

GRADIENT ESTIMATES FOR A NONLINEAR PARABOLIC EQUATION WITH POTENTIAL UNDER GEOMETRIC FLOW

ABIMBOLA ABOLARINWA

ABSTRACT. Let (M, g) be an n dimensional complete Riemannian manifold. In this article we prove local Li-Yau type gradient estimates for all positive solutions to the nonlinear parabolic equation

$$(\partial_t - \Delta_g + \mathcal{R})u(x, t) = -au(x, t) \log u(x, t)$$

along the generalised geometric flow on M . Here $\mathcal{R} = \mathcal{R}(x, t)$ is a smooth potential function and a is an arbitrary constant. As an application we derive a global estimate and a space-time Harnack inequality.

1. PRELIMINARIES AND MAIN RESULTS

Gradient and Harnack estimates are fundamental tools to tackle classical and modern problems in geometric analysis. These methods applied to parabolic equations were first studied by Li and Yau in their celebrated paper [12]. They have been applied successfully to the setting of various geometric flows, for more details see [1, 2, 3, 4, 5, 6, 9, 10, 11, 13] and the references therein. In this article we derive various gradient estimates for the following nonlinear parabolic equation with potential

$$\left(\frac{\partial}{\partial t} - \Delta + \mathcal{R}\right)u(x, t) = -au(x, t) \log u(x, t), \quad (1.1)$$

in a more general setting of geometric flow. Here the symbol $\Delta = \Delta_g$ is the Laplace-Beltrami operator acting on functions in space with respect to metric $g(t)$ in time, a is a constant and $\mathcal{R} : M \times [0, T] \rightarrow \mathbb{R}$ is a C^∞ -function on M . For instance if we take \mathcal{R} to be the scalar curvature of the manifold and we allow g to evolve by the Ricci flow, $\partial_t g = -2Rc$, where Rc is the Ricci curvature tensor, it then reduces to the study of gradient Ricci soliton. Taking $f = \log u$, a standard calculation yields

$$\left(\frac{\partial}{\partial t} - \Delta\right)f = |\nabla f|^2 - af - \mathcal{R}. \quad (1.2)$$

The study of gradient estimates on M can be reduced to the study of the properties of the solution f of (1.2) and it is related to gradient soliton equation [5, 6]. Let

2000 *Mathematics Subject Classification*. 35K55, 53C21, 53C44, 58J35.

Key words and phrases. Gradient estimates; Harnack inequalities; parabolic equations; geometric flows.

©2015 Texas State University - San Marcos.

Submitted December 8, 2014. Published January 8, 2015.

$(M, g(t)), 0 \leq t \leq T$, be an n -dimensional complete Riemannian manifold whose metric $g(t)$ evolves by the geometric flow

$$\frac{\partial}{\partial t} g_{ij}(x, t) = 2h_{ij}(x, t), \quad (x, t) \in M \times [0, T], \quad (1.3)$$

where h_{ij} is a general time-dependent symmetric $(0, 2)$ -tensor and $0 < T < T_\varepsilon$ is taken to be the maximum time of existence for the flow; i.e., T_ε is the first time where the flow blows-up. In [1] we obtained local gradient estimates for

$$\left(\frac{\partial}{\partial t} - \Delta_g + \mathcal{R}\right)u(x, t) = 0 \quad (1.4)$$

coupled to (1.3). In this paper we extend the results to the case of (1.1) under the assumption that the geometry of the manifold remains uniformly bounded throughout the evolution. In particular, our results here can be generalised to Ricci flow and some other geometric flows on complete manifolds. Indeed, Ricci flow is a nice setting because of contracted second Bianchi identity that makes the divergence of Ricci tensor to be equal to the half gradient of scalar tensor.

We will impose boundedness condition on the Ricci curvature of the metric $g(t)$. We notice that when the metric evolves by the Ricci flow, boundedness and sign assumptions are preserved as long as the flow exists, so it follows that the metrics are uniformly equivalent. Precisely, if $-K_1g \leq Rc \leq K_2g$, where $g(t), t \in [0, T]$ is a Ricci flow, then

$$e^{-K_1T}g(0) \leq g(t) \leq e^{K_2T}g(0). \quad (1.5)$$

To see the bounds (1.5) we consider the evolution of a vector form $|X|_g = g(X, X)$, $X \in T_xM$. By the equation of the Ricci flow $\partial_t g(X, X) = -2Rc(X, X)$, $0 \leq t_1 \leq t_2 \leq T$ and by the boundedness of the Ricci curvature we have $|\partial_t g(X, X)| \leq K_2g(X, X)$, which implies (by integrating from t_1 to t_2)

$$\left| \log \frac{g(t_2)(X, X)}{g(t_1)(X, X)} \right| \leq K_2 t \Big|_{t_1}^{t_2}.$$

Taking the exponential of this estimate with $t_1 = 0$ and $t_2 = T$ yields $|g(t)| \leq e^{k_2T}g(0)$ from which the uniform boundedness of the metric follows. See [7] and [8] for details on the theory of the Ricci flow. Similarly, if there holds boundedness assumption $-cg \leq h \leq Cg$, the metric $g(t)$ are uniformly bounded below and above for all time $0 \leq t \leq T$ under the geometric flow (1.3). Then, it does not matter what metric we use in the argument that follows.

We now state the general local space-time gradient estimate corresponding to those of [1, Theorem 3.2]

Theorem 1.1 (Local gradient estimates). *Let $(M, g(t)), t \in [0, T]$ be a complete solution to the geometric flow (1.3) in some time interval $[0, T]$. Suppose there exist some nonnegative constants k_1, k_2 , and k_3 , such that $R_{ij}(g) \geq -k_1g$ and $-k_2g \leq h \leq k_3g$ for all $t \in [0, T]$. Let $u \in C^{2,1}(M \times [0, T])$ be any smooth positive solution to (1.1) in the geodesic ball $\mathcal{B}_{2\rho, T}$. Suppose $|\nabla h|, |\mathcal{R}|, |\nabla \mathcal{R}|$ and $|\Delta \mathcal{R}|$ are uniformly bounded*

on $M \times [0, T]$. Then, the following estimate holds

$$\begin{aligned} & \sup_{x \in \mathcal{B}_{2\rho}} \{ |\nabla f|^2 - \alpha f_t - \alpha a f - \alpha \mathcal{R} \} \\ & \leq \frac{\alpha n p}{2t} + \frac{\alpha n p}{4(\alpha - 1)} C_8 + \frac{\alpha n}{2} (k_2 + k_3) \varphi \sqrt{pq} \\ & \quad + \frac{\alpha n p}{2} \left\{ \frac{C_9}{\rho^2} \left(\frac{\alpha p}{\alpha - 1} + \rho \sqrt{k_1} + \rho^2 (k_2 + k_3)^2 \right) - a \right\} \end{aligned} \quad (1.6)$$

for all $(x, t) \in \mathcal{B}_{2\rho, T}$, $t > 0$ and some constants C_8 and C_9 depending only on n, α and uniform bounds for $|\nabla h|$, $|\nabla \mathcal{R}|$ and $|\Delta \mathcal{R}|$, where $f = \log u$ and $\alpha > 1$ are given such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}$ for any real numbers $p, q > 0$.

As an application of the above result we obtain global gradient estimates (Cf. Remark 3.1, equation (3.9)). We then apply the global estimates obtained to derive classical Harnack inequalities by integrating along a space-time path joining any two points in M .

The rest of the paper is as follows; in the next section we state and prove an important lemma that will be applied to prove the theorem above. The last section is devoted to the descriptions of the cut-off function needed in the proof and the detail of the proof of Theorem 1.1 itself and its application to Harnack inequality (Cf. Corollary 3.2).

2. IMPORTANT LEMMA

We first prove the following technical lemma which is a generalization of [1, Lemma 3.1]. It is originally proved for heat equation on static metric by Li and Yau [12]. This is very crucial to derivation of both local and global estimate of Li-Yau type.

Lemma 2.1. *Let $(M, g(t))$ be a complete solution to the generalized flow (1.3) in some time interval $[0, T]$. Suppose there exist some nonnegative constants k_1, k_2, k_3 , and k_4 such that $R_{ij}(g) \geq -k_1 g$, $-k_2 g \leq h \leq k_3 g$ and $|\nabla h| \leq k_4$ for all $t \in [0, T]$. For any smooth positive solution $u \in C^{2,1}(M \times [0, T])$ to equation (1.1) in the geodesic ball $\mathcal{B}_{2\rho, T}$, it holds that*

$$\begin{aligned} (\Delta - \partial_t)F & \geq -2\langle \nabla f, \nabla F \rangle - \frac{2\alpha t}{np} (\Delta f)^2 - \frac{F}{t} - 3\alpha n^{1/2} k_4 t |\nabla f| \\ & \quad - ((\alpha - 1)(2k_3 + a) + 2k_1)t |\nabla f|^2 - \alpha t \Delta \mathcal{R} \\ & \quad - 2t(\alpha - 1)\langle \nabla f, \nabla \mathcal{R} \rangle - \frac{\alpha n q}{2} t (k_2 + k_3)^2 + aF, \end{aligned} \quad (2.1)$$

where $f = \log u$, $F = t(|\nabla f|^2 - \alpha \partial_t f - \alpha \mathcal{R} - \alpha a f)$ and $\alpha \geq 1$ are given such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}$ for any real numbers $p, q > 0$.

Proof. Recall from [1, Lemma 2.1] the following evolutions under the flow

$$(|\nabla f|^2)_t = -2h_{ij} f_i f_j + 2f_i f_{ti}, \quad (2.2)$$

$$(\Delta f)_t = \Delta(f_t) - 2h_{ij} f_{ij} - 2\langle \operatorname{div} h, \nabla f \rangle + \langle \nabla H, \nabla f \rangle, \quad (2.3)$$

where div is the divergence operator, i.e., $(\operatorname{div} h)_k = g^{ij} \nabla_i h_{jk}$. Notice also that $f_t = \Delta f + |\nabla f|^2 - \mathcal{R} - a f$. Taking covariant derivative of F we have

$$F_i = t(2f_j f_{ji} - \alpha f_{ti} - \alpha \mathcal{R}_i - \alpha a f_i)$$

and with Bochner-Weitzenböck's formula

$$\Delta|\nabla f|^2 = 2|f_{ij}|^2 + 2f_j f_{jji} + 2R_{ij} f_i f_j \quad (2.4)$$

we have

$$\Delta F = \sum_{i=1}^n F_{ii} = t \left(2f_{ij}^2 + 2f_j f_{jji} + 2R_{ij} f_i f_j - \alpha \Delta(f_t) - \alpha \Delta \mathcal{R} - \alpha a \Delta f \right).$$

Using (2.3) we obtain

$$\begin{aligned} \Delta F &= t \left[2f_{ij}^2 + 2f_j f_{jji} + 2R_{ij} f_i f_j - \alpha(\Delta f)_t - 2\alpha h_{ij} f_{ij} \right. \\ &\quad \left. - 2\alpha(\operatorname{div} h)_i f_j + \alpha H_i f_j - \alpha \Delta \mathcal{R} - \alpha a \Delta f \right] \\ &= t(2f_{ij}^2 - 2\alpha h_{ij} f_{ij}) + 2t \langle \nabla f, \nabla(f_t + af + \mathcal{R} - |\nabla f|^2) \rangle \\ &\quad - \alpha t(f_t + af + \mathcal{R} - |\nabla f|^2)_t - 2\alpha t(\operatorname{div} h)_i f_j \\ &\quad + \alpha t H_i f_j - \alpha t \Delta \mathcal{R} - \alpha a t \Delta f + 2t R_{ij} f_i f_j. \end{aligned}$$

Notice that

$$\begin{aligned} -\alpha t(f_t + af + \mathcal{R} - |\nabla f|^2)_t &= t(\alpha|\nabla f|^2 - \alpha f_t - \alpha af - \alpha t \mathcal{R})_t \\ &= t \left(|\nabla f|^2 - \alpha f_t - \alpha af - \alpha \mathcal{R} + (\alpha - 1)|\nabla f|^2 \right)_t \\ &= t \left(\frac{F}{t} + (\alpha - 1)|\nabla f|^2 \right)_t \\ &= F_t - \frac{F}{t} + t(\alpha - 1)(|\nabla f|^2)_t, \end{aligned} \quad (2.5)$$

$$\begin{aligned} &2t \langle \nabla f, \nabla(f_t + af + \mathcal{R} - |\nabla f|^2) \rangle + t(\alpha - 1)(|\nabla f|^2)_t \\ &= 2t \langle \nabla f, \nabla(f_t + af + \mathcal{R} - |\nabla f|^2) \rangle + 2t(\alpha - 1) \langle \nabla f, \nabla(f_t) \rangle - 2t(\alpha - 1) h_{ij} f_i f_j \\ &= 2t \langle \nabla f, \nabla(\alpha f_t + af + \mathcal{R} - |\nabla f|^2) \rangle - 2t(\alpha - 1) h_{ij} f_i f_j \\ &= -2t \langle \nabla f, \nabla \left(\frac{F}{t} + (\alpha - 1)(af + \mathcal{R}) \right) \rangle - 2t(\alpha - 1) h_{ij} f_i f_j \\ &= -2 \langle \nabla f, \nabla F \rangle - 2t(\alpha - 1) \langle \nabla f, \nabla \mathcal{R} \rangle - 2t(\alpha - 1) a |\nabla f|^2 - 2t(\alpha - 1) h_{ij} f_i f_j, \end{aligned} \quad (2.6)$$

$$\begin{aligned} -\alpha a t \Delta f &= at(\alpha|\nabla f|^2 - \alpha f_t - \alpha af - \alpha \mathcal{R}) \\ &= aF + t(\alpha - 1)(|\nabla f|^2). \end{aligned} \quad (2.7)$$

From (2.5)–(2.7) we obtain

$$\begin{aligned} \Delta F - F_t &= t \left(2f_{ij}^2 - 2\alpha h_{ij} f_{ij} \right) - 2 \langle \nabla f, \nabla F \rangle - \frac{F}{t} - 2t(\alpha - 1) h_{ij} f_i f_j \\ &\quad - \alpha t(2(\operatorname{div} h)_i f_j - H_i f_j) - 2t(\alpha - 1) \langle \nabla f, \nabla \mathcal{R} \rangle - \alpha t \Delta \mathcal{R} \\ &\quad + 2t R_{ij} f_i f_j - 2t(\alpha - 1) a |\nabla f|^2 + t(\alpha - 1) a |\nabla f|^2 + aF. \end{aligned} \quad (2.8)$$

We now choose any two real numbers $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}$ so that we can write

$$\begin{aligned} 2f_{ij}^2 - 2\alpha h_{ij} f_{ij} &= \frac{2\alpha}{p} f_{ij}^2 + 2\alpha \left(\frac{1}{q} f_{ij}^2 - h_{ij} f_{ij} \right) \\ &\geq \frac{2\alpha}{p} f_{ij}^2 - \frac{\alpha q}{2} h_{ij}^2, \end{aligned}$$

where we have used completing the square method to arrive at the last inequality. Also by Cauchy-Schwarz inequality we have $f_{ij}^2 \geq \frac{1}{n}(\Delta f)^2$. We can also write the boundedness condition on h_{ij} as $-(k_2 + k_3)g \leq h_{ij} \leq (k_2 + k_3)g$ so that

$$\sup_M |h_{ij}|^2 \leq n(k_2 + k_3)^2$$

since h_{ij} is a symmetric tensor. Therefore,

$$t\left(2f_{ij}^2 - 2\alpha h_{ij}f_{ij}\right) \geq \frac{2\alpha t}{np}(\Delta f)^2 - \frac{\alpha nq}{2}t(k_2 + k_3)^2. \tag{2.9}$$

Notice also that

$$\begin{aligned} \alpha t\left(2(\operatorname{div} h)_i f_j - H_i f_j\right) &= 2\alpha t\left(\operatorname{div} h - \frac{1}{2}\nabla H\right) f_j \\ &= 2\alpha t\left(g^{kl}\nabla_k h_{li} - \frac{1}{2}g^{kl}\nabla_i h_{kl}\right)\nabla_j f \\ &\leq 2\alpha t\left(\frac{3}{2}|g||\nabla h|\right)|\nabla f| \\ &\leq 3\alpha t n^{\frac{1}{2}}k_4|\nabla f|. \end{aligned}$$

Putting together the last inequality, (2.9) and (2.8) with the assumption that $R_{ij} \geq -k_1g$, we arrive at

$$\begin{aligned} (\Delta - \partial_t)F &\geq -2\langle \nabla f, \nabla F \rangle - \frac{2\alpha t}{np}(\Delta f)^2 - \frac{F}{t} + aF - 2t(\alpha - 1)k_3|\nabla f|^2 \\ &\quad - 2tk_1|\nabla f|^2 - 3\alpha t n^{\frac{1}{2}}k_4|\nabla f| - \frac{\alpha nq}{2}t(k_2 + k_3)^2 - \alpha t\Delta \mathcal{R} \\ &\quad - 2t(\alpha - 1)\langle \nabla f, \nabla \mathcal{R} \rangle - t(\alpha - 1)a|\nabla f|^2. \end{aligned}$$

Our calculation is valid in the ball $\mathcal{B}_{2\rho, T}$. Hence the desired claim follows. \square

3. PROOF OF THEOREM 1.1

To prove Theorem 1.1 we use the lemma above and the assumptions that the sectional curvature, $\|\nabla h\|$, $|\mathcal{R}|$, $|\nabla \mathcal{R}|$ and $|\Delta \mathcal{R}|$ are uniformly bounded on $M \times [0, T]$. Then we write equation (2.1) as

$$\begin{aligned} (\Delta - \partial_t)F &\geq -2\langle \nabla f, \nabla F \rangle - \frac{2\alpha t}{np}(\Delta f)^2 - \frac{F}{t} + aF - C_1 t|\nabla f|^2 \\ &\quad - C_2 t|\nabla f| - 2k_1 t|\nabla f|^2 - \frac{\alpha nq}{2}t(k_2 + k_3)^2, \end{aligned} \tag{3.1}$$

where constants $C_1 > 0$ depends on α , $\max\{a, 0\}$, $\sup |h|$ and $\|\nabla h\|$, and $C_2 > 0$ depends on α , n and the space-time bounds of $\|\nabla h\|$, $|\nabla \mathcal{R}|$, $|\Delta \mathcal{R}|$. We have used the following inequality

$$3\alpha n^{1/2}k_4 t|\nabla f| \leq 2tk_4|\nabla f|^2 + 2\alpha^2 ntk_4.$$

Furthermore, by using

$$-C_2 t|\nabla f| \geq -\delta^{-1}tC_2^2 - \delta t|\nabla f|^2$$

for any number $\delta > 0$, we have

$$\begin{aligned} (\Delta - \partial_t)F &\geq -2\langle \nabla f, \nabla F \rangle - \frac{2\alpha t}{np}(\Delta f)^2 - \frac{F}{t} + aF - C_3 t|\nabla f|^2 \\ &\quad - C_4 t - 2k_1 t|\nabla f|^2 - \frac{\alpha nq}{2}t(k_2 + k_3)^2, \end{aligned} \tag{3.2}$$

where $C_3 > 0$ depends on C_1 and δ and C_4 depends on C_2 and δ .

Estimating the cut-off function. A natural function that will be defined on M is the distance function from a given point. Namely, let $y \in M$ and define $d(x, y)$ for all $x \in M$, where $d(\cdot, \cdot)$ is the geodesic distance. Note that d is everywhere continuous except on the cut locus of y and on the point where x and y coincide. It is then easy to see that $|\nabla d| = g^{ij} \partial_i d \partial_j d = 1$ on $M \setminus \{\{y\} \cup \text{cut}(y)\}$. Let $d(x, y, t)$ be the geodesic distance between x and y with respect to the metric $g(t)$, we define a smooth cut-off function $\varphi(x, t)$ with support in the geodesic ball

$$\mathcal{B}_{2\rho, T} := \{(x, t) \in M \times (0, T] : d(x, y, t) \leq 2\rho\}.$$

For any C^2 -function $\psi(s)$ on $[0, +\infty)$ with $\psi(s) = 1$ on $0 \leq s \leq 1$ and $\psi(s) = 0$ on $2 \leq s \leq +\infty$ such that $-C_5 \leq \psi'(s) \leq 0$, $-C_6 \leq \psi''(s) \leq C_6$ and $-C_6\psi \leq |\psi'|^2 \leq C_6\psi$, where C_5, C_6 are absolute constants. Let $\rho \geq 1$ and define a smooth function

$$\varphi(x, t) = \psi\left(\frac{d(x, p, t)}{\rho}\right) \quad \text{and} \quad \varphi|_{\mathcal{B}_{2\rho, T}} = 1.$$

We will apply maximum principle and invoke Calabi's trick to assume everywhere smoothness of $\varphi(x, t)$ since $\psi(s)$ is in general Lipschitz (see the argument of Li-Yau in [12]). We need Laplacian comparison theorem [14] to do some calculation on $\varphi(x, t)$. Let M be a complete n -dimensional Riemannian manifold whose Ricci curvature is bounded from below by $Rc \geq -(n-1)k_1$ for some constant $k_1 \in \mathbb{R}$, then the Laplacian of the distance function satisfies

$$\Delta d(x, y) \leq (n-1)\sqrt{|k_1|} \coth(\sqrt{|k_1|}\rho), \quad \forall x \in M \quad d(x, y) \geq \rho.$$

We need the following calculation

$$\frac{|\nabla \varphi|^2}{\varphi} = \frac{|\psi'|^2 \cdot |\nabla d|^2}{\rho^2 \psi} \leq \frac{C_6}{\rho^2}$$

and by the Laplacian comparison theorem we have

$$\begin{aligned} \Delta \varphi &= \frac{\psi' \Delta d}{\rho} + \frac{\psi'' |\nabla d|^2}{\rho^2} \\ &\geq -\frac{C_5}{\rho} (n-1)\sqrt{|k_1|} \coth(\sqrt{|k_1|}\rho) - \frac{C_6}{\rho^2} \\ &\geq -\frac{C_5 \sqrt{|k_1|}}{\rho} - \frac{C_6}{\rho^2}. \end{aligned}$$

Next is to estimate time derivative of φ : consider a fixed smooth path $\gamma : [a, b] \rightarrow M$ whose length at time t is given by $d(\gamma) = \int_a^b |\gamma'(s)|_{g(t)} ds$, where s is the arc length along the path. Differentiating we get

$$\frac{\partial}{\partial t}(d(\gamma)) = \frac{1}{2} \int_a^b |\gamma'(s)|_{g(t)}^{-1} \frac{\partial g}{\partial t}(\gamma'(s), \gamma'(s)) ds = \int_{\gamma} h_{ij}(X, X) ds,$$

where X is the unit tangent vector to the path γ . Now

$$\begin{aligned} \frac{\partial}{\partial t} \varphi &= \psi' \frac{1}{\rho} \frac{d}{dt}(d(t)) = \psi' \frac{1}{\rho} \int_{\gamma} h_{ij}(X, X) ds \\ &\leq \frac{\sqrt{C_6} \psi^{1/2}}{\rho} (k_2 + k_3)^2 \int_{\gamma} dr \leq \sqrt{C_6} (k_2 + k_3)^2 \end{aligned}$$

by choosing $0 \leq s \leq 1$, $\rho \geq 1$ and fixing the path to be of length not more than unit so that it always stays inside the geodesic ball $\mathcal{B}_{2\rho, T}$. Hence we denote

$$(\Delta - \partial_t)\varphi \geq \left(-\frac{C_5\sqrt{k_1}}{\rho} - \frac{C_6}{\rho^2} - \sqrt{C_6}(k_2 + k_3)^2 \right) =: C_7.$$

which will be used in the proof of our result.

Proof of Theorem 1.1. Using the same notation as in the previous lemma, we write $\tilde{K} = (k_2 + k_3)^2$. For a fixed $\tau \in (0, T]$ and a smooth cut-off function $\varphi(x, t)$ (chosen as before), we now estimate the inequality (3.2) at the point $(x_0, t_0) \in \mathcal{B}_{2\rho, T} \subset (M \times [0, T])$ such that $d(x, x_0, t) < 2\rho$. The argument follows from the identity

$$(\Delta - \partial_t)(\varphi F) = 2\nabla\varphi\nabla F + \varphi(\Delta - \partial_t)F + F(\Delta - \partial_t)\varphi. \quad (3.3)$$

Suppose (φF) attains its maximum value at $(x_0, t_0) \in M \times [0, T]$, for $t_0 > 0$. If $(\varphi F)(x_0, t_0) \leq 0$ for any $\rho \geq 1$, then the result holds trivially in $M \times [0, T]$ and we are done. Hence we may assume without loss of generality that there exists $(\varphi F)(x_0, t_0) > 0$. Then since $(\varphi F)(x, 0) = 0$ for all $x \in M$, we have by the maximum principle that

$$\nabla(\varphi F)(x_0, t_0) = 0, \quad \frac{\partial}{\partial t}(\varphi F)(x_0, t_0) \geq 0, \quad \Delta(\varphi F)(x_0, t_0) \leq 0, \quad (3.4)$$

where the function (φF) is being considered with support on $\mathcal{B}_{2\rho} \times [0, T]$ and we have assumed that $(\varphi F)(x_0, t_0) > 0$ for $t_0 > 0$. By (3.4) we notice that

$$(\Delta - \partial_t)(\varphi F)(x_0, t_0) \leq 0.$$

Hence we have by using the inequality (3.2) and equation (3.3):

$$\begin{aligned} 0 &\geq (\Delta - \partial_t)(\varphi F) \\ &\geq 2\nabla\varphi\nabla F + C_7F + \varphi \left\{ \frac{2\alpha}{np}t_0(\Delta f)^2 - 2\langle \nabla f, \nabla F \rangle - \frac{F}{t_0} + aF \right. \\ &\quad \left. - C_3t_0|\nabla f|^2 - C_4t_0 - 2k_1t_0|\nabla f|^2 - \frac{\alpha nq}{2}t_0(k_2 + k_3)^2 \right\}. \end{aligned} \quad (3.5)$$

The above inequality holds in the part of $\mathcal{B}_{2\rho, T}$ where $\varphi(x, t)$ is strictly positive ($0 < \varphi(x, t) \leq 1$). Notice that since $\nabla(\varphi F) = 0$, the product rule tells us that we can always replace $-F\nabla\varphi$ with $\varphi\nabla F$ at the maximum point (x_0, t_0) . Indeed, the following equalities hold

$$\begin{aligned} 2\nabla\varphi\nabla F &= 2\varphi\frac{\nabla\varphi}{\varphi}\nabla F = 2\frac{\nabla\varphi}{\varphi}(-F\nabla\varphi) = -2F\frac{C_6}{\rho^2}, \\ -2\varphi\nabla F \cdot \nabla f &= 2F\nabla\varphi \cdot \nabla f = 2F|\nabla f|\varphi\frac{|\nabla\varphi|}{\varphi} \geq -2\frac{\sqrt{C_6}}{\rho}|\nabla f|\varphi^{1/2}F. \end{aligned}$$

Multiplying (3.5) by $(t_0\varphi)$, after some simple calculations involving the last two identities at the maximum point we obtain

$$\begin{aligned} 0 &\geq -2t_0\frac{C_6}{\rho^2}\varphi F - \varphi^2F - 2t_0\frac{\sqrt{C_6}}{\rho}|\nabla f|\varphi^{\frac{3}{2}}F + C_7t_0\varphi F + a\varphi^2t_0F \\ &\quad + \varphi\frac{2t_0^2}{n}\left(\frac{\alpha}{p}(\varphi|\nabla f|^2 - \varphi(f_t + af + \varphi\mathcal{R}))\right)^2 - \frac{\alpha nq}{2}t_0^2\tilde{K}\varphi^2 \\ &\quad - C_3t_0^2\varphi^2|\nabla f|^2 - C_4\varphi^2t_0^2 - 2k_1t_0^2\varphi^2|\nabla f|^2 \end{aligned}$$

$$\begin{aligned} &\geq -2t_0 \frac{C_6}{\rho^2} \varphi F - \varphi^2 F - 2t_0 \frac{\sqrt{C_6}}{\rho} |\nabla f| \varphi^{\frac{3}{2}} F + C_7 t_0 \varphi F + a \varphi^2 t_0 F \\ &\quad + \varphi \frac{2t_0^2}{n} \left(\frac{\alpha}{p} (\varphi |\nabla f|^2 - \varphi (f_t + af + \varphi \mathcal{R})) \right)^2 - \frac{\alpha n q}{2} t_0^2 \tilde{K} \varphi^2 - C_8 t_0^2 \varphi^2 |\nabla f|^2, \end{aligned}$$

where C_8 depends on C_3 , C_4 and k_1 . Using a similar technique as in Li-Yau paper [12], when $t_0 > 0$, let $y = \varphi |\nabla f|^2$ and $z = \varphi (f_t + af + \mathcal{R})$ to obtain $\varphi^2 |\nabla f|^2 = \varphi y \leq y$, $y^{1/2} (y - \alpha z) = \frac{1}{t_0} |\nabla f| \varphi^{\frac{3}{2}} F$ and $\varphi F = t_0 (y - \alpha z)$. We obtain

$$\begin{aligned} 0 &\geq \frac{2t_0^2}{n} \left(\frac{\alpha}{p} (y - z)^2 - \frac{C_8}{2} n y - \frac{n \sqrt{C_6}}{\rho} y^{1/2} (y - \alpha z) \right) \\ &\quad - \frac{\alpha n q}{2} t_0^2 \tilde{K} \varphi^2 + \left(C_7 t_0 - 2t_0 \frac{C_6}{\rho^2} - 1 + a \varphi t_0 \right) (\varphi F). \end{aligned} \quad (3.6)$$

Notice that by direct calculations,

$$\begin{aligned} (y - z)^2 &= \left[\frac{1}{\alpha} (y - \alpha z) + \frac{\alpha - 1}{\alpha} y \right]^2 \\ &= \frac{1}{\alpha^2} (y - \alpha z)^2 + \frac{(\alpha - 1)^2}{\alpha^2} y^2 + \frac{2(\alpha - 1)}{\alpha^2} y (y - \alpha z). \end{aligned}$$

Then, the first term in the right hand side of inequality (3.6) can be simplified as follows:

$$\begin{aligned} &\frac{2t_0^2}{n} \left\{ \frac{\alpha}{p} \left[(y - z)^2 - \frac{C_8 n p}{2\alpha} y - \frac{n p \sqrt{C_6}}{\alpha \rho} y (y - \alpha z) \right] \right\} \\ &= \frac{2t_0^2}{n} \left\{ \frac{\alpha}{p} \left[\frac{1}{\alpha^2} (y - \alpha z)^2 + \left(\frac{(\alpha - 1)^2}{\alpha^2} y^2 - \frac{C_8 n p}{2\alpha} y \right) \right. \right. \\ &\quad \left. \left. + \left(\frac{2(\alpha - 1)}{\alpha^2} y - \frac{n p \sqrt{C_6}}{\alpha \rho} y^{1/2} \right) (y - \alpha z) \right] \right\} \\ &\geq \frac{2t_0^2}{n} \left\{ \frac{1}{\alpha p} (y - \alpha z)^2 - \frac{C_8^4 \alpha n^2 p}{16(\alpha - 1)^2} - \frac{C_6 \alpha n^2 p}{8\rho^2(\alpha - 1)} (y - \alpha z) \right\} \\ &= \frac{2}{\alpha n p} (\varphi F)^2 - \frac{C_8^2 \alpha n p}{8(\alpha - 1)^2} t_0^2 - \frac{C_6 \alpha n p}{4\rho^2(\alpha - 1)} t_0 (\varphi F). \end{aligned}$$

We have used the inequality of the form $ax^2 - bx \geq -\frac{b^2}{4a}$, ($a, b > 0$), to compute

$$\begin{aligned} \frac{(\alpha - 1)^2}{\alpha^2} y^2 - \frac{C_8^2 n p}{2\alpha} y &\geq -\frac{C_8^4 n^2 p^2}{16(\alpha - 1)^2}, \\ \frac{2(\alpha - 1)}{\alpha^2} y - \frac{n p \sqrt{C_6}}{\alpha \rho} y^{1/2} &\geq -\frac{C_6 n^2 p^2}{8(\alpha - 1)\rho^2}. \end{aligned}$$

Therefore, putting all these together into (3.6), we get a quadratic polynomial in (φF)

$$\begin{aligned} 0 &\geq \frac{2}{\alpha n p} (\varphi F)^2 + \left(C_7 t_0 - 2t_0 \frac{C_6}{\rho^2} - 1 + a t_0 - \frac{C_6 \alpha n p}{4\rho^2(\alpha - 1)} t_0 \right) (\varphi F) \\ &\quad - \left(\frac{C_8^2 \alpha n p}{8(\alpha - 1)^2} t_0^2 + \frac{\alpha n q}{2} t_0^2 \tilde{K} \varphi^2 \right). \end{aligned}$$

Then we develop a formula for quadratic inequality of the form $ax^2 + bx + c \leq 0$, for $x \in \mathbb{R}$. Note that when $a > 0$ and $c < 0$, then $b^2 - 4ac > 0$ and we have an

upper bound

$$x \leq \frac{-b + \sqrt{b^2 - 4ac}}{2a} \leq \frac{1}{a} \left\{ -b + \sqrt{-ac} \right\}. \tag{3.7}$$

The next is to make more explicit the term

$$\begin{aligned} b &:= \left(C_7 t_0 - 2t_0 \frac{C_6}{\rho^2} - 1 + at_0 - \frac{C_6 \alpha n p}{4\rho^2(\alpha - 1)} t_0 \right) \\ &= \left(-\frac{C_5 \sqrt{k_1}}{\rho} t_0 - \frac{C_6}{\rho^2} t_0 - \sqrt{C_6 \tilde{K}} t_0 - 2t_0 \frac{C_6}{\rho^2} - 1 + at_0 - \frac{C_6 \alpha n p}{4\rho^2(\alpha - 1)} t_0 \right) \\ &= -\left(\frac{C_9}{\rho^2} t_0 \left(\frac{\alpha p}{\alpha - 1} + \rho \sqrt{k_1} + \rho^2 \tilde{K} \right) - at_0 + 1 \right), \end{aligned}$$

where $C_9 > 0$ depends on C_6 and n . Hence, we have by applying (3.7)

$$\begin{aligned} \varphi F &\leq \frac{\alpha n p}{2} + \frac{\alpha n p}{2} \left\{ \frac{C_9}{\rho^2} t_0 \left(\frac{\alpha p}{\alpha - 1} + \rho \sqrt{k_1} + \rho^2 (k_2 + k_3)^2 \right) - at_0 \right\} \\ &\quad + \frac{\alpha n p}{4(\alpha - 1)} C_8 t_0 + \frac{\alpha n}{2} (k_2 + k_3) t_0 \varphi \sqrt{pq}. \end{aligned}$$

To obtain the required bound on $F(x, \tau)$ for an appropriate range of $x \in M$, we take $\varphi(x, \tau) \equiv 1$ whenever $d(x, x_0, \tau) < 2\rho$ and since (x_0, t_0) is the maximum point for (φF) in $\mathcal{B}_{2\rho, T}$, we have

$$F(x, \tau) = (\varphi F)(x, \tau) \leq (\varphi F)(x_0, t_0)$$

for all $x \in M$, such that $d(x, x_0, \tau) < \rho$ and $\tau \in (0, T]$ was arbitrarily chosen, then we have the conclusion in a more compact way, that

$$\sup_{x \in \mathcal{B}_{2\rho}} \left\{ |\nabla f|^2 - \alpha f_t - \alpha a f - \alpha \mathcal{R} \right\} \leq \frac{\alpha n p}{2t} (1 - at) + C_{10}, \tag{3.8}$$

where C_{10} depends on $\alpha, \tau, \rho, k_1, k_2, k_3, n, p$ and q . This completes the proof. \square

Remark 3.1. Global estimate follows by letting $\rho \rightarrow \infty$ for all $t > 0$. For instance, if we set $p = 2\alpha = q$ and allow ρ goes to infinity, we have the estimate

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} - \alpha a \log u - \alpha \mathcal{R} \leq \frac{\alpha^2 n}{t} + C_{11} \tag{3.9}$$

where C_{11} is an absolute constant depending on n, τ, α and the upper bounds of $|Rc|, |\nabla \mathcal{R}|, |\Delta \mathcal{R}|, |h|, |\nabla h|$ and $-\min\{a, 0\}$.

As an application of the global gradient estimates derived in Theorem 1.1, we obtain the following result for the corresponding Harnack estimates.

Corollary 3.2 (Hanarck estimates). *With the same assumption as in Theorem 1.1. The estimate*

$$\frac{u(x_1, t_1)}{u(x_2, t_2) e^{-\alpha(t_2 - t_1)}} \leq \left(\frac{t_2}{t_1} \right)^{\alpha n} \exp \left\{ \frac{d^2(x_1, x_2)}{4(t_2 - t_1)} + C_{12}(t_2 - t_1) \right\} \tag{3.10}$$

holds for all $(x, t) \in M \times (0, T]$, where C_{12} is an absolute constant depending on n, τ, α and the upper bounds of $|Rc|, |\mathcal{R}|, |\nabla \mathcal{R}|, |\Delta \mathcal{R}|, |h|, |\nabla h|$ and $-\min\{a, 0\}$. Here $d(x_1, x_2)$ is the geodesic distance between points x_1 and x_2 . The space-time path $\gamma : [t_1, t_2] \rightarrow M$ connects points $x_1 = \gamma(t_1)$ and $x_2 = \gamma(t_2)$ in M . Denote $|\dot{\gamma}(t)| = d(x_1, x_2)/(t_2 - t_1)$, where the norm $|\cdot|$ depends on t .

Proof. Equation (3.9) implies

$$f_t \geq \frac{1}{\alpha} |\nabla f|^2 - \frac{\alpha n}{t} - af - \mathcal{R} - \frac{1}{\alpha} C_{11}. \quad (3.11)$$

Straight computation yields

$$\begin{aligned} & e^{at_2} f(x_2, t_2) - e^{at_1} f(x_1, t_1) \\ &= \int_{t_1}^{t_2} \frac{d}{dt} \left(e^{at} f(\gamma(t), t) \right) dt \\ &= \int_{t_1}^{t_2} \left\{ e^{at} \left(f_t + \langle \nabla f(\gamma(t), t), \dot{\gamma}(t) \rangle \right) + a e^{at} f \right\} dt \\ &\geq \int_{t_1}^{t_2} \left\{ e^{at} \left(\frac{1}{\alpha} |\nabla f|^2 - \frac{\alpha n}{t} - \mathcal{R} - \frac{1}{\alpha} C_{11} + \langle \nabla f(\gamma(t), t), \dot{\gamma}(t) \rangle \right) \right\} dt \\ &\geq -e^{at_1} \left(\int_{t_1}^{t_2} \left\{ \frac{\alpha |\dot{\gamma}(t)|^2}{4} + \frac{\alpha n}{t} + \frac{1}{\alpha} C_{11} \right\} dt \right) \\ &= -e^{at_1} \left(\int_{t_1}^{t_2} \frac{\alpha d^2(x_1, x_2)}{4(t_2 - t)^2} dt + \log \left(\frac{t_2}{t_1} \right)^{\alpha n} + C_{12}(t_2 - t_1) \right). \end{aligned}$$

In the computation above we have used (3.11) to arrive at the inequality in the third line, whereas the quantity e^{at} coming up in the first line helped to get rid of the term af in (3.11). We also used an inequality of the form $Ay^2 + By \geq -B^2/4A$ to obtain the inequality in the fourth line and denoted $|\dot{\gamma}(t)| = d(x_1, x_2)/(t_2 - t_1)$ by defining a curve η in $M \times (0, T_\varepsilon)$, $\eta : [t_1, t_2] \rightarrow M \times (0, T_\varepsilon)$, by $\eta(s) = (\gamma(s), s)$. Positive constant C_{12} depends on α, C_{11} and the uniform bound for $|\mathcal{R}|$. Now, multiplying both sides by e^{-at_1} the expression in the left hand side becomes

$$f(x_1, t_1) - e^{a(t_2 - t_1)} f(x_2, t_2) = \log \left(\frac{u(x_1, t_1)}{u(x_2, t_2) e^{a(t_2 - t_1)}} \right).$$

By exponentiation we arrive at

$$u(x_1, t_1) \leq u(x_2, t_2) e^{a(t_2 - t_1)} \left(\frac{t_2}{t_1} \right)^{\alpha n} \exp \left\{ \int_{t_1}^{t_2} \frac{d^2(x_1, x_2)}{4(t_2 - t)^2} dt + C_{12}(t_2 - t_1) \right\},$$

which concludes the proof of the corollary. \square

Acknowledgements. The author wishes to thank the anonymous referees for their useful comments and for informing him that reference [3] will appear in *Advances in Geometry*. His research is supported by the TETFund of Federal Government of Nigeria and University of Sussex, United Kingdom.

REFERENCES

- [1] A. Abolarinwa; *Gradient estimates for heat-type equations on evolving manifolds*, Journal of Nonlinear Evolution Equation and Applications. To appear.
- [2] A. Abolarinwa; *Differential Harnack inequalities for nonlinear parabolic equation on time-dependent metrics*, Adv. Th. Appl. Math., **9**(2) (2014), 155-166.
- [3] M. Băileşteanu; *Gradient estimates for the heat equation under the Ricci-harmonic map flow*, arXiv:1309.0139.
- [4] M. Băileşteanu, X. Cao, A. Pulemotov; *Gradient estimates for the heat equation under the Ricci flow*, J. Funct. Anal., **258** (2010), 3517–3542.
- [5] X. Cao, R. S. Hamilton; *Differential Harnack estimates for time-dependent heat equations with potentials*, Geom. Funct. Anal., **19**(4)(2009), 989–1000.

- [6] X. Cao, Z. Zhang; *Differential Harnack estimates for parabolic equations*, Com. and Diff. Geom. Springer Proceedings in Mathematics **8**, (2011), 87–98.
- [7] B. Chow, S.-c Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, L. Peng; Luo, N. Feng; L. Ni; *The Ricci Flow: Techniques and Applications. Parts I–III, Geometric-analytic Aspects*, Math. Surveys and Monographs, 135, 144, 163, AMS, Providence, RI, (2007, 2008, 2010).
- [8] B. Chow, P. Lu, L. Ni; *Hamilton’s Ricci Flow: An Introduction*. American Mathematics Society, (2006).
- [9] G. Huang, Z. Huang, H. Li; *Gradient estimates and differential Harnack inequalities for a nonlinear parabolic equation on Riemannian manifolds*, Annals of Global Analysis and Geometry, **23**(3) (1993), 209–232.
- [10] G. Huang, B. Ma; *Gradient estimates for a nonlinear parabolic equation on Riemannian manifolds*, Arch. Math. 94 (2010), 265–275.
- [11] S. Kuang, Q. S. Zhang; *A gradient estimate for all positive solutions of the conjugate heat equation under Ricci flow*, J. Funct. Anal., **255**(4) (2008), 1008–1023.
- [12] P. Li, S.-T. Yau; *On the parabolic kernel of the Schrödinger operator*, Acta Mathematica **156**(1) (1986), 153–201.
- [13] L. Ni; *A matrix Li-Yau-Hamilton estimate for Kähler-Ricci flow*, J. Differential Geom., **75**(2) (2007), 303–358.
- [14] R. Schoen, S.-T. Yau; *Lectures on differential geometry*. International Press, Cambridge, MA, (1994).

ABIMBOLA ABOLARINWA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SUSSEX, BRIGHTON, BN1 9QH, UNITED KINGDOM
E-mail address: A.Abolarinwa@sussex.ac.uk