_0)\cap\{x_1 > x_{01}\}$. Then from (2.15), it follows that

$$\begin{aligned} \int_{B_r^-} a(x)\nabla u\nabla\xi dx &= -\int_{B_r^-} \chi h(x)\xi_{x_2} dx - \int_{B_r^+} \chi h(x)\xi_{x_2} dx \\ &= -\int_{B_r^-} h(x)\xi_{x_2} dx - \int_{B_r^+} \frac{\beta(x,\varphi)}{\nu_2}\xi_{x_2} dx. \end{aligned} \tag{2.16}$$

Integrating by parts, we have

$$\int_{B_r^+} \frac{\beta(x,\varphi)}{\nu_2}\xi_{x_2} dx = \int_{B_r^+} \big(\frac{\beta(x,\varphi)}{\nu_2}\xi\big)_{x_2} dx = 0. \tag{2.17}$$

It follows from (2.16)-(2.17) and taking into account (1.5) that

$$\int_{B_r(x_0)} a(x)\nabla u\nabla\xi dx = -\int_{B_r^-} h\xi_{x_2} = \int_{B_r^-} h_{x_2}\xi \geq 0.$$

We deduce from the maximum principle that either $u > 0$ in $B_r(x_0)$ or $u = 0$ in $B_r(x_0)$, which both contradict (ii).

The proof (iii) is similar to the proof of (ii), an it is omitted. □

Theorem 2.10. *Let (u,χ) be a solution of* (P1)*. Then we have*

$$\chi \geq \kappa = \min\Big(1, \frac{\beta(x,\varphi)}{h\nu_2}\Big) \quad \text{a.e. in } \Omega.$$

To prove the above theorem we need the following lemma.

Lemma 2.11. *Let $(a,b) \subset (a_0,b_0)$, y_0 such that $(a,b)\times\{y_0\} \subset \Omega$ and let $D = ((a,b)\times(y_0,\infty))\cap\Omega$. Then we have*

$$\int_D h(\kappa-\chi)^+\xi_{x_2} dx \leq 0, \quad \forall\xi \in H^1(D),\ \xi \geq 0,\ \xi = 0 \text{ on } (\partial D)\cap\Omega.$$

Proof. Let $\Gamma' = \{(x_1,\gamma(x_1)) : x_1 \in (a,b)\}$, and $\xi \in H^1(D)$ such that $\xi \geq 0$ and $\xi = 0$ on $(\partial D)\cap\Omega$. Using $\pm\chi(D)(H_\delta(u)-1)\xi$ as test functions for (P1), we obtain

$$\int_D \big(a(x)\nabla u + \chi he\big)\nabla\big((H_\delta(u)-1)\xi\big) dx = \int_{\Gamma'} \beta(x,\varphi-u)(H_\delta(u)-1)\xi d\sigma(x)$$

which can be written as

$$\begin{aligned} &\int_D H_\delta'(u)a(x)\nabla u\cdot\nabla u.\xi dx + \int_D (H_\delta(u)-1)a(x)\nabla u\nabla\xi dx \\ &+ \int_D \chi h[H_\delta(u)\xi]_{x_2} dx - \int_D \chi h(x)\xi_{x_2} dx \\ &\leq \int_{\Gamma'} \beta(x,\varphi-u)(H_\delta(u)-1)\xi d\sigma(x)\,. \end{aligned}$$

By taking into account the monotonicity of H_δ, integrating by parts, and using the ellipticity of $a(x)$, we have

$$\begin{aligned}
-\int_D \chi h\xi_{x_2}dx &\le \int_D (1-H_\delta(u))a(x)\nabla u\nabla\xi dx - \int_D h[H_\delta(u)\xi]_{x_2}dx\\
&\quad + \int_{\Gamma'}\beta(x,\varphi-u)(H_\delta(u)-1)\xi d\sigma(x)\\
&= \int_D (1-H_\delta(u))a(x)\nabla u\cdot\nabla\xi dx + \int_D H_\delta(u)h_{x_2}\xi dx\\
&\quad + \int_{\Gamma'}(H_\delta(u)-1)[\beta(x,\varphi-u)-h\nu_2]\xi d\sigma(x) - \int_{\Gamma'} h\nu_2\xi d\sigma(x).
\end{aligned} \tag{2.18}$$

Next, integrating by parts again, we have

$$\int_D \kappa h\xi_{x_2}dx = \int_{\Gamma'}\kappa h\nu_2\xi d\sigma(x) - \int_D (\kappa h)_{x_2}\xi dx. \tag{2.19}$$

Adding (2.18) and (2.19), and using the fact that $(\kappa h)_{x_2} = h_{x_2}\chi_{\{\beta(x,\varphi)\ge h\cdot\nu_2\}} \ge 0$, we obtain

$$\begin{aligned}
\int_D (\kappa-\chi)h\xi_{x_2}dx &\le \int_D (1-H_\delta(u))a(x)\nabla u\cdot\nabla\xi\, dx + \int_D H_\delta(u)h_{x_2}(x)\xi dx\\
&\quad + \int_{\Gamma'}(H_\delta(u)-1)[\beta(x,\varphi-u)-h\nu_2]\xi d\sigma(x)\\
&\quad + \int_{\Gamma'}(\kappa-1)h\nu_2\xi d\sigma(x).
\end{aligned} \tag{2.20}$$

Letting δ go to 0 in (2.20), we obtain

$$\begin{aligned}
\int_D h(\kappa-\chi)\xi_{x_2}dx &\le -\int_{\Gamma'\cap\{u=0\}}(\beta(x,\varphi)-h\nu_2]\xi d\sigma(x)\\
&\quad + \int_D \chi_{\{u>0\}}h_{x_2}\xi dx + \int_{\Gamma'}(\kappa-1)h\nu_2\xi d\sigma(x)
\end{aligned}$$

which leads to

$$\begin{aligned}
&\int_D h(\kappa-\chi)\xi_{x_2}dx\\
&\le \int_D \chi_{\{u>0\}}h_{x_2}\xi dx + \int_{\Gamma'\cap\{u=0\}}\Big(\kappa - \frac{\beta(x,\varphi)}{h\nu_2}\Big)h\nu_2\xi d\sigma(x)\\
&\quad + \int_{\Gamma'\cap\{u>0\}}(\kappa-1)h\nu_2\xi d\sigma(x)\\
&\le \int_D \chi_{\{u>0\}}h_{x_2}\xi dx \quad \forall\xi\in H^1(D),\quad \xi\ge 0,\ \xi=0 \text{ on } \partial D\cap\Omega.
\end{aligned}$$

Hence we arrive at

$$\int_D (h\kappa - h\chi)\xi_{x_2}dx \le \int_D \chi_{\{u>0\}}h_{x_2}\xi dx, \tag{2.21}$$

for all $\xi\in H^1(D)$, $\xi\ge 0$, $\xi=0$ on $\partial D\cap\Omega$.

Now we extend the functions $h\kappa$, $h\chi$, and $\chi_{\{u>0\}}h_{x_2}$ by 0 and denote the extensions respectively by $\overline{\kappa}$, $\overline{\chi}$, and θ. In particular (2.21) holds for any $\xi\in C^{0,1}(\mathbb{R}^2)$, $\xi\ge$

0, with ξ having a compact support in $D\cup\Gamma'$. Let ξ be such a function and let

$$\epsilon_0=\operatorname{dist}(\operatorname{supp}(\xi_{|\overline{D}}),\partial D\cap\Omega).$$

Then for each $y\in B_\epsilon(O)$, and $\epsilon\in(0,\epsilon_0/2)$, the function $x\to\xi(x+y)$ is nonnegative, belongs to $C^{0,1}(\mathbb{R}^2)$, and has a compact support in $D\cup\Gamma'$. Therefore we obtain from (2.21)

$$\int_{\mathbb{R}^2}(\overline{\kappa}-\overline{\chi})\xi_{x_2}(x+y)dx\le\int_{\mathbb{R}^2}\theta(x)\xi(x+y)dx$$

which leads to

$$\int_{\mathbb{R}^2}\rho_\epsilon(y)\Big(\int_{\mathbb{R}^2}(\overline{\kappa}-\overline{\chi}).\xi_{x_2}(x+y)dx\Big)dy\le\int_{\mathbb{R}^2}\rho_\epsilon(y)\Big(\int_{\mathbb{R}^2}\theta(x)\xi(x+y)dx\Big)dy$$

where ρ_ϵ is a smooth function satisfying $\rho_\epsilon\ge0, supp\rho_\epsilon\subset B_\epsilon(O)$ and $\int_{\mathbb{R}^2}\rho_\epsilon=1$.

Writing $f_\epsilon=\rho_\epsilon*f$ for a function f, we obtain

$$\int_{\mathbb{R}^2}(\overline{\kappa}_\epsilon(x)-\overline{\chi}_\epsilon(x))\xi_{x_2}dx\le\int_{\mathbb{R}^2}\theta_\epsilon(x)\xi dx\quad\forall\xi\in C^{0,1}(\mathbb{R}^2),\ \xi\ge0,;\operatorname{supp}(\xi)\subset D\cup\Gamma'.$$

In particular, we obtain for the function $\xi=\min\big(1,\frac{(\overline{\kappa}_\epsilon-\overline{\chi}_\epsilon)^+}{\delta}\big)\zeta$, with $\delta>0$, $\zeta\in C^{0,1}(\mathbb{R}^2)$, $\zeta\ge0$, $\operatorname{supp}(\zeta)\subset D\cup\Gamma'$,

$$\begin{aligned}&\int_{\mathbb{R}^2}(\overline{\kappa}_\varepsilon-\overline{\chi}_\varepsilon)\Big(\min\Big(1,\frac{(\overline{\kappa}_\varepsilon-\overline{\chi}_\varepsilon)^+}{\delta}\Big)\zeta\Big)_{x_2}dx\\&\le\int_{\mathbb{R}^2}\theta_\varepsilon(x)\min\Big(1,\frac{(\overline{\kappa}_\varepsilon-\overline{\chi}_\varepsilon)^+}{\delta}\Big)\zeta dx\\&\le\int_{\mathbb{R}^2}\theta_\varepsilon(x)\zeta dx.\end{aligned}\tag{2.22}$$

Note that

$$\begin{aligned}&\int_{\mathbb{R}^2}(\overline{\kappa}_\varepsilon-\overline{\chi}_\varepsilon)\Big(\min\Big(1,\frac{(\overline{\kappa}_\varepsilon-\overline{\chi}_\varepsilon)^+}{\delta}\Big)\zeta\Big)_{x_2}dx\\&=\int_{\mathbb{R}^2}\min\Big(1,\frac{(\overline{\kappa}_\epsilon-\overline{\chi}_\epsilon)^+}{\delta}\Big)(\overline{\kappa}_\epsilon-\overline{\chi}_\epsilon)\zeta_{x_2}dx\\&\quad+\int_{\mathbb{R}^2}(\overline{\kappa}_\epsilon-\overline{\chi}_\epsilon)\Big(\min\Big(1,\frac{(\overline{\kappa}_\epsilon-\overline{\chi}_\epsilon)^+}{\delta}\Big)\Big)_{x_2}\zeta dx.\end{aligned}\tag{2.23}$$

As $\delta\to0$, we have

$$\int_{\mathbb{R}^2}\min\Big(1,\frac{(\overline{\kappa}_\epsilon-\overline{\chi}_\epsilon)^+}{\delta}\Big)(\overline{\kappa}_\epsilon-\overline{\chi}_\epsilon)\zeta_{x_2}dx\to\int_{\mathbb{R}^2}(\overline{\kappa}_\epsilon-\overline{\chi}_\epsilon)^+\zeta_{x_2}dx\tag{2.24}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^2}(\overline{\kappa}_\epsilon-\overline{\chi}_\epsilon)\Big(\min\Big(1,\frac{(\overline{\kappa}_\epsilon-\overline{\chi}_\epsilon)^+}{\delta}\Big)\Big)_{x_2}\zeta dx\\&=\int_{\mathbb{R}^2\cap\{0<\overline{\kappa}_\epsilon-\overline{\chi}_\epsilon<\delta\}}\frac{\zeta}{\delta}(\overline{\kappa}_\epsilon-\overline{\chi}_\epsilon)^+(\overline{\kappa}_\varepsilon-\overline{\chi}_\varepsilon)_{x_2}dx\\&=\int_{\mathbb{R}^2}\frac{\zeta}{2\delta}((\min(\delta,(\overline{\kappa}_\epsilon-\overline{\chi}_\epsilon)^+))^2)_{x_2}dx\\&=-\int_{\mathbb{R}^2}(\min(\delta,(\overline{\kappa}_\epsilon-\overline{\chi}_\epsilon)^+))^2\frac{\zeta_{x_2}}{2\delta}dx\to0.\end{aligned}\tag{2.25}$$

It follows from (2.22)-(2.25) that
$$\int_{\mathbb{R}^2}(\overline{\kappa}_\epsilon-\overline{\chi}_\epsilon)^+\zeta_{x_2}dx\le\int_{\mathbb{R}^2}\theta_\epsilon\zeta dx$$
which by letting $\epsilon\to 0$ leads to
$$\int_D(\kappa-\chi)^+h\zeta_{x_2}dx\le\int_D\chi_{\{u>0\}}h_{x_2}\zeta dx \tag{2.26}$$
Taking $\zeta=(1-H_\delta(u))\zeta$ with $\zeta\in H^1(\mathbb{R}^2)$, $\zeta\ge 0$, $\operatorname{supp}(\zeta)\subset D\cup\Gamma'$ in (2.26), we obtain
$$\begin{aligned}&\int_D h(\kappa-\chi)^+\zeta_{x_2}dx\\&\le\int_D(1-H_\delta(u))\chi_{\{u>0\}}h_{x_2}\zeta dx+\int_D\zeta h(\kappa-\chi)^+(H_\delta(u))_{x_2}dx\end{aligned}\tag{2.27}$$
Since $\chi=1$ a.e. in $\{u>0\}$ and $\kappa\le 1$, we have
$$\int_D\zeta h(\kappa-\chi)^+(H_\delta(u))_{x_2}dx=\int_{D\cap\{u>0\}}\zeta h(\kappa-1)^+(H_\delta(u))_{x_2}dx=0. \tag{2.28}$$
Then we deduce from (2.27)–(2.28) that
$$\int_D h(\kappa-\chi)^+\zeta_{x_2}dx\le\int_D(1-H_\delta(u))\chi_{\{u>0\}}h_{x_2}\zeta dx. \tag{2.29}$$
Letting δ go to 0 in (2.29), we obtain
$$\int_D h(\kappa-\chi)^+\zeta_{x_2}dx\le 0\quad\forall\zeta\in H^1(D),\ \zeta\ge 0,\ \zeta=0\text{ on }\partial D\cap\Omega$$
which completes the proof of the lemma. □

Proof of Theorem 2.10. Let D be a domain below Γ given by $D=((a,b))\times(y_0,\infty))\cap\Omega$, with $\{y_0\}\times(a,b)\subset\Omega$. By Lemma 2.11, we have for $\zeta=l(x_1)(y-y_0)\chi(D)$, where $l(x_1)=(x_1-a)(b-x_1)$
$$\int_D h(\kappa-\chi)^+(x_1-a)(b-x_1)dx\le 0$$
which leads to $(\kappa-\chi)^+=0$ a.e. in D and $\kappa\le\chi$ a.e. in D. Since this holds for all such domains D, we obtain $\chi\ge\kappa$ a.e in Ω. □

Corollary 2.12. *Let (u,χ) be a solution of* (P1) *and let $\Gamma'=\{(x_1,\gamma(x_1)),x_1\in(a,b)\}\subset\Gamma$, with $a_0<a<b<b_0$. Assume that*
$$\frac{\beta(x,\varphi)}{h\nu_2}>1\qquad a.e.\ in\ \Gamma'.$$
Then we have $u>0$ and $\chi=1$ below Γ'.

Proof. Let $D=((a,b))\times(y_0,\infty))\cap\Omega$ such that $\{y_0\}\times(a,b)\subset\Omega$. We deduce from Theorem 2.10 and the assumption that $\chi=1$ a.e. in D. Using (P1)(ii) and (1.5), we obtain $\operatorname{div}(a(x)\nabla u)=-h_{x_2}\le 0$ in D, which leads by the maximum principle to either $u=0$ in D or $u>0$ in D.

Now, assume that $u=0$ in D and let $\zeta\in H^1(D)$ with $\zeta\ge 0$ in D and $\zeta=0$ on $\partial D\cap\Omega$. Since $\pm\chi(D)\xi$ are a test functions for (P1), we have
$$\int_D h\zeta_{x_2}=\int_{\Gamma'}\beta(x,\varphi)\zeta d\sigma(x)$$

which becomes after integrating by parts and taking into account (1.5)

$$\int_{\Gamma'} (\beta(x,\varphi) - h\nu_2)\zeta d\sigma(x) = -\int_D \zeta h_{x_2} \le 0.$$

Since ζ is arbitrary, we obtain $\beta(x,\varphi) - h\nu_2 \le 0$ a.e. in Γ', which contradicts the assumption. $\square$

3. A barrier function

For the rest of this article, we assume that

$$a \in C^{0,\alpha}_{\rm loc}(\Omega), \quad \alpha \in (0,1), \tag{3.1}$$

$$h_{x_2} \le c_0 \quad \text{in } \mathcal{D}'(\Omega). \tag{3.2}$$

In this section, we construct a barrier function that will be used in Section 4 to establish the continuity of the function ϕ.

Theorem 3.1. *Let (u,χ) be a solution of* (P1). *Then $u \in C^{0,1}_{\rm loc}(\Omega)$.*

Proof. We refer to [13] when $H(x) = h(x)e$, and to [14] or [22] for more general situations.

Let $(x_{11}, \underline{x}_2)$, $(x_{12}, \underline{x}_2) \in \Omega$ such that $x_{11} < x_{12}$. We assume that $\epsilon = x_{12} - x_{11}$ is small enough to guarantee that

$$(x_{11} - \epsilon, x_{12} + \epsilon) \times (\underline{x}_2, \underline{x}_2 + 2\epsilon) \subset\subset \Omega.$$

Let $Z = (x_{11} - \epsilon, x_{12} + \epsilon) \times (\underline{x}_2, \underline{x}_2 + \epsilon)$. We denote by v the unique solution in $H^1(Z)$ of the problem

$$\begin{gathered} \operatorname{div}(a(x)\nabla v) = -h_{x_2} \quad \text{in } Z \\ v = \varphi(x) = \epsilon(\underline{x}_2 + \epsilon - x_2)^+ \quad \text{on } \partial Z. \end{gathered} \tag{3.3}$$

$\square$

Remark 3.2. Under the above assumptions on a and h, we deduce from (3.3) (see remark subsequent to [17, Corollary 8.36 p. 212]) that $v \in C^{1,\alpha}_{\rm loc}\big(Z \cup (x_{11} - \epsilon, x_{12} + \epsilon) \times \{\underline{x}_2 + \epsilon\}\big)$. Moreover, we have for a positive constant C independent of ϵ, the following gradient estimate as in [13]:

$$|\nabla v(x)| \le C\epsilon^{(1-\frac{2}{p})} \quad \forall x \in T = [x_{11}, x_{12}] \times \{\underline{x}_2 + \epsilon\}. \tag{3.4}$$

Let us extend the function v by 0 to $D = \big((x_{11}, x_{12}) \times (\underline{x}_2, +\infty)\big) \cap \Omega$, and let $\theta = \chi(\{v > 0\}) + \frac{\beta(x,\varphi)}{h\nu_2}\chi(\{v = 0\})$. The main result of this section is the following Lemma.

Lemma 3.3. *Assume that for some positive number μ we have*

$$h(x) - \frac{\beta(x,\varphi(x))}{\nu_2} > \mu \quad \text{on } T. \tag{3.5}$$

Then for $\epsilon > 0$ small enough we have

$$\begin{gathered} \int_D \Big(a(x)\nabla v + \theta h(x)e\Big) \cdot \nabla\zeta \ge \int_\Gamma \beta(x,\varphi) d\sigma(x) \\ \forall \zeta \in H^1(D),\ \zeta \ge 0,\ \zeta = 0 \text{ on } \partial D \setminus \Gamma. \end{gathered} \tag{3.6}$$

Proof. Let ν be the outward unit normal vector to D. First we have by (1.1), (3.4) and (3.5) for ϵ small enough

$$a(x)\nabla v.\nu + h(x) - \frac{\beta(x,\varphi)}{\nu_2} \geq -\Lambda C.\epsilon^{(1-\frac{2}{p})} + \mu \geq 0 \quad \text{on } T. \tag{3.7}$$

Next, for $\zeta \in H^1(D)$, $\zeta \geq 0$, $\zeta = 0$ on $\partial D \setminus \Gamma$, by integrating by parts and using (3.5) and (3.7) we obtain

$$\begin{aligned}
&\int_D \Big(a(x)\nabla v + \theta h(x)e\Big) \cdot \nabla\zeta dx \\
&= \int_{D\cap[v>0]} \Big(a(x)\nabla v + \theta h(x)e\Big) \cdot \nabla\zeta dx + \int_{D\cap[v=0]} \frac{\beta(x,\varphi)}{\nu_2}\zeta_{x_2} dx \\
&= -\int_{D\cap[v>0]} \Big(div(a(x)\nabla v) + h_{x_2}(x)\Big)\zeta dx + \int_\Gamma \beta(x,\varphi)\zeta d\sigma(x) \\
&\quad + \int_T \Big(a(x)\nabla v\cdot\nu + h(x) - \frac{\beta(x,\varphi)}{\nu_2}\Big)\zeta d\sigma \\
&\geq \int_\Gamma \beta(x,\varphi)\zeta d\sigma(x).
\end{aligned}$$

□

4. Continuity of the free boundary

This last section is devoted to the upper semi-continuity of ϕ. We assume that

$$\gamma \in C^{1,\alpha}_{\rm loc}(a_0, b_0), \tag{4.1}$$

$$a \in C^{0,\alpha}_{\rm loc}(\Omega \cup \Gamma), \tag{4.2}$$

$$\beta(x,\varphi) - h\nu_2 \in C^0_{\rm loc}(\Gamma). \tag{4.3}$$

The main result is the following theorem.

Theorem 4.1. *Let $x_{01} \in (a_0, b_0)$ such that $(x_{01}, \Phi(x_{01})) \in \Omega$ and*

$$\frac{\beta(x_{01}, \gamma(x_{01}), \varphi(x_{01}, \gamma(x_{01})))}{\nu_2(x_{01}, \gamma(x_{01}))} < h(x_{01}, \Phi(x_{01})). \tag{4.4}$$

Then Φ is continuous at x_{01}.

The proof of Theorem 4.1 is based on the following two lemmas and follows the steps of the one of [13, Theorem 5.1].

Lemma 4.2. *Let $x_0 = (x_{01}, x_{02}) \in \Omega$ such that $u(x_0) = 0$ and*

$$\frac{\beta(x_{01}, \gamma(x_{01}), \varphi(x_{01}, \gamma(x_{01})))}{\nu_2(x_{01}, \gamma(x_{01}))} < h(x_{01}, \Phi(x_{01})).$$

Then one of the following situations holds:

(i) *There exists $x_{11} < x_{01}$ such that $u(x_1, \gamma(x_1)) = 0$ for all $x_1 \in (x_{11}, x_{01})$.*
(ii) *There exists $x_{12} > x_{01}$ such that $u(x_1, \gamma(x_1)) = 0$ for all $x_1 \in (x_{01}, x_{12})$.*

Proof. Let $\mu > 0$ small enough and $Z_\mu = \big((x_{01} - \mu, x_{01} + \mu) \times (x_{02}, +\infty)\big) \cap \Omega$. We denote by v the unique solution in $H^1(Z_\mu)$ of

$$\begin{gathered}
\operatorname{div}(a(x)\nabla v) = -h_{x_2} \quad \text{in } Z_\mu \\
v = \varphi(x) = \gamma(x_1) - x_2 \quad \text{on } \partial Z_\mu.
\end{gathered} \tag{4.5}$$

We have (see [17, p. 212]) $v \in C^{1,\alpha}_{\rm loc}\big(Z_\mu \cup \Gamma_\mu\big)$, where $\Gamma_\mu = \{(x_1, \gamma(x_1)) \ / \ x_1 \in (x_{01}-\mu, x_{01}+\mu)\ \}$. Moreover, given the sign of the functions $\gamma(x_1) - x_2$ and h_{x_2}, we have by the maximum principle $v > 0$ in Z_μ.

Since $\beta(x,\varphi) < h(x)\nu_2$ at $\overline{x}_0 = (x_{01}, \gamma(x_{01})$, there exists by (4.2)-(4.3) a positive real number λ_ϵ such that

$$\beta(x,\varphi) < h(x)\nu_2 + \lambda_\epsilon a(x)(\nabla v)\cdot \nu \quad \text{on } \Gamma_{\mu/2}. \tag{4.6}$$

Since $u(x_0) = 0$, and u is continuous, there exists $\mu_1 \in (0, \mu/2)$ such that

$$u(x_1, x_2) \le \lambda_\epsilon \min_{x \in \overline{B}_{\mu/2}(x_0)} v(x) \quad \forall x \in \overline{B}_{\mu_1}(x_0). \tag{4.7}$$

By Proposition 2.9 one of the following situations holds

(i) there exists $(x_{11}, x_{21}) \in B_{\mu_1}(x_0)$ such that $x_{11} < x_{01}$ and $u(x_{11}, x_{21}) = 0$.

(ii) there exists $(x_{12}, x_{22}) \in B_{\mu_1}(x_0)$ such that $x_{11} > x_{01}$ and $u(x_{12}, x_{22}) = 0$.

Let us assume for example that (i) holds. Set $\underline{x}_2 = \max(x_{02}, x_{21})$, $Z = \big((x_{11}, x_{01}) \times (\underline{x}_2, +\infty)\big) \cap \Omega$, $v_\epsilon = \lambda_\epsilon v$ and assume that μ_1 is small enough. Since $(x_{11}, x_{01}) \times \{\underline{x}_2\} \subset \overline{B}_{\mu_1}(x_0)$, we have by (4.7)

$$u(x_1, \underline{x}_2) \le v_\epsilon(x_1, \underline{x}_2), \quad \forall x_1 \in (x_{11}, x_{01}) \tag{4.8}$$

Moreover, since $u(x_{11}, \underline{x}_2) = u(x_{01}, \underline{x}_2) = 0$, we obtain by Proposition 2.4 (ii) that

$$u(x_{11}, x_2) = u(x_{01}, x_2) = 0, \quad \forall x_2 \ge \underline{x}_2 \tag{4.9}$$

Taking into account (4.8)-(4.9), we have $u \le v_\epsilon$ on $(\partial Z) \cap \Omega$. Therefore we obtain by using $(u - v_\epsilon)^+ \chi_Z \in H^1(\Omega)$ as a test function in (P1) (ii) and (4.5)

$$\begin{aligned} &\int_Z (a(x)\nabla u + \chi h(x) e)\nabla(u - v_\epsilon)^+ dx \\ &= \int_{(\partial Z)\cap\Gamma} \beta(x, \varphi - u)(u - v_\epsilon)^+ d\sigma(x). \end{aligned} \tag{4.10}$$

$$\begin{aligned} &\int_Z (a(x)\nabla u + h(x) e)\nabla(u - v_\epsilon)^+ dx \\ &= \int_{(\partial Z)\cap\Gamma} (\lambda_\epsilon a(x)(\nabla v)\cdot\nu + h(x)\nu_2)(u - v_\epsilon)^+ d\sigma(x). \end{aligned} \tag{4.11}$$

Subtracting (4.11) from (4.10) and using (4.6) and the fact that $\chi h(x) e \nabla(u - v_\epsilon)^+ = h(x) e \nabla(u - v_\epsilon)^+$ a.e. in Z, we obtain

$$\begin{aligned} &\int_Z a(x)\nabla(u - v_\epsilon)^+ \cdot \nabla(u - v_\epsilon)^+ dx \\ &= \int_{(\partial Z)\cap\Gamma} \big(\beta(x, \varphi - u) - \lambda_\epsilon a(x)(\nabla v)\cdot\nu - h(x)\nu_2\big)(u - v_\epsilon)^+ d\sigma(x) \\ &\le \int_{(\partial Z)\cap\Gamma} \big(\beta(x, \varphi) - \lambda_\epsilon a(x)(\nabla v)\cdot\nu - h(x)\nu_2\big)(u - v_\epsilon)^+ d\sigma(x) \le 0. \end{aligned} \tag{4.12}$$

We deduce from (4.12) and (1.2) that $\nabla(u - v_\epsilon)^+ = 0$ a.e. in Z. Since $(u - v_\epsilon)^+ = 0$ on $(\partial Z) \cap \Omega$, we obtain $u \le v_\epsilon$ in Z and in particular $u(x_1, \gamma(x_1)) = 0 \quad \forall x_1 \in (x_{11}, x_{01})$.

If (ii) holds, in a similar way we obtain that $u(x_1, \gamma(x_1)) = 0 \quad \forall x_1 \in (x_{01}, x_{12})$. $\square$

Lemma 4.3. *Let $\epsilon > 0$ small enough and let v be the barrier function defined by* (3.3). *Assume that (u, χ) is a solution of* (P1) *such that:*

$$u(x_1, \underline{x}_2) \le v(x_1, \underline{x}_2) \forall x_1 \in (x_{11}, x_{12}) \tag{4.13}$$

$$u(x_{i1}, \underline{x}_2) = 0 \quad i = 1, 2 \tag{4.14}$$

$$u(x_1, \gamma(x_1)) = 0 \quad \forall x_1 \in (x_{11}, x_{12}). \tag{4.15}$$

Then for $\delta > 0$ and $D_\delta = D \cap \{v > 0\} \cap \{0 < u - v < \delta\}$ we have

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{D_\delta} a(x) \nabla (u - v)^+ . \nabla (u - v)^+ dx = 0.$$

Proof. Let $\delta, \eta > 0$. We consider the function $d_\eta(x_2) = H_\eta(x_2 - \overline{x}_2)$, where $H_\eta(s)$ was defined in (2.6) and $\overline{x}_2 = \underline{x}_2 + \varepsilon$. Then we introduce the non-negative function $\zeta = H_\delta(u - v) + d_\eta(1 - H_\delta(u))$ which belongs to $H^1(D) \cap L^\infty(D)$. Using the fact that $d_\eta(\underline{x}_2) = 0$ and (4.13), we see that ζ vanishes on $\{x = \underline{x}_2\}$. Therefore by Lemma 2.11 we have

$$\begin{aligned} &\int_D (a(x)\nabla u + \chi h(x) e) \nabla (H_\delta(u - v)) dx \\ &\le \int_\Gamma \beta(x, \varphi - u)(H_\delta(u - v) + d_\eta(1 - H_\delta(u))) d\sigma(x) \\ &\quad - \int_D (a(x)\nabla u + \chi h(x) e) \nabla (d_\eta(1 - H_\delta(u))) dx. \end{aligned} \tag{4.16}$$

Given that $u(x_{1i}, \underline{x}_2) = 0$ for $i = 1, 2$, from Proposition 2.4 (ii) we deduce that $u(x_{1i}, x_2) = 0, \forall x_2 \ge \underline{x}_2, i = 1, 2$. This leads to

$$u(x_{1i}, x_2) \le v(x_{1i}, x_2), \quad \forall x_2 \ge \underline{x}_2,\ i = 1, 2. \tag{4.17}$$

Combining (4.13) and (4.17), we obtain $u \le v$ on $\partial D \cap \Omega$, and therefore since $H_\delta(s) = 0$ for $s \le 0$, we obtain $H_\delta(u - v) = 0$ on $\partial D \cap \Omega = \partial D \setminus \Gamma$. So using $H_\delta(u - v)$ as a test function in (3.6) we obtain

$$-\int_D (a(x)\nabla v + \theta h(x) e) \nabla (H_\delta(u - v)) dx \le -\int_\Gamma \beta(x, \varphi) H_\delta(u - v) d\sigma(x) = 0. \tag{4.18}$$

Adding (4.16) and (4.18), we obtain by taking into account the fact that $u = 0$ on $\Gamma \cap \partial D$,

$$\begin{aligned} &\int_D a(x) \nabla (u - v) \nabla (H_\delta(u - v)) dx \\ &\le \int_D (\theta - \chi) h(x) (H_\delta(u - v))_{x_2} dx \\ &\quad - \int_D (a(x)\nabla u + \chi h(x) e) \nabla (d_\eta(1 - H_\delta(u - v))) dx \\ &\quad + \int_\Gamma \beta(x, \varphi) d_\eta (1 - H_\delta(u)) d\sigma(x)\,. \end{aligned}$$

Taking into account that $u=0$ on $\Gamma\cap\partial D$ and $d_\eta=0$ in $\{v>0\}$ the above inequality becomes

$$\begin{aligned}
&\int_{D\cap\{v>0\}} H'_\delta(u)a(x)\nabla(u-v)\nabla(u-v)dx\\
&\le -\int_{D\cap\{v=0\}} H'_\delta(u)a(x)\nabla u\nabla u - \int_{D\cap\{v=0\}} \chi h(x)(H_\delta(u))_{x_2}dx\\
&\quad+\int_{D\cap\{v=0\}} \frac{\beta(x,\varphi)}{\nu_2}(H_\delta(u))_{x_2}dx\\
&\quad+\int_{D\cap\{v=0\}} (a(x)\nabla u+\chi h(x)e)\nabla((1-d_\eta)(1-H_\delta(u)))dx\\
&\quad+\int_{D\cap\{v=0\}} (a(x)\nabla u+\chi h(x)e)\nabla(H_\delta(u))dx\\
&= I_1^\delta+I_2^\delta+I_3^\delta+I_4^\delta+I_5^\delta.
\end{aligned}\tag{4.19}$$

We observe that $I_1^\delta+I_2^\delta+I_5^\delta=0$. Moreover, integrating by parts, we have since $u=0$ on $\Gamma\cap\partial D$

$$\begin{aligned}
I_3^\delta &= \int_\Gamma \beta(x,\varphi)H_\delta(u)d\sigma(x) - \int_{\{x_2=\overline{x}_2\}} \frac{\beta(x,\varphi)}{\nu_2}(H_\delta(u))dx_1\\
&= -\int_{\{x_2=\overline{x}_2\}} \frac{\beta(x,\varphi)}{\nu_2}(H_\delta(u))dx_1 \le 0.
\end{aligned}\tag{4.20}$$

From (4.19)-(4.20) we obtain

$$\begin{aligned}
&\int_{D\cap\{v>0\}} H'_\delta(u)a(x)\nabla(u-v)\nabla(u-v)dx\\
&\le \int_{D\cap\{v=0\}} (a(x)\nabla u+\chi h(x)e)\nabla((1-d_\eta)(1-H_\delta(u)))dx.
\end{aligned}$$

At this point the proof follows step by step the one of [13, Lemma 5.1]. □

Proof of Theorem 4.1. Let $\epsilon>0$ be small enough. Let $x_{01}\in(a_0,b_0)$. Set $x_0=(x_{01},\phi(x_{01}))=(x_{01},x_{02})$ and assume that $x_0\in\Omega$. Using the continuity of $\beta(x,\varphi)-h(x)\nu_2$ at $\overline{x}_0=(x_{01},\gamma(x_{01}))$, there exists for ϵ small enough a positive number μ_ϵ such that

$$h(x)\nu_2-\beta(x,\varphi)>\mu_\epsilon \quad \text{on } \Gamma_\epsilon=\{(x_1,\gamma(x_1)) : x_1\in(x_{01}-\epsilon,x_{01}+\epsilon)\}. \tag{4.21}$$

Since $u(x_0)=0$ and u is continuous, there exists $\eta_1\in(0,\epsilon)$ such that

$$u(x_1,x_2)\le\epsilon^2 \quad \forall (x_1,x_2)\in B_{\eta_1}(x_0). \tag{4.22}$$

By Proposition 2.9, one of the following situations holds

(i) There exists $x_n=(x_{n1},x_{n2})\in B_{\eta_1}(x_0)$ such that $x_{n1}<x_{01}$, $u(x_{n1},x_{n2})=0$ and $\lim_{n\to\infty}x_n=x_0$.
(ii) there exists $x_n=(x_{n1},x_{n2})\in B_{\eta_1}(x_0)$ such that $x_{n1}>x_{01}$, $u(x_{n1},x_{n2})=0$ and $\lim_{n\to\infty}x_n=x_0$.

Let us assume that $i)$ holds. Then there exists by Lemma 4.2 a positive integer n large enough such that

$$u(x_1,\gamma(x_1))=0 \quad \forall x_1\in(x_{n1},x_{01}). \tag{4.23}$$

Set $\underline{x}_2 = \max(\phi(x_{01}), x_{n2})$ and assume that ϵ is small enough so that

$$(x_{n1} - \epsilon, x_{01} + \epsilon) \times (\underline{x}_2 - 2\epsilon, \underline{x}_2 + 2\epsilon) \Subset \Omega.$$

Let v_1 be the barrier function defined by (3.3) in the set $Z_1 = (x_{n1} - \epsilon, x_{01} + \epsilon) \times (\underline{x}_2, \underline{x}_2 + 2\epsilon)$. We consider the extension by 0 of v_1 to $D_1 = \big((x_{n1}, x_{01}) \times (\underline{x}_2, +\infty)\big) \cap \Omega$. Taking into account (4.4), we see that v_1 satisfies (3.6).

Now since $(x_{n1}, x_{01}) \times \{\underline{x}_2\} \subset B_{\eta_1}(x_0)$, by (4.22) we have

$$u(x_1, \underline{x}_2) \le \epsilon^2 = v_1(x_1, \underline{x}_2) \quad \forall x_1 \in (x_{n1}, x_{01}). \tag{4.24}$$

Moreover since $u(x_{n1}, \underline{x}_2) = u(x_{01}, \underline{x}_2) = 0$, by Proposition 2.4 (ii) we obtain

$$u(x_{n1}, x_2) = u(x_{01}, x_2) = 0 \quad \forall x_2 \ge \underline{x}_2. \tag{4.25}$$

Combining (4.23)-(4.25), we see that Lemma 4.3 holds for $D_1 = (x_{n1}, x_{01}) \times (\underline{x}_2, \underline{x}_2 + \epsilon)$. Then we can argue as in [13] to obtain for $\Delta_1 = (x_{n1}, x_{01}) \times (\underline{x}_2 - \epsilon, \underline{x}_2 + \epsilon)$

$$\int_{\Delta_1} a(x)\nabla(u - v_1)^+ . \nabla\zeta dx \,=\, 0 \quad \forall \zeta \in \mathcal{D}(\Delta_1)$$

which by (4.24) and the strong maximum principle leads to $(u - v_1)^+ \equiv 0$ in Δ_1. Consequently we have $u \le v_1$ in D_1 and in particular $u(x_1, \underline{x}_2 + \epsilon) = 0$ $\forall x_1 \in (x_{n1}, x_{01})$. Therefore

$$u(x_1, x_2) = 0 \quad \forall x_2 \ge \underline{x}_2 + \epsilon = \bar{x}_2, \quad \forall x_1 \in [x_{n1}, x_{01}].$$

Now, by continuity of u there exists $\eta_2 \in (0, x_{01} - x_{n1})$ such that

$$u(x_1, x_2) \le \epsilon^2 \quad \forall (x_1, x_2) \in B_{\eta_2}(x_{01}, \bar{x}_2).$$

By Proposition 2.9, there exists $(x_{m1}, x_{m2}) \in B_{\eta_2}(x_{01}, \bar{x}_2)$ such that

$$x_{m2} > \bar{x}_2, \quad x_{m1} > x_{01}, \quad u(x_{m1}, x_{m2}) = 0.$$

Set $\underline{x}'_2 = x_{m2}$ and assume that ϵ is small enough so that

$$(x_{n1} - \epsilon, x_{m1} + \epsilon) \times (\underline{x}'_2, \underline{x}'_2 + 2\epsilon) \Subset \Omega.$$

Let v_2 be the barrier function defined by (3.2) in the set $Z_2 = (x_{n1} - \epsilon, x_{m1} + \epsilon) \times (\underline{x}'_2, \underline{x}'_2 + \epsilon)$. Clearly the extension by 0 of v_2 to $D_2 = \big((x_{01}, x_{m1}) \times (\underline{x}'_2, +\infty)\big) \cap \Omega$ satisfies (3.6). Then, since $(x_{01}, x_{m1}) \times \{\underline{x}'_2\} \subset B_{\eta_2}(x_{01}, \bar{x}_2)$, we have

$$u(x_1, \underline{x}'_2) \le \epsilon^2 = v_2(x_1, \underline{x}'_2) \quad \forall x_1 \in (x_{01}, x_{m1}).$$

Arguing as above, we show that $(u - v_2)^+ \equiv 0$ in $D_2 \cap [v_2 > 0]$, which leads to

$$u(x_1, x_2) \equiv 0 \quad \forall x_2 \ge \underline{x}'_2 + \epsilon, \quad \forall x_1 \in [x_{01}, x_{m1}].$$

Hence we have

$$u(x_1, x_2) \equiv 0 \quad \forall x_2 \ge \underline{x}'_2 + \epsilon, \quad \forall x_1 \in [x_{n1}, x_{m1}].$$

Note that if (ii) holds, we argue similarly to obtain the same conclusion.

We have proved that for all $x_2 \in (x_{n1}, x_{m1})$,

$$\phi(x_1) \le \underline{x}'_2 + \epsilon < \bar{x}_2 + \eta_2 + \epsilon = \underline{x}_2 + \epsilon + \eta_2 + \epsilon < x_{02} + \eta_1 + \eta_2 + 2\epsilon < \phi(x_{01}) + 4\epsilon$$

which is the upper semi-continuity of ϕ at x_{01}. $\square$

References

[1] H. W. Alt; *Strömungen durch inhomogene poröse Medien mit freiem Rand*. Journal für die Reine und Angewandte Mathematik 305, (1979), 89-115.
[2] H. W. Alt; *The fluid flow Through Porous Media. Regularity of the free Surface*. Manuscripta Math. 21, (1977), 255-272.
[3] S. J. Alvarez, J. Carrillo; *A free boundary problem in theory of lubrication*. Comm. Partial Differential Equations 19 (1994), no. 11-12, 17431761.
[4] G. Bayada, M. Chambat; *Existence and uniqueness for a lubrication problem with nonregular conditions on the free boundary*. Boll. Un. Mat. Ital. B (6) 3 (1984), no. 2, 543-557.
[5] A. Bermúdez, M. C. Muñiz, P. Quintela; *Existence and uniqueness for a free boundary problem in aluminum electrolysis*. J. Math. Anal. Appl. 191, No. 3, 497-527, (1995).
[6] J. Carrillo, A. Alonso; *A unified formulation for the boundary conditions in some convection-diffusion problem*. Elliptic and parabolic problems (Pont-à-Mousson, 1994), 5163, Pitman Res. Notes Math. Ser., 325, Longman Sci. Tech., Harlow, 1995.
[7] J. Carrillo, M. Chipot; *On the Dam Problem*. J. Differential Equations 45, (1982), 234 - 271.
[8] J. Carrillo, M. Chipot; *The Dam Problem with Leaky Boundary conditions*. Applied Mathematics and Optimisation 28, (1993), 57-85.
[9] M. Chipot, A. Lyaghfouri; *The dam problem with linear Darcy's law and nonlinear leaky boundary conditions*. Advances in Differential Equations Vol. 3, No. 1, 1-50, (1998).
[10] M. Chipot, A. Lyaghfouri; *The dam problem with nonlinear Darcy's law and leaky boundary conditions*. Mathematical Methods in the Applied Sciences Vol. 20, No. 12, 1045-1068, (1997).
[11] M. Chipot; *On the Continuity of the Free Boundary in some Class of Dimensional Problems*. Interfaces and Free Boundaries. Vol. 3, No. 1, 81-99, (2001).
[12] M. Challal, A. Lyaghfouri; *A Filtration Problem through a Heterogeneous Porous Medium*. Interfaces and Free Boundaries 6, 55-79 (2004).
[13] S. Challal, A. Lyaghfouri; *On the Continuity of the Free Boundary in Problems of type* $\operatorname{div}(a(x)\nabla u) = -(\chi(u)h(x))_{x_1}$. Nonlinear Analysis : Theory, Methods & Applications, Vol. 62, No. 2, 283-300 (2005).
[14] S. Challal, A. Lyaghfouri; *On a class of Free Boundary Problems of type* $\operatorname{div}(a(X)\nabla u) = -\operatorname{div}(H(X)\chi(u))$. Differential and Integral Equations, Vol. 19, No. 5, 481-516 (2006).
[15] S. Challal, A. Lyaghfouri; *Continuity of the Free Boundary in the Problem* $\Delta_p u = -(h(x,y)\chi(u))_x$ *with* $h_x \geqslant 0$. Applicable Analysis, Vol. 86, No. 9, 1177-1184 (2007).
[16] S. Challal, A. Lyaghfouri; *The Heterogeneous Dam problem with Leaky Boundary Condition*. Communications in Pure and Applied Analysis. Vol. 10, No. 1, 93-125 (2011).
[17] D. Gilbarg, N.S. Trudinger; *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag 1983.
[18] A. Lyaghfouri; *A unified formulation for the dam problem*. Rivista di Matematica della Università di Parma. (6) 1, 113-148, (1998).
[19] A. Lyaghfouri; *On the uniqueness of the solution of a nonlinear filtration problem through a porous medium*. Calculus of Variations and Partial Differential Equations Vol. 6, No. 1, 67-94, (1998).
[20] A. Lyaghfouri; *A free boundary problem for a fluid flow in a heterogeneous porous medium*. Annali dell' Universita di Ferrara-Sez. VII-Sc. Mat., Vol. IL, 209-262 (2003).
[21] A. Lyaghfouri; *The Inhomogeneous Dam Problem with Linear Darcy's Law and Dirichlet Boundary Conditions*. Mathematical Models and Methods in Applied Sciences 8(6), (1996), 1051-1077.
[22] A. Lyaghfouri; *On the Lipschitz Continuity of the Solutions of a Class of Elliptic Free Boundary Problems*. Journal of Applied Analysis, Vol. 14, No. 2, 165-181 (2008).
[23] A. Lyaghfouri; *The dam Problem*. Handbook of Differential Equations, Stationary Partial Differential Equations, Vol. 3, ch. 06, 465-552 (2006).

Abderachid Saadi
Ecole Normale Supérieure, 16050 Kouba, Algiers, Algeria.
Université de M'sila, B.P 166 ICHBELIA, M'sila, Algeria
E-mail address: `rachidsaadi81@gmail.com`