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EXISTENCE OF INFINITELY MANY SYMMETRIC SOLUTIONS TO PERTURBED ELLIPTIC EQUATIONS WITH DISCONTINUOUS NONLINEARITIES IN $\mathbb{R}^N$

SHAPOUR HEIDARKHANI, FARIBA GHAREHGAZLOUEI, AREZOO SOLIMANINIA

Abstract. In this article we study the existence of infinitely many radially symmetric solutions for a class of perturbed elliptic equations with discontinuous nonlinearities in $\mathbb{R}^N$. We determine open intervals of positive parameters for which the problem admits infinitely many symmetric solutions. Our proofs are based on variational methods.

1. Introduction

We consider the perturbed elliptic problem

$$-\Delta_p u+|u|^{p-2}u=\lambda f(|x|,u)+\mu g(|x|,u), \quad x\in\mathbb{R}^N, \quad u\in W_r^{1,p}(\mathbb{R}^N) \tag{1.1}$$

where $\Delta_p u:=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\lambda>0$, $\mu\geq 0$, $2\leq N<p<+\infty$, the functions $f,g:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ are continuous almost everywhere. We recall that f is continuous almost everywhere if the set $D_f=\cup_{x\in\mathbb{R}^N}\{z\in\mathbb{R}: f(|x|,.)$ is discontinuous at $z\}$ has measure zero.

Since many free boundary problems and obstacle problems may be reduced to partial differential equations with discontinuous nonlinearities, as it arises in physics problems, such as nonlinear elasticity theory, mechanics and engineering topics, the existence of multiple solutions for Dirichlet boundary value problems with discontinuous nonlinearities has been widely investigated in recent years. Chang [4] extended the variational methods to a class of non-differentiable functionals, and applied directly the variational methods for non-differentiable functionals to prove some existence theorems for PDE with discontinuous nonlinearities. Later, Hu et al. in [6] obtained the existence of two solutions for an eigenvalue Dirichlet problem involving the p-Laplacian with discontinuous nonlinearities. Next, Motreanu and Panagiotopoulos [13, Chapter 3] studied the critical point theory for non-smooth functionals and in this framework, very recently, Marano and Motreanu [11] obtained an infinitely many critical points theorem, which extends the Variational Principle of Ricceri [15] to non-smooth functionals, and applies the result to variational-hemivariational inequalities and semilinear elliptic eigenvalue problems with discontinuous nonlinearities. In [2] Bonanno and Molica Bisci presented a more

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precise version of the infinitely many critical points theorem of Marano and Motreanu, and as an application of their result, they ensured the existence of infinitely many solutions for a two-point boundary value problem with the Sturm-Liouville equation having discontinuous nonlinear term. In [7] the authors employing the same critical points theorem of Marano and Motreanu, investigated the existence of infinitely many radially symmetric solutions for a class of differential inclusion problems. In [17] the authors using a three critical points theorem for a non-differentiable functional and a Sobolev embedding result, established the existence of three radially symmetric solutions for the problem (1.1), in the case $\mu = 0$.

In the present paper, under some appropriate hypotheses on the behavior of the potential of f, under a condition on the potential of g, at infinity, we ensure the existence of infinitely many radially symmetric solutions for the problem (1.1); this is done in Theorem 3.1. We also list some special cases of Theorem 3.1. Further, replacing the conditions at infinity of the potentials of f and g, by a similar one at zero, the same results hold and, in addition, the sequence of symmetric solutions uniformly converges to zero; this is done in Theorem 3.8. The abstract approach is fully based on the critical point theorem proved in [2]. Our approach here is in the one dimensional setting and is different from that employed in [7] in which the author directly discussed the existence of infinitely many solutions for the original differential inclusion problem, while here by setting $\rho = |x|$ and treating (1.1) as an ordinary differential equation we establish the existence of infinitely many solutions for the ordinary differential equation which will be observed later (see (3.2)), and since the solutions of the ordinary differential equation are the solutions of the problem (1.1), we have the results for the problem (1.1).

A special case of our main result is the following theorem.

Theorem 1.1. *Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be continuous almost everywhere, and assume that for each $\delta > 0$ there is a constant M_δ such that*

$$\sup_{|z|\le\delta} |f(\rho, z)| \le M_\delta,$$

where $\rho = |x|$, and that for all $z \in D(f)$ the condition $f^-(\rho, z) \le 0 \le f^+(\rho, z)$ implies $f(\rho, z) = 0$, where

$$\begin{aligned} f^-(\rho, z) &= \lim_{\delta\to 0^+} \operatorname{ess\,inf}_{|z-\zeta|<\delta} f(\rho, \zeta),\\ f^+(\rho, z) &= \lim_{\delta\to 0^+} \operatorname{ess\,sup}_{|z-\zeta|<\delta} f(\rho, \zeta). \end{aligned} \tag{1.2}$$

Put

$$F(\rho, t) = \int_0^t f(\rho, s)ds, \quad \rho \in \mathbb{R}^+ \cup \{0\}, \quad t \in \mathbb{R}.$$

Assume that

$$\liminf_{\xi\to+\infty} \frac{\int_0^{+\infty} \sup_{|t|\le\xi} F(\rho, t)\rho^{N-1}d\rho}{\xi^p} = 0,$$

$$\limsup_{\xi\to+\infty} \frac{\int_0^{\frac{D}{2}} F(\rho, \xi)\rho^{N-1}d\rho}{\xi^p} = +\infty \quad \text{for some } D > 0.$$

Then, the problem

$$-\Delta_p u + |u|^{p-2}u = f(|x|, u), \quad x \in \mathbb{R}^N, \quad u \in W_r^{1,p}(\mathbb{R}^N)$$

admits a sequence of symmetric solutions.

2. Basic definitions and preliminary results

For basic notation and definitions on the subject, we refer the reader to [1, 3, 12, 14]. Let $(X, \|\cdot\|_X)$ be a real Banach space. We denote by X^* the dual space of X, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X^* and X. A function $\varphi : X \to \mathbb{R}$ is called locally Lipschitz if, for all $u \in X$, there exist a neighborhood U of u and a real number $L > 0$ such that

$$|\varphi(v) - \varphi(w)| \le L\|v - w\|_X \quad \text{for all } v, w \in U.$$

If φ is locally Lipschitz and $u \in X$, the generalized directional derivative of φ at u along the direction $v \in X$ is

$$\varphi^\circ(u; v) := \limsup_{w\to u,\, \tau\to 0^+} \frac{\varphi(w + \tau v) - \varphi(w)}{\tau}.$$

The generalized gradient of φ at u is the set

$$\partial\varphi(u) := \{u^* \in X^* : \langle u^*, v\rangle \le \varphi^\circ(u; v) \text{ for all } v \in X\}.$$

So $\partial\varphi : X \to 2^{X^*}$ is a multifunction. We say that φ has compact gradient if $\partial\varphi$ maps bounded subsets of X into relatively compact subsets of X^*.

Lemma 2.1 ([13, Proposition 1.1]). *Let φ be a functional in $C^1(X)$. Then φ is locally Lipschitz and*

$$\varphi^\circ(u; v) = \langle\varphi'(u), v\rangle \quad \text{for all } u, v \in X;$$
$$\partial\varphi(u) = \{\varphi'(u)\} \quad \text{for all } u \in X.$$

Lemma 2.2 ([13, Proposition 1.3]). *Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional. Then $\varphi^\circ(u; \cdot)$ is subadditive and positively homogeneous for all $u \in X$, and*

$$\varphi^\circ(u; v) \le L\|v\| \quad \text{for all } u, v \in X,$$

with $L > 0$ being a Lipschitz constant for φ with respect to u.

Lemma 2.3 ([5]). *Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional. Then $\varphi^\circ : X \times X \to \mathbb{R}$ is upper semicontinuous and for all $\lambda \ge 0$, $u, v \in X$,*

$$(\lambda\varphi)^\circ(u; v) = \lambda\varphi^\circ(u; v).$$

Moreover, if $\varphi, \psi : X \to \mathbb{R}$ are locally Lipschitz functionals, then

$$(\varphi + \psi)^\circ(u; v) \le \varphi^\circ(u; v) + \psi^\circ(u; v) \quad \text{for all } u, v \in X.$$

Lemma 2.4 ([13, Proposition 1.6]). *Let $\varphi, \psi : X \to \mathbb{R}$ be locally Lipschitz functionals. Then*

$$\partial(\lambda\varphi)(u) = \lambda\partial\varphi(u) \text{ for all } u \in X,\ \lambda \in \mathbb{R}, \text{and}$$
$$\partial(\varphi + \psi)(u) \subseteq \partial\varphi(u) + \partial\psi(u) \text{ for all } u \in X.$$

We say that $u \in X$ is a (generalized) critical point of a locally Lipschitz functional φ if $0 \in \partial\varphi(u)$, i.e.,

$$\varphi^\circ(u; v) \ge 0 \quad \text{for all } v \in X.$$

When a non-smooth functional, $g : X \to (-\infty, +\infty)$, is expressed as a sum of a locally Lipschitz function, $\varphi : X \to \mathbb{R}$, and a convex, proper, and lower semicontinuous function, $j : X \to (-\infty, +\infty)$, that is, $g := \varphi + j$, a (generalized) critical point of g is every $u \in X$ such that

$$\varphi^\circ(u; v - u) + j(v) - j(u) \ge 0$$

for all $v \in X$ (see [13, Chapter 3]).

Let the space

$$W^{1,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N)\},$$

be equipped with the norm

$$\|u\|_{W^{1,p}(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} (|\nabla u(x)|^p + |u(x)|^p) dx \Big)^{1/p}.$$

The action of the orthogonal group $O(N)$ on $W^{1,p}(\mathbb{R}^N)$ can be defined by $gu(x) = u(g^{-1}x)$ for every $g \in O(N)$, $u \in W^{1,p}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$ (see [16]), and we can define the subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$ by

$$W_r^{1,p}(\mathbb{R}^N) = \{u \in W^{1,p}(\mathbb{R}^N) : gu = u, \forall g \in O(N)\}$$

equipped with the norm

$$\|u\|_{W_r^{1,p}(\mathbb{R}^N)} = \Big(\int_0^{+\infty} (|u'(\rho)|^p + |u(\rho)|^p) \rho^{N-1} d\rho \Big)^{1/p}.$$

As pointed out in [7, Theorem 3.1], since $2 \leq N < p < +\infty$, $W_r^{1,p}(\mathbb{R}^N)$ is compactly embedded in $L^\infty(\mathbb{R}^N)$. In particular, there exists a positive constant $k > 0$ such that

$$\sup_{\rho \in [0,+\infty]} |u(\rho)| \leq k \|u\|_{W_r^{1,p}(\mathbb{R}^N)} \tag{2.1}$$

for each $u \in W_r^{1,p}(\mathbb{R}^N)$.

Hereafter, we assume that X is a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous functional, $\Upsilon : X \to \mathbb{R}$ is a sequentially weakly upper semicontinuous functional, λ is a positive parameter, $j : X \to (-\infty, +\infty)$ is a convex, proper, and lower semicontinuous functional, and $D(j)$ is the effective domain of j. Write

$$\Psi := \Upsilon - j, \qquad I_\lambda := \Phi - \lambda \Psi = (\Phi - \lambda \Upsilon) + \lambda j.$$

We also assume that Φ is coercive and

$$D(j) \cap \Phi^{-1}(-\infty, r) \neq \emptyset \tag{2.2}$$

for all $r > \inf_X \Phi$. Moreover, owing to (2.2) and provided $r > \inf_X \Phi$, we can define

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty,r)} \frac{\big(\sup_{v \in \Phi^{-1}(-\infty,r)} \Psi(v)\big) - \Psi(u)}{r - \Phi(u)},$$
$$\gamma := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

When Φ and Υ are locally Lipschitz functionals the following result is proved in [2, Theorem 2.1]; it is a more precise version of [11, Theorem 1.1] (see also [15]), which is the main tool to prove our results.

Theorem 2.5. *Under the above assumptions on X, Φ and Ψ, one has*

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional $I_\lambda = \Phi - \lambda \Psi$ to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(b) *If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*

(b1) *I_λ possesses a global minimum, or*

(b2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n \to +\infty} \Phi(u_n) = +\infty$.*

(c) *If* $\delta < +\infty$, *then for each* $\lambda \in (0, 1/\delta)$, *the following alternative holds: either*

(c1) *there is a global minimum of* Φ *which is a local minimum of* I_λ, *or*

(c2) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points (local minima) of* I_λ, *with* $\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$, *which weakly converges to a global minimum of* Φ.

3. Main results

Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be continuous almost everywhere and assume that for each $\delta_1 > 0$ there is a constant M_{δ_1} such that

$$\sup_{|z|\le\delta_1} |f(|x|, z)| \le M_{\delta_1}. \tag{3.1}$$

Since x is away from the origin, we set $\rho = |x|$ and treat (1.1) as an ordinary differential equation. Thus we write $u(\rho)$ instead of $u(x)$, and the problem (1.1) corresponds exactly to

$$-(\rho^{N-1}\phi(u'))' + \rho^{N-1}\phi(u) = \lambda\rho^{N-1} f(\rho, u) + \mu\rho^{N-1} g(\rho, u) \tag{3.2}$$

where $'$ denotes $\frac{d}{d\rho}$ and $\phi(s) = |s|^{p-2}s$. Put

$$F(\rho, t) = \int_0^t f(\rho, s)ds, \ \rho \in \mathbb{R}^+ \cup \{0\}, \quad t \in \mathbb{R}.$$

Pick $D > 0$ such that $S(0, D) \subseteq \mathbb{R}^N$ where $S(0, D)$ denotes the ball with center at 0 and radius of D, and let ω_N be the volume of the N-dimensional unit ball.

Our main result is stated using the following assumptions:

(A1) $F(\rho, t) \ge 0$ for all $(\rho, t) \in [\frac{D}{2}, +\infty) \times (\mathbb{R}^+ \cup \{0\})$;

(A2)
$$\begin{aligned}
&\liminf_{\xi\to+\infty} \frac{\int_0^{+\infty} (\sup_{|t|\le\xi} F(\rho, t))\rho^{N-1} d\rho}{\xi^p} \\
&< \frac{1}{k^p \omega_N D^N (\frac{2^p}{D^p}(1 - \frac{1}{2^N}) + 1)} \limsup_{\xi\to+\infty} \frac{\int_0^{\frac{D}{2}} F(\rho, \xi)\rho^{N-1} d\rho}{\xi^p};
\end{aligned}$$

(A3) for all $z \in D(f)$ the condition $f^-(\rho, z) \le 0 \le f^+(\rho, z)$ implies $f(\rho, z) = 0$, where $f^-(\rho, z)$ and $f^+(\rho, z)$ are given as in (1.2).

Put

$$\lambda_1 := \frac{\omega_N D^N (\frac{2^p}{D^p}(1 - \frac{1}{2^N}) + 1)}{p \limsup_{\xi\to+\infty} \frac{\int_0^{\frac{D}{2}} F(\rho,\xi)\rho^{N-1} d\rho}{\xi^p}}$$

$$\lambda_2 := \Big(pk^p \liminf_{\xi\to+\infty} \frac{\int_0^{+\infty} (\sup_{|t|\le\xi} F(\rho, t))\rho^{N-1} d\rho}{\xi^p}\Big)^{-1}.$$

Suppose that $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous almost everywhere, and for $\delta_2 > 0$ there is a constant M_{δ_2} such that

$$\sup_{|z|\le\delta_2} |g(|x|, z)| \le M_{\delta_2}, \tag{3.3}$$

(A4) for all $z \in D(g)$ the condition $g^-(\rho,z) \le 0 \le g^+(\rho,z)$ implies $g(\rho,z)=0$, where

$$g^-(\rho,z)=\lim_{\delta\to 0^+}\operatorname{ess\,inf}_{|z-\zeta|<\delta} g(\rho,\zeta),\quad g^+(\rho,z)=\lim_{\delta\to 0^+}\operatorname{ess\,sup}_{|z-\zeta|<\delta} g(\rho,\zeta),$$

whose potential $G(\rho,t)=\int_0^t g(\rho,s)ds$, $\rho\in\mathbb{R}^+\cup\{0\}$, $t\in\mathbb{R}$, is a non-negative function satisfying the condition

$$g_\infty := \lim_{\xi\to+\infty}\frac{\int_0^{+\infty}(\sup_{|t|\le\xi} G(\rho,t))\rho^{N-1}d\rho}{\xi^p} < +\infty\,. \tag{3.4}$$

Set

$$\mu_{g,\lambda} := \frac{1}{pk^p g_\infty}\Big(1-\lambda pk^p \liminf_{\xi\to+\infty}\frac{\int_0^{+\infty}(\sup_{|t|\le\xi} F(\rho,t))\rho^{N-1}d\rho}{\xi^p}\Big).$$

Theorem 3.1. *Under assumptions* (A1)–(A4)*, for each* $\lambda\in]\lambda_1,\lambda_2[$ *and for every* $\mu\in[0,\mu_{g,\lambda}[$*, problem* (1.1) *has an unbounded sequence of symmetric solutions.*

Proof. To apply Theorem 2.5 to our problem, we take $X = W_r^{1,p}(\mathbb{R}^N)$. Fix $\overline{\lambda}\in]\lambda_1,\lambda_2[$ and let g be an almost everywhere continuous function satisfying the condition (3.4). Arguing as in [1], we follow the proof in the case $\mu>0$. Since, $\overline{\lambda}<\lambda_2$, one has

$$\mu_{g,\overline{\lambda}} := \frac{1}{pk^p g_\infty}\Big(1-\overline{\lambda}\, pk^p \liminf_{\xi\to+\infty}\frac{\int_0^{+\infty}(\sup_{|t|\le\xi} F(\rho,t))\rho^{N-1}d\rho}{\xi^p}\Big) > 0.$$

Fix $\overline{\mu}\in]0,\mu_{g,\overline{\lambda}}[$ and set $\nu_1:=\lambda_1$ and $\nu_2:=\frac{\lambda_2}{1+pk^p\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2 g_\infty}$. If $g_\infty=0$, clearly, $\nu_1=\lambda_1$, $\nu_2=\lambda_2$ and $\lambda\in]\nu_1,\nu_2[$. If $g_\infty\neq 0$, since $\overline{\mu}<\mu_{g,\overline{\lambda}}$, we obtain

$$\frac{\overline{\lambda}}{\lambda_2}+pk^p\overline{\mu}g_\infty<1,$$

and so

$$\frac{\lambda_2}{1+pk^p\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2 g_\infty}>\overline{\lambda},$$

namely, $\overline{\lambda}<\nu_2$. Hence, since $\overline{\lambda}>\lambda_1=\nu_1$, one has $\overline{\lambda}\in]\nu_1,\nu_2[$. We now set

$$\Phi(u)=\frac{1}{p}\|u\|^p_{W_r^{1,p}(\mathbb{R}^N)},\quad \Upsilon(u)=\int_0^{+\infty}[F(\rho,u)+\frac{\overline{\mu}}{\overline{\lambda}}G(\rho,u)]\rho^{N-1}d\rho,$$
$$j(u)=0,\quad \Psi(u)=\Upsilon(u)-j(u)=\Upsilon(u)$$

for each $u\in X$. Clearly, the functional Φ is locally Lipschitz and weakly sequentially lower semi-continuous. Put $I_{\overline{\lambda}}:=\Phi-\overline{\lambda}\Psi$. Since f and g satisfy (3.1) and (3.3), respectively, and $W_r^{1,p}(\mathbb{R}^N)$ is compactly embedded in $L^\infty(\mathbb{R}^N)$, the assertion remains true regarding Ψ too (see [8, 9]). By a simple computation, we obtain

$$\frac{d\Phi(u)}{du}=\int_0^{+\infty}[-(|u'|^{p-2}u')'+|u|^{p-2}u]\rho^{N-1}d\rho.$$

From Chang [4, Theorem 2.1], we have

$$\partial\Psi(u)=[(f^-(\rho,u)+\frac{\overline{\mu}}{\overline{\lambda}}g^-(\rho,u))\rho^{N-1},(f^+(\rho,u)+\frac{\overline{\mu}}{\overline{\lambda}}g^+(\rho,u))\rho^{N-1}].$$

So the critical point of the functional $I_{\overline{\lambda}}$ is precisely the solution of the differential inclusion

$$\begin{aligned} &-(\rho^{N-1}\phi(u'))' + \rho^{N-1}\phi(u) \\ &\in \overline{\lambda}[(f^-(\rho,u) + \frac{\overline{\mu}}{\overline{\lambda}}g^-(\rho,u))\rho^{N-1}, (f^+(\rho,u) + \frac{\overline{\mu}}{\overline{\lambda}}g^+(\rho,u))\rho^{N-1}] \end{aligned} \tag{3.5}$$

for $\rho \in [0,+\infty]\backslash(u^{-1}(D_f)\bigcup u^{-1}(D_g))$.

Since $m(D_f) = m(D_g) = 0$, we can obtain $-(\rho^{N-1}\phi(u'))' + \rho^{N-1}\phi(u)) = 0$ for almost all $\rho \in u^{-1}(D_f)\cap u^{-1}(D_g)$. On the other hand, in view of Assumptions (A3) and (A4), we obtain $f(\rho, u(\rho)) = 0$ for almost all $\rho \in u^{-1}(D_f)$ and $g(\rho, u(\rho)) = 0$ for almost all $\rho \in u^{-1}(D_g)$, respectively, i.e.

$$-(\rho^{N-1}\phi(u'))' + \rho^{N-1}\phi(u) = \overline{\lambda}\rho^{N-1}f(\rho,u) + \overline{\mu}\rho^{N-1}g(\rho,u) \tag{3.6}$$

for almost all $\rho \in u^{-1}(D_f)\cap u^{-1}(D_g)$. Combining (3.5) and (3.6), we can obtain that the solutions of the problem (3.2) are exactly the critical points of the functional $I_{\overline{\lambda}}$. Now, we claim that $\gamma < +\infty$.

Let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \to +\infty$ as $n \to \infty$ and

$$\begin{aligned} &\lim_{n\to\infty} \frac{\int_0^{+\infty} (\sup_{|t|\le\xi_n}[F(\rho,t) + \frac{\overline{\mu}}{\overline{\lambda}}G(\rho,t)])\rho^{N-1}d\rho}{\xi_n^p} \\ &= \liminf_{\xi\to+\infty} \frac{\int_0^{+\infty} (\sup_{|t|\le\xi}[F(\rho,t) + \frac{\overline{\mu}}{\overline{\lambda}}G(\rho,t)])\rho^{N-1}d\rho}{\xi^p}. \end{aligned}$$

Put $r_n = \frac{1}{p}(\frac{\xi_n}{k})^p$ for all $n \in \mathbb{N}$. Bearing in mind (2.1), we have

$$\begin{aligned} \Phi^{-1}(-\infty, r_n) &= \{u \in X; \Phi(u) < r_n\} \\ &= \{u \in X; \|u\|^p_{W_r^{1,p}(\mathbb{R}^N)} < pr_n\} \\ &\subseteq \{u \in X; |u(\rho)| \le \xi_n \text{ for all } \rho \in [0,+\infty]\}. \end{aligned}$$

Hence, taking into account that $\inf_X \Phi(0) = 0$ and $\Psi(0) = 0$ for every n large enough, one has

$$\begin{aligned} \varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{(\sup_{v\in\Phi^{-1}(-\infty,r_n)}\Psi(v)) - \Psi(u)}{r_n - \Phi(u)} \\ &\le \frac{\sup_{v\in\Phi^{-1}(-\infty,r_n)}\Psi(v)}{r_n} \\ &\le \frac{\int_0^{+\infty} (\sup_{|t|\le\xi_n}\Big[F(\rho,t) + \frac{\overline{\mu}}{\overline{\lambda}}G(\rho,t)\Big])\rho^{N-1}d\rho}{\frac{1}{p}(\frac{\xi_n}{k})^p} \\ &\le \frac{\int_0^{+\infty} (\sup_{|t|\le\xi_n}F(\rho,t))\rho^{N-1}d\rho}{\frac{1}{p}(\frac{\xi_n}{k})^p} + \frac{\overline{\mu}}{\overline{\lambda}} \frac{\int_0^{+\infty} (\sup_{|t|\le\xi_n}G(\rho,t))\rho^{N-1}d\rho}{\frac{1}{p}(\frac{\xi_n}{k})^p}. \end{aligned}$$

From Assumption (A2) and the condition (3.4) one has

$$\begin{aligned} &\lim_{n\to\infty} \frac{\int_0^{+\infty} (\sup_{|t|\le\xi_n}F(\rho,t))\rho^{N-1}d\rho}{\frac{1}{p}(\frac{\xi_n}{k})^p} + \lim_{n\to\infty} \frac{\overline{\mu}}{\overline{\lambda}} \frac{\int_0^{+\infty} (\sup_{|t|\le\xi_n}G(\rho,t))\rho^{N-1}d\rho}{\frac{1}{p}(\frac{\xi_n}{k})^p} \\ &< +\infty, \end{aligned}$$

from which it follows that

$$\lim_{n\to\infty}\frac{\int_0^{+\infty}(\sup_{|t|\le\xi_n}[F(\rho,t)+\frac{\overline{\mu}}{\overline{\lambda}}G(\rho,t)])\rho^{N-1}d\rho}{\frac{1}{p}(\frac{\xi_n}{k})^p}<+\infty.$$

Therefore,

$$\gamma\le\liminf_{n\to+\infty}\varphi(r_n)\le\lim_{n\to\infty}\frac{\int_0^{+\infty}(\sup_{|t|\le\xi_n}[F(\rho,t)+\frac{\overline{\mu}}{\overline{\lambda}}G(\rho,t)])\rho^{N-1}d\rho}{\frac{1}{p}(\frac{\xi_n}{k})^p}<+\infty. \tag{3.7}$$

Since

$$\begin{aligned}
&\frac{\int_0^{+\infty}(\sup_{|t|\le\xi}[F(\rho,t)+\frac{\overline{\mu}}{\overline{\lambda}}G(\rho,t)])\rho^{N-1}d\rho}{\frac{1}{p}(\frac{\xi}{k})^p}\\
&\le\frac{\int_0^{+\infty}(\sup_{|t|\le\xi}F(\rho,t))\rho^{N-1}d\rho}{\frac{1}{p}(\frac{\xi}{k})^p}+\frac{\overline{\mu}}{\overline{\lambda}}\frac{\int_0^{+\infty}(\sup_{|t|\le\xi}G(\rho,t))\rho^{N-1}d\rho}{\frac{1}{p}(\frac{\xi}{k})^p},
\end{aligned}$$

taking into account (3.4), one has

$$\begin{aligned}
&\liminf_{\xi\to+\infty}\frac{\int_0^{+\infty}(\sup_{|t|\le\xi}[F(\rho,t)+\frac{\overline{\mu}}{\overline{\lambda}}G(\rho,t)])\rho^{N-1}d\rho}{\xi^p}\\
&\le\liminf_{\xi\to+\infty}\frac{\int_0^{+\infty}(\sup_{|t|\le\xi}F(\rho,t))\rho^{N-1}d\rho}{\xi^p}+\frac{\overline{\mu}}{\overline{\lambda}}g_\infty.
\end{aligned}\tag{3.8}$$

Since G is nonnegative, we obtain

$$\limsup_{\xi\to+\infty}\frac{\int_0^{\frac{D}{2}}[F(\rho,\xi)+\frac{\overline{\mu}}{\overline{\lambda}}G(\rho,\xi)]\rho^{N-1}d\rho}{\xi^p}\ge\limsup_{\xi\to+\infty}\frac{\int_0^{\frac{D}{2}}F(\rho,\xi)\rho^{N-1}d\rho}{\xi^p}. \tag{3.9}$$

Therefore, in view of (3.8) and (3.9), we have

$$\begin{aligned}
\overline{\lambda}&\in]\nu_1,\nu_2[\\
&\subseteq\Big]\frac{\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)}{p\limsup_{\xi\to+\infty}\frac{\int_0^{\frac{D}{2}}[F(\rho,\xi)+\frac{\overline{\mu}}{\overline{\lambda}}G(\rho,\xi)]\rho^{N-1}d\rho}{\xi^p}},\\
&\quad\frac{1}{pk^p\liminf_{\xi\to+\infty}\frac{\int_0^{+\infty}(\sup_{|t|\le\xi}[F(\rho,t)+\frac{\overline{\mu}}{\overline{\lambda}}G(\rho,t)])\rho^{N-1}d\rho}{\xi^p}}\Big[\\
&\subseteq]0,\frac{1}{\gamma}[,
\end{aligned}$$

where we used (A2) and (3.7).

For a fixed $\overline{\lambda}$, inequality (3.7) implies that the condition (b) of Theorem 2.5 can be applied and either $I_{\overline{\lambda}}$ has a global minimum or there exists a sequence $\{u_n\}$ of solutions of the problem (3.2) such that $\lim_{n\to\infty}\|u_n\|=+\infty$.

The other step is to show that for the fixed $\overline{\lambda}$ the functional $I_{\overline{\lambda}}$ has no global minimum. Let us verify that the functional $I_{\overline{\lambda}}$ is unbounded from below. Since

$$\frac{1}{\overline{\lambda}}<\frac{p}{\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)}\limsup_{\xi\to+\infty}\frac{\int_0^{\frac{D}{2}}F(\rho,\xi)\rho^{N-1}d\rho}{\xi^p}$$

$$\le \frac{p}{\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)} \limsup_{\xi\to+\infty} \frac{\int_0^{\frac{D}{2}}[F(\rho,\xi)+\frac{\overline{\mu}}{\overline{\lambda}}G(\rho,\xi)]\rho^{N-1}d\rho}{\xi^p},$$

we can consider a real sequence $\{d_n\}$ and a positive constant τ such that $d_n \to +\infty$ as $n \to \infty$ and

$$\frac{1}{\overline{\lambda}} < \tau < \frac{p}{\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)} \frac{\int_0^{\frac{D}{2}}[F(\rho,d_n)+\frac{\overline{\mu}}{\overline{\lambda}}G(\rho,d_n)]\rho^{N-1}d\rho}{d_n} \tag{3.10}$$

for each $n \in \mathbb{N}$ large enough. Let $\{w_n\}$ be a sequence in X defined by

$$w_n(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}^N \setminus S(0,D), \\ \frac{2d_n}{D}(D-|x|) & \text{if } x \in S(0,D) \setminus S(0,\frac{D}{2}), \\ d_n & \text{if } x \in S(0,\frac{D}{2}). \end{cases} \tag{3.11}$$

For any fixed $n \in \mathbb{N}$, it is easy to see that $w_n \in X$ and, in particular, one has

$$\|w_n\|^p_{W^{1,p}_r(\mathbb{R}^N)} \le d_n^p \omega_N D^N\big(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1\big). \tag{3.12}$$

On the other hand, since $0 \le w_n(x) \le d_n$ for every $x \in \mathbb{R}^N$, from (A1) and since G is nonnegative, from the definition of Ψ, we infer

$$\Psi(w_n) \ge \int_0^{\frac{D}{2}} [F(\rho,d_n)+\frac{\overline{\mu}}{\overline{\lambda}}G(\rho,d_n)]\rho^{N-1}d\rho. \tag{3.13}$$

So, according to (3.10), (3.12) and (3.13), we obtain

$$\begin{aligned} I_{\overline{\lambda}}(w_n) &\le \frac{1}{p} d_n^p \omega_N D^N\big(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1\big) - \overline{\lambda}\int_0^{\frac{D}{2}} [F(\rho,d_n)+\frac{\overline{\mu}}{\overline{\lambda}}G(\rho,d_n)]\rho^{N-1}d\rho \\ &< \frac{1}{p} d_n^p \omega_N D^N\big(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1\big)(1-\overline{\lambda}\tau) \end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Since $\overline{\lambda}\tau > 1$ and $\lim_{n\to+\infty} d_n = +\infty$ we have

$$\lim_{n\to+\infty} I_{\overline{\lambda}}(w_n) = -\infty\,.$$

Hence, the functional $I_{\overline{\lambda}}$ is unbounded from below, and it follows that $I_{\overline{\lambda}}$ has no global minimum. Therefore, applying Theorem 2.5 we deduce that there is a sequence $\{u_n\} \subset X$ of critical points of $I_{\overline{\lambda}}$ such that $\lim_{n\to\infty} \|u_n\|_{W^{1,p}_r(\mathbb{R}^N)} = +\infty$. Hence, since the critical points of the functional $I_{\overline{\lambda}}$ are exactly the solutions of the problem (3.2), and then they are the solutions of the problem (1.1), the conclusion is achieved. □

Remark 3.2. We notice that instead of Assumption (A2) in Theorem 3.1 we are allowed to assume the more general condition

(A5) there exist two sequence $\{\alpha_n\}$ and $\{\beta_n\}$ with

$$\Big(\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)\Big)^{1/p}\alpha_n < \frac{\beta_n}{k}$$

for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty} \beta_n = +\infty$ such that

$$\lim_{n\to+\infty} \frac{\int_0^{+\infty}(\sup_{|t|\le\beta_n} F(\rho,t))\rho^{N-1}d\rho - \int_0^{\frac{D}{2}} F(\rho,\alpha_n)\rho^{N-1}d\rho}{(\frac{\beta_n}{k})^p - \omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)\alpha_n^p}$$

$$< \frac{1}{\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)} \limsup_{\xi\to+\infty} \frac{\int_0^{\frac{D}{2}} F(\rho,\xi)\rho^{N-1}d\rho}{\xi^p}.$$

Obviously, Assumption (A2) follows from Assumption (A5), by choosing $\alpha_n = 0$ for all $n \in \mathbb{N}$. Moreover, if we assume (A5) instead of (A2) and set $r_n = \frac{1}{p}(\frac{\beta_n}{k})^p$ for all $n \in \mathbb{N}$, by the same reasoning as in Theorem 3.1, we obtain

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{(\sup_{v\in\Phi^{-1}(-\infty,r_n)}\Psi(v)) - \Psi(u)}{r_n - \Phi(u)} \\
&\le \frac{\sup_{v\in\Phi^{-1}(-\infty,r_n)}\Psi(v) - \int_0^{+\infty} F(\rho,w_n(x))\rho^{N-1}d\rho}{r_n - \frac{1}{p}\|w_n\|^p_{W_r^{1,p}(\mathbb{R}^N)}} \\
&\le \frac{\int_0^{+\infty}(\sup_{|t|\le\in\xi} F(\rho,t))\rho^{N-1}d\rho - \int_0^{\frac{D}{2}} F(\rho,\alpha_n)\rho^{N-1}d\rho}{\frac{1}{p}(\frac{\beta_n}{k})^p - \frac{1}{p}\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)\alpha_n^p},
\end{aligned}$$

where $w_n(x)$ is defined as given in (3.11), for $x \in \mathbb{R}^N$ with α_n instead of d_n. We then have the same conclusion as in Theorem 3.1 with λ_2 replaced by

$$\lambda_2' := \Big(p \lim_{n\to+\infty} \frac{\int_0^{+\infty}(\sup_{|t|\le\xi} F(\rho,t))\rho^{N-1}d\rho - \int_0^{\frac{D}{2}} F(\rho,\alpha_n)\rho^{N-1}d\rho}{(\frac{\beta_n}{k})^p - \omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)\alpha_n^p}\Big)^{-1}$$

The following result is a special case of Theorem 3.1 with $\mu = 0$.

Theorem 3.3. *Assume that* (A1)–(A3) *hold. Then, for each*

$$\lambda \in \Lambda_1 := \Big]\frac{\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)}{p\limsup_{\xi\to+\infty}\frac{\int_0^{\frac{D}{2}} F(\rho,\xi)\rho^{N-1}d\rho}{\xi^p}}, \frac{1}{pk^p \liminf_{\xi\to+\infty}\frac{\int_0^{+\infty}(\sup_{|t|\le\xi} F(\rho,t))\rho^{N-1}d\rho}{\xi^p}}\Big[$$

the problem

$$-\Delta_p u + |u|^{p-2}u = \lambda f(|x|,u), \quad x\in\mathbb{R}^N, \quad u \in W_r^{1,p}(\mathbb{R}^N) \tag{3.14}$$

has an unbounded sequence of symmetric solutions.

Here we point out the following consequence of Theorem 3.3.

Corollary 3.4. *Assume that* (A1) *and* (A3) *hold. Also assume that:*

(A6)
$$\liminf_{\xi\to+\infty} \frac{\int_0^{+\infty}(\sup_{|t|\le\xi} F(\rho,t))\rho^{N-1}d\rho}{\xi^p} < \frac{1}{pk^p};$$

(A7)
$$\limsup_{\xi\to+\infty} \frac{\int_0^{D/2} F(\rho,\xi)\rho^{N-1}d\rho}{\xi^p} > \frac{1}{p}\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1).$$

Then the problem

$$-\Delta_p u + |u|^{p-2}u = f(|x|,u), \quad x\in\mathbb{R}^N, \quad u \in W_r^{1,p}(\mathbb{R}^N)$$

has an unbounded sequence of symmetric solutions.

Remark 3.5. Theorem 1.1 is an immediately consequence of Corollary 3.4.

Now, we point out a special situation of Theorem 3.3 when the nonlinear term has separable variables. To be precise, let α be a continuous function such that $\alpha(|x|) \geq 0$ a.e. $x \in \mathbb{R}^N$, $\alpha \not\equiv 0$, and let $h : \mathbb{R} \to \mathbb{R}$ be non-negative and continuous almost everywhere; namely, $m(D_h) = 0$ where $D_h = \{z \in \mathbb{R}, h(z) \text{ is discontinuous at } z\}$. We also assume that for each $\iota > 0$ there is a constant M_ι such that

$$\sup_{|z|\leq\iota} |h(z)| \leq M_\iota.$$

Put $H(t) = \int_0^t h(s)ds$, $t \in \mathbb{R}$. Then, we have the following consequence of Theorem 3.1.

Theorem 3.6. *Assume that*

(A8)

$$\liminf_{\xi\to+\infty} \frac{H(\xi)}{\xi^p} < \frac{\int_0^{\frac{D}{2}} \alpha(\rho)\rho^{N-1}d\rho}{k^p\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)(\int_0^{+\infty} \alpha(\rho)\rho^{N-1}d\rho)} \limsup_{\xi\to+\infty} \frac{H(\xi)}{\xi^p};$$

(A9) *for all $z \in D(h)$ the condition $h^-(z) \leq 0 \leq h^+(z)$ implies $h(z) = 0$, where*

$$h^-(z) = \lim_{\delta\to0^+} \operatorname{ess\,inf}_{|z-\zeta|<\delta} h(\zeta), \quad h^+(z) = \lim_{\delta\to0^+} \operatorname{ess\,sup}_{|z-\zeta|<\delta} h(\zeta).$$

Put

$$\begin{aligned}\Lambda_2 := \Big]&\frac{\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)}{p(\int_0^{\frac{D}{2}} \alpha(\rho)\rho^{N-1}d\rho)\limsup_{\xi\to+\infty}\frac{H(\xi)}{\xi^p}},\\ &\frac{1}{pk^p(\int_0^{+\infty} \alpha(\rho)\rho^{N-1}d\rho)\liminf_{\xi\to+\infty}\frac{H(\xi)}{\xi^p}}\Big[.\end{aligned}$$

Suppose that $g : \mathbb{R}^N\times\mathbb{R} \to \mathbb{R}$ is an almost everywhere continuous function such that for $\delta_2 > 0$ there is a constant M_{δ_2} such that (3.3) *holds, and satisfies (A4), whose potential $G(\rho,t) = \int_0^t g(\rho,s)ds$, $\rho \in \mathbb{R}^+\cup\{0\}$, $t \in \mathbb{R}$, is a non-negative function satisfying the condition* (3.4)*. Set*

$$\mu'_{g,\lambda} := \frac{1}{pk^p g_\infty}\Big(1-\lambda pk^p(\int_0^{+\infty} \alpha(\rho)\rho^{N-1}d\rho)\liminf_{\xi\to+\infty}\frac{H(\xi)}{\xi^p}\Big).$$

Then, for each $\lambda \in \Lambda_2$ and for every $\mu \in [0, \mu'_{g,\lambda}[$ the problem

$$-\Delta_p u + |u|^{p-2}u = \lambda\alpha(|x|)h(u) + \mu g(|x|,u), \quad x \in \mathbb{R}^N, \quad u \in W_r^{1,p}(\mathbb{R}^N) \qquad (3.15)$$

has an unbounded sequence of symmetric solutions.

Next we give an example where the hypotheses of Theorem 3.6 are satisfied.

Example 3.7. Let $2 \leq N < p < +\infty$ and $h : \mathbb{R} \to \mathbb{R}$ be defined by

$$h(z) = \begin{cases} e^z, & z < 2,\\ 0, & z = 2,\\ z^2, & z > 2.\end{cases}$$

The function h has only one discontinuity point at $z_0 = 2$ where $h(z_0) = 0$. Hence, the condition (A9) is satisfied. A direct calculation shows that

$$H(z) = \begin{cases} e^z - 1, & z < 2,\\ 0, & z = 2,\\ z^3/3, & z > 2.\end{cases}$$

Therefore,

$$\liminf_{\xi\to+\infty}\frac{\sup_{|t|\leq\xi}H(t)}{\xi^p}=0,\quad \limsup_{\xi\to+\infty}\frac{H(\xi)}{\xi^p}=+\infty,$$

and we observe that (A8) is fulfilled. Hence, using Theorem 3.6, the problem

$$-\Delta_p u+|u|^{p-2}u=\lambda\frac{h(u)}{(1+|x|^2)^2}+\mu\frac{g_1(u)}{1+|x|^2},\quad x\in\mathbb{R}^N,$$

where

$$g_1(z)=\begin{cases}e^z, & z<2,\\ 0, & z\geq 2.\end{cases}$$

for every $(\lambda,\mu)\in]0,+\infty[\times[0,+\infty[$ admits an unbounded sequence of radially symmetric solutions in $W_r^{1,p}(\mathbb{R}^N)$.

Arguing as in the proof of Theorem 3.1, but using conclusion (c) of Theorem 2.5 instead of (b), the following result holds.

Theorem 3.8. *Assume that* (A1) *and* (A3) *hold and*

(B1)

$$\begin{aligned}&\liminf_{\xi\to0^+}\frac{\int_0^{+\infty}(\sup_{|t|\leq\xi}F(\rho,t))\rho^{N-1}d\rho}{\xi^p}\\ &<\frac{1}{k^p\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)}\limsup_{\xi\to0^+}\frac{\int_0^{\frac{D}{2}}F(\rho,\xi)\rho^{N-1}d\rho}{\xi^p}.\end{aligned}$$

Put

$$\lambda_3:=\frac{\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)}{p\limsup_{\xi\to0^+}\frac{\int_0^{\frac{D}{2}}F(\rho,\xi)\rho^{N-1}d\rho}{\xi^p}},$$

$$\lambda_4:=\frac{1}{pk^p\liminf_{\xi\to0^+}\frac{\int_0^{+\infty}(\sup_{|t|\leq\xi}F(\rho,t))\rho^{N-1}d\rho}{\xi^p}}.$$

Suppose that $g:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is continuous almost everywhere, and that for $\delta_2>0$ there is a constant M_{δ_2} such that (3.3) *holds, and satisfies (A4), whose potential $G(\rho,t)=\int_0^t g(\rho,s)ds$, $\rho\in\mathbb{R}^+\cup\{0\}$, $t\in\mathbb{R}$ is a non-negative function satisfying the condition*

$$g_0:=\lim_{\xi\to0^+}\frac{\int_0^{+\infty}(\sup_{|t|\leq\xi}G(\rho,t))\rho^{N-1}d\rho}{\xi^p}<+\infty \tag{3.16}$$

and set

$$\bar{\mu}_{g,\lambda}:=\frac{1}{pk^pg_0}\Big(1-\lambda pk^p\liminf_{\xi\to0^+}\frac{\int_0^{+\infty}(\sup_{|t|\leq\xi}F(\rho,t))\rho^{N-1}d\rho}{\xi^p}\Big).$$

Then for each $\lambda\in]\lambda_3,\lambda_4[$ and for every $\mu\in[0,\bar{\mu}_{g,\lambda}[$, problem (1.1) *has a sequence of symmetric solutions, which strongly converges to* 0 *in $W_r^{1,p}(\mathbb{R}^N)$.*

Proof. We take X, Φ, Υ, j, Ψ and I_λ as in the proof of Theorem 3.1. By a similar way as in the proof of Theorem 3.1 we show that $\delta<+\infty$. For this, let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n\to0^+$ as $n\to+\infty$ and

$$\lim_{n\to+\infty}\frac{\int_0^{+\infty}(\sup_{|t|\leq\xi_n}[F(\rho,t)+\frac{\mu}{\lambda}G(\rho,t)])\rho^{N-1}d\rho}{\xi_n^p}<+\infty.$$

Setting $r_n = \frac{1}{p}(\frac{\xi_n}{k})^p$ for all $n \in \mathbb{N}$, arguing as in the proof of Theorem 3.1, it follows that $\delta < +\infty$. Fix $\lambda \in]\lambda_3, \lambda_4[$. The functional I_λ does not have a local minimum at zero. Indeed, let $\{d_n\}$ be a sequence of positive numbers and $\tau > 0$ such that $d_n \to 0^+$ as $n \to \infty$ and

$$\frac{1}{\lambda} < \tau < \frac{p}{\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)} \frac{\int_0^{\frac{D}{2}}[F(\rho,d_n)+\frac{\mu}{\lambda}G(\rho,d_n)]\rho^{N-1}d\rho}{d_n} \tag{3.17}$$

for each $n \in \mathbb{N}$ large enough. Let $\{w_n\}$ be a sequence in $W_r^{1,p}(\mathbb{R}^N)$ defined as given in (3.11). According to (3.12), (3.13) and (3.17), we obtain

$$\begin{aligned} I_\lambda(w_n) &\le \frac{1}{p} d_n^p \omega_N D^N\big(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1\big) - \lambda \int_0^{\frac{D}{2}} [F(\rho,d_n)+\frac{\mu}{\lambda}G(\rho,d_n)]\rho^{N-1}d\rho \\ &< \frac{1}{p} d_n^p \omega_N D^N\big(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1\big)(1-\lambda\tau) < 0 \end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Since $I_\lambda(0) = 0$, this means the functional I_λ does not have a local minimum at zero. Hence, the part (c) of Theorem 2.5 concludes that there exists a sequence $\{u_n\}$ in X of critical points of I_λ such that $\|u_n\|_{W_r^{1,p}(\mathbb{R}^N)} \to 0$ as $n \to \infty$, and the proof is complete. □

Remark 3.9. Note that Assumption (B1) in Theorem 3.8 could be replaced by the more general condition

(B2) there exist two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with

$$\Big(\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)\Big)^{1/p}\alpha_n < \frac{\beta_n}{k}$$

for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty}\beta_n = 0$ such that

$$\begin{aligned} &\lim_{n\to+\infty} \frac{\int_0^{+\infty}(\sup_{|t|\le\beta_n} F(\rho,t))\rho^{N-1}d\rho - \int_0^{\frac{D}{2}} F(\rho,\alpha_n)\rho^{N-1}d\rho}{(\frac{\beta_n}{k})^p - \omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)\alpha_n^p} \\ &< \frac{1}{\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)} \limsup_{\xi\to 0^+} \frac{\int_0^{\frac{D}{2}} F(\rho,\xi)\rho^{N-1}d\rho}{\xi^p}. \end{aligned}$$

Remark 3.10. We observe that in Theorem 3.3, Corollary 3.4 and Theorem 3.6 by Theorem 3.8 and replacing $\xi \to +\infty$ with $\xi \to 0^+$, by the same reasoning, we have the conclusions, $\xi \to +\infty$ replaced by $\xi \to 0^+$, but the sequences of symmetric solutions strongly converge to 0 in $W_r^{1,p}(\mathbb{R}^N)$, instead.

We here give the following example to illustrate our results.

Example 3.11. *Put $N = 2$ and $p = 3$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(z) = \begin{cases} 1, & (z-1) \in [0,1] \setminus C, \\ 0, & \textit{otherwise} \end{cases}$$

where C is the "middle third set" of Cantor. Clearly, $m(D_f) = m(1+C) = 0$ and for each $z \in D_f$ one has $f(z) = 0$. A direct calculation shows

$$F(z) = \begin{cases} z, & (z-1) \in [0,1] \setminus C, \\ 0, & \textit{otherwise.} \end{cases}$$

Therefore,

$$\liminf_{\xi\to 0^+}\frac{\sup_{|t|\le\xi}F(t)}{\xi^3}=0,\quad \limsup_{\xi\to 0^+}\frac{F(\xi)}{\xi^3}=+\infty.$$

Hence, taking Remark 3.10 into account, by the similar result to Theorem 3.8, for a fixed continuous almost everywhere function $g:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ satisfying the required assumptions in Theorem 3.8, the problem

$$-\Delta_3 u+|u|u=\lambda f(u)+\mu g(|x|,u),\quad x\in\mathbb{R}^2,\quad u\in W_r^{1,3}(\mathbb{R}^2),$$

for every $\lambda\in]0,+\infty[$ and μ lying in a convenient interval, admits a sequence of symmetric solutions, which converges strongly to 0 in $W_r^{1,3}(\mathbb{R}^2)$.

We now consider the problem

$$-\Delta_p u+|u|^{p-2}u=\lambda\alpha(x)f(u)+\mu\beta(x)g(u),\quad x\in\mathbb{R}^N,\quad u\in W^{1,p}(\mathbb{R}^N)\tag{3.18}$$

where $\lambda>0$ and $\mu\ge 0$ are two parameters, $\alpha,\beta\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ are radially symmetric, $\alpha,\beta\ge 0$, $\alpha,\beta\not\equiv 0$, $f,g:\mathbb{R}\to\mathbb{R}$ are non-negative continuous almost everywhere, namely, $m(D_f)=0$ where $D_f=\{z\in\mathbb{R}: f(z)\text{ is discontinuous at }z\}$, and $m(D_g)=0$ where $D_g=\{z\in\mathbb{R}, g(z)\text{ is discontinuous at }z\}$. We also assume that for each $\iota_1>0$ there is a constant M_{ι_1} such that

$$\sup_{|z|\le\iota_1}|f(z)|\le M_{\iota_1}.\tag{3.19}$$

Let k_∞ be the embedding constant of $W^{1,p}(\mathbb{R}^N)\subset L^\infty(\mathbb{R}^N)$; we obtain

$$\sup_{x\in\mathbb{R}^N}|u(x)|\le k_\infty\|u\|_{W^{1,p}(\mathbb{R}^N)},$$

and $k_\infty\le 2p(p-N)^{-1}$ (see [7]). Put

$$F(t)=\int_0^t f(s)ds,\quad t\in\mathbb{R}.$$

Next we have an existence result under the following assumptions:

(A10)
$$\liminf_{\xi\to+\infty}\frac{\|\alpha\|_{L^1(\mathbb{R}^N)}F(\xi)}{\xi^p}<\frac{\|\alpha\|_{L^1(S(0,\frac{D}{2}))}}{k_\infty^p\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)}\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p};$$

(A11) for all $z\in D(f)$ the condition $f^-(z)\le 0\le f^+(z)$ implies $f(z)=0$, where
$$f^-(z)=\lim_{\delta\to 0^+}\operatorname{ess\,inf}_{|z-\zeta|<\delta}f(\zeta),\ f^+(z)=\lim_{\delta\to 0^+}\operatorname{ess\,sup}_{|z-\zeta|<\delta}f(\zeta).$$

Put

$$\lambda_5:=\frac{\omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)}{p\|\alpha\|_{L^1(S(0,\frac{D}{2}))}\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}},$$
$$\lambda_6:=\frac{1}{pk_\infty\|\alpha\|_{L^1(\mathbb{R}^N)}\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}}.$$

Suppose that $g:\mathbb{R}\to\mathbb{R}$ is a non-negative continuous almost everywhere function such that for each $\iota_2>0$ there is a constant M_{ι_2} such that

$$\sup_{|z|\le\iota_2}g(z)\le M_{\iota_2},\tag{3.20}$$

(A12) for all $z \in D(g)$ the condition $g^-(z) \le 0 \le g^+(z)$ implies $g(z) = 0$, where $g^-(z) = \lim_{\delta\to 0^+} \operatorname{ess\,inf}_{|z-\zeta|<\delta} g(\zeta), g^+(z) = \lim_{\delta\to 0^+} \operatorname{ess\,sup}_{|z-\zeta|<\delta} g(\zeta)$, whose potential $G(t) = \int_0^t g(s)ds,\ t \in \mathbb{R}$, is a non-negative function satisfying the condition

$$g'_\infty := \|\beta\|_{L^1(\mathbb{R}^N)} \lim_{\xi\to+\infty} \frac{G(\xi)}{\xi^p} < +\infty \tag{3.21}$$

and set

$$\bar{\mu}'_{g,\lambda} := \frac{1}{pk_\infty^p g'_\infty}\Big(1 - \lambda p k_\infty^p \|\beta\|_{L^1(\mathbb{R}^N)} \liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^p}\Big).$$

Theorem 3.12. *Under assumptions* (A10)–(A12)*, for each* $\lambda \in]\lambda_5, \lambda_6[$ *and for every* $\mu \in [0, \bar{\mu}'_{g,\lambda}[$*, problem* (3.18) *has an unbounded sequence of symmetric solutions in* $W_r^{1,p}(\mathbb{R}^N)$*.*

We remark that no symmetry requirements on the nonlinear terms f and g are needed.

Proof of Theorem 3.12. Fix λ and μ as in the conclusion. Take $X = W^{1,p}(\mathbb{R}^N)$ and define the functionals

$$\Phi(u) = \frac{1}{p}\|u\|^p_{W^{1,p}(\mathbb{R}^N)}, \quad \Upsilon(u) = \int_{\mathbb{R}^N} [\alpha(x)F(u(x)) + \frac{\mu}{\lambda}\beta(x)G(u(x))]dx,$$
$$j(u) = 0, \quad \Psi(u) = \Upsilon(u) - j(u) = \Upsilon(u)$$

for each $u \in X$. Put $I_\lambda = \Phi - \lambda\Psi$. Clearly, the functional Φ is locally Lipschitz and weakly sequentially lower semi-continuous. Since f and g satisfy (3.19) and (3.20), respectively, and $W_r^{1,p}(\mathbb{R}^N)$ is compactly embedded in $L^\infty(\mathbb{R}^N)$, the assertion remains true regarding I_λ too. Moreover, like for Theorem 3.1, we obtain that any critical point $u \in W^{1,p}(\mathbb{R}^N)$ of the functional I_λ is a solution of the problem (3.18). Thanks to a non-smooth version of the principle of symmetric criticality introduced by Krawcewicz and Marzantowicz [10], we can obtain any critical point of $I_\lambda^r = I_\lambda|_{W_r^{1,p}(\mathbb{R}^N)}$ will be also a critical point of I_λ. Consider a real sequence $\{d_n\}$ such that $d_n \to +\infty$ as $n \to \infty$. Let $\{w_n\}$ be a sequence in $W^{1,p}(\mathbb{R}^N)$ defined as in (3.11). It is easy to verify that $w_n \in W^{1,p}(\mathbb{R}^N)$ and it is radially symmetric. Since $0 \le w_n(x) \le d_n$ for every $x \in \mathbb{R}^N$, and f and α are non-negative, one has

$$\int_{S(0,D)\setminus S(0,\frac{D}{2})} \alpha(x)F(w_n(x))dx \ge 0.$$

Hence, one has

$$\begin{aligned}
\int_{\mathbb{R}^N} \alpha(x)F(w_n(x))dx &= \int_{S(0,\frac{D}{2})} \alpha(x)F(w_n(x))dx + \int_{S(0,D)\setminus S(0,\frac{D}{2})} \alpha(x)F(w_n(x))dx \\
&\ge \int_{S(0,\frac{D}{2})} \alpha(x)F(w_n(x))dx \\
&= \omega_N(\frac{D}{2})^N \|\alpha\|_{L^1(S(0,\frac{D}{2}))} F(d_n).
\end{aligned}$$

Then, from (A10) we have

$$\liminf_{\xi\to+\infty} \frac{\|\alpha\|_{L^1(\mathbb{R}^N)}F(\xi)}{\xi^p} < \frac{\|\alpha\|_{L^1(S(0,\frac{D}{2}))}}{k_\infty^p \omega_N D^N(\frac{2^p}{D^p}(1-\frac{1}{2^N})+1)} \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^p}$$

$$\leq \frac{\int_{\mathbb{R}^N} \alpha(x)dx}{k_\infty^p \omega_N D^N(\frac{2p}{D^p}(1-\frac{1}{2^N})+1)} \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^p}.$$

As in Theorem 3.1, we can prove that the functional $I_{\overline{\lambda}}$ is unbounded from below, and it follows that $I_{\overline{\lambda}}$ has no global minimum. Therefore, applying Theorem 2.5 we deduce that there is a sequence $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^N)$ of critical points of $I_{\overline{\lambda}}$ such that $\lim_{n\to\infty} \|u_n\|_{W_r^{1,p}(\mathbb{R}^N)} = +\infty$. Hence, we have the conclusion. □

Remark 3.13. We also observe that in Theorem 3.12 by Theorem 3.8 and replacing $\xi \to +\infty$ with $\xi \to 0^+$, by the same reasoning, we have the conclusions, $\xi \to +\infty$ replaced by $\xi \to 0^+$, but the sequences of symmetric solutions strongly converge to 0 in $W_r^{1,p}(\mathbb{R}^N)$, instead.

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Shapour Heidarkhani
Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran
E-mail address: s.heidarkhani@razi.ac.ir

Fariba Gharehgazlouei
Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran
E-mail address: f.gharehgazloo@yahoo.com

Arezoo Solimaninia
Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran
E-mail address: a.solimaninia@yahoo.com