Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 117, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

GLOBAL WELL-POSEDNESS OF DAMPED MULTIDIMENSIONAL GENERALIZED BOUSSINESQ EQUATIONS

YI NIU, XIUYAN PENG, MINGYOU ZHANG

Abstract. We study the Cauchy problem for a sixth-order Boussinesq equations with the generalized source term and damping term. By using Galerkin approximations and potential well methods, we prove the existence of a global weak solution. Furthermore, we study the conditions for the damped coefficient to obtain the finite time blow up of the solution.

1. Introduction

In this article, we consider the Cauchy problem for damped multidimensional generalized Boussinesq equations

$$u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u_{tt} - k\Delta u_t = \Delta f(u), \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where k is a positive constant, and $f(u)$ satisfies

(A1) $f(u) = -|u|^{p-1}u$, $\frac{n+2}{n} \le p < \frac{n+2}{n-2}$ for $n \ge 3$, $1 < p < \infty$ for $n = 1, 2$.

Boussinesq [1] first derived the equation

$$u_{tt} = -\gamma u_{xxxx} + u_{xx} + (u^2)_{xx}, \tag{1.3}$$

to describe the propagation of small amplitude long waves on the surface of shallow water. Later, Makhankov [3] obtained that the improved Boussinesq equation (IBq),

$$u_{tt} - \Delta u - \Delta u_{tt} = \Delta(u^2),\ x \in \mathbb{R}^n,\ t > 0, \tag{1.4}$$

which can be derived by using the exact hydro-dynamical set of equations in plasma. A modification of the IBq equation analogous to the modified Korteweg-de Vries equation yields

$$u_{tt} - \Delta u - \Delta u_{tt} = \Delta(u^3). \tag{1.5}$$

This equation is the so-called IMBq (modified IBq) equation. Wang and Chen [7, 8] considered the existence of local and global solutions, the nonexistence of solutions, and the existence of global small amplitude solutions for (1.5) with a general source term $\Delta f(u)$.

2010 *Mathematics Subject Classification.* 35A01, 35B44, 35L75.
Key words and phrases. Cauchy problem; global solution; finite time blow up; damping term.

Submitted June 3, 2014. Published April 30, 2015.

Through investigating the water wave problem, Schneider [6] improved the model (1.3) as follows

$$u_{tt} - u_{xx} - u_{xxtt} - \mu u_{xxxx} + u_{xxxxtt} = (u^2)_{xx}, \tag{1.6}$$

where $x, t, \mu \in \mathbb{R}$ and $u(x,t) \in \mathbb{R}$. This nonlinear wave equation not only models the water wave problem with surface tension, but can also be formally derived from the two-dimensional water wave problem. Because of the linear instability, equation (1.6) with $\mu > 0$ is known as the "bad" Boussinesq equation. For the case $\mu = -1$, Wang and Mu [10] showed that (1.6) has blow up and scattering solution. By using contracting mapping principle, Wang and Guo [9] proved the existence and uniqueness for the Cauchy problem (1.6) with $\mu = -1$. Furthermore, they gave the sufficient conditions of blowup of the solution for the problem in finite time. For the multidimensional case (1.6) and the special case of nonlinear term like u^p, the Littlewood-Paley dyadic decomposition guarantees the global existence and scattering results of solution with the small initial data [11].

Xu and Liu [14] considered the initial boundary value problem of the generalized Pochhammer-Chree equation

$$u_{tt} - u_{xx} - u_{xxt} - u_{xxtt} = f(u)_{xx}, \tag{1.7}$$

where $x \in \Omega = (0,1)$. By using the contract mapping principle, they established the existence of local solutions. After modifying the source term $f(u)_{xx}$, they discussed the $W^{k,p}$ global solution and global nonexistence of generalized IMBq equations. Necat Polat [4, 5] studied the Cauchy problem of the generalized damped multidimensional Boussinesq equation with double dispersive term

$$u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u - k\Delta u_t = \Delta f(u). \tag{1.8}$$

First, by starting with the contraction mapping principle, the authors pointed out the locally well-posedness of the Cauchy problem. Then the authors obtain the necessary a priori bound. Thanks to the a priori bound, every local solution is indeed global in time. Finally, by using the concavity method, the authors proved that the local solution of the Cauchy problem blows up in finite time with negative and nonnegative initial energy. Unfortunately, it is much less known for the sixth order equations with strong damping term.

In this article, we study the Cauchy problem (1.1), (1.2), which is not only the multidimensional generalized sixth order Boussinesq equation, but also includes both nondecreasing source term and strong damping term. To deal with such problem, we refer to the papers [13, 12, 14, 15, 2, 16] especially the work by Xu and Liu [15] who proved that the Cauchy problem (1.1), (1.2) for the multidimensional sixth order equation with the generalized source term has the global H^m solution and the finite time blow up solution. However, the method they employed can not be used directly to solve the sixth order Boussinesq equation with strong damping term considered in this article. So we improve the standard concavity method and exploit further the character of the Nehari manifold in order to give a threshold result of global existence and nonexistence of solutions, and point that the solution blows up in correspondence of the sufficiently small damping coefficient. This paper is organized as follows.

In Section 2 we give some preliminary lemmas and local existence theorem. In Section 3 we give the sufficient conditions for existence and nonexistence of global weak solution for problem (1.1), (1.2), and provide the proofs of the main Theorems. In Section 4 we give some remarks on the main results ofthis article.

2. Preliminary lemmas and local existence

To prove the existence of local solutions, and the main results of this article, we provide some preliminary lemmas. Firstly, we denote $L^p(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$ by L^p and H^s respectively, with the norm $\|\cdot\|_p = \|\cdot\|_{L^p(\mathbb{R}^n)}$, $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^n)}$ and the inner product $(u,v) = \int_{\mathbb{R}^n} uv\mathrm{d}x$. We also define the space

$$H = \{u \in H^1 : (-\Delta)^{-1/2}u \in L^2\},$$

with the norm

$$\|u\|_H^2 = \|u\|_{H^1}^2 + \|(-\Delta)^{-1/2}u\|_{L^2}^2,$$

where $(-\Delta)^{-\alpha}v = \mathscr{F}^{-1}\left(|\xi|^{-2\alpha}\mathscr{F}v\right)$, $\mathscr{F}$ and $\mathscr{F}^{-1}$ are the Fourier transformation and the inverse Fourier transformation respectively. For problem (1.1)-(1.2) we introduce the functionals

$$J(u) = \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^n} F(u)dx, \quad F(u) = \int_0^u f(s)\mathrm{d}s,$$

$$I(u) = \|u\|^2 + \int_{\mathbb{R}^n} uf(u)dx,$$

$$d = \inf_{u\in\mathcal{N}} J(u), \quad \mathcal{N} = \{u \in H^1 \mid I(u) = 0, \|u\| \neq 0\}.$$

And we define the following subsets of $H^1(\mathbb{R}^n)$:

$$W = \{u \in H^1 : I(u) > 0,\ J(u) < d\} \cup \{0\};$$

$$V = \{u \in H^1 : I(u) < 0,\ J(u) < d\}, W' = \{u \in H^1 : I(u) > 0\} \cup \{0\};$$

$$V' = \{u \in H^1 : I(u) < 0\}.$$

Definition 2.1. We call $u(x,t)$ a weak solution of problem (1.1), (1.2) on $\mathbb{R}^n \times [0,T)$, if $u \in L^\infty(0,T;H^1)$, $u_t \in L^\infty(0,T;H)$ satisfy

(i) for all $v \in H$ and all $t \in [0,T)$,

$$\begin{aligned}&\Big((-\Delta)^{-1/2}u_t, (-\Delta)^{-1/2}v\Big) + (\nabla u_t, \nabla v) + (u_t, v) + k(u,v)\\&+ \int_0^t ((u,v) + (f(u),v))\,\mathrm{d}\tau\\&= \Big((-\Delta)^{-1/2}u_1, (-\Delta)^{-1/2}v\Big) + (\nabla u_1, \nabla v) + k(u_0,v)\,.\end{aligned} \tag{2.1}$$

(ii) There holds $u(x,0) = u_0(x)$ in H^1 and

$$u_t(x,0) = u_1(x) \quad \text{in } H. \tag{2.2}$$

(iii) for all $t \in [0,T)$,

$$E(t) + k\int_0^t \|u_\tau\|^2 \mathrm{d}\tau \le E(0) \tag{2.3}$$

where

$$E(t) = \frac{1}{2}\|u_t\|_H^2 + \frac{1}{2}\|u\| +^2 \int_{\mathbb{R}^n} F(u)\mathrm{d}x, \quad F(u) = \int_0^u f(s)\mathrm{d}s.$$

We present the following theorem about local existence [6, 4].

Theorem 2.2. *Suppose that $f(x)$ satisfies* (A1) *and $u_0(x), u_1(x) \in H$. Then* (1.1)-(1.2) *admits a unique local solution $u(x,t) \in H$.*

Let $u_0 \in H^1$, $u_1 \in H$, $\{w_j\}_{j=1}^{\infty}$ be a basis function system in H. We construct the approximate solutions of problem (1.1), (1.2)

$$u_m(x,t) = \sum_{j=1}^{m} g_{jm}(t) w_j(x),\ m = 1, 2, \dots, \tag{2.4}$$

satisfying

$$\begin{aligned} &\Big((-\Delta)^{-1/2} u_{mtt}, (-\Delta)^{-1/2} w_s\Big) + (u_m, w_s) + (u_{mtt}, w_s) \\ &+ (\nabla u_{mtt}, \nabla w_s) + k(u_{mt}, w_s) + (f(u_m), w_s) = 0, \quad s = 1, 2, \dots, m, \end{aligned} \tag{2.5}$$

$$u_m(x,0) = \sum_{j=1}^{m} a_{jm} w_j(x) \to u_0(x) \quad \text{in } H^1, \tag{2.6}$$

$$u_{mt}(x,0) = \sum_{j=1}^{m} b_{jm} w_j(x) \to u_1(x) \quad \text{in } H. \tag{2.7}$$

Multiplying (2.5) by $g'_{sm}(t)$ and summing for s we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} E_m(t) + k\|u_{mt}\|^2 = 0$$

and

$$E_m(t) + k\int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau = E_m(0), \tag{2.8}$$

where

$$E_m(t) = \frac{1}{2}\|u_{mt}\|_H^2 + \frac{1}{2}\|u_m\|^2 + \int_{\mathbb{R}^n} F(u_m)\mathrm{d}x, \quad F(u) = \int_0^u f(s)\mathrm{d}s. \tag{2.9}$$

Lemma 2.3. *Let $f(u)$ satisfy* (A1) *and $u \in H^1$. We have*

(i) $\lim_{\lambda\to 0} J(\lambda u) = 0$.
(ii) $I(\lambda u) = \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} J(\lambda u)$, $\forall\ \lambda > 0$. *Furthermore if $\int_{\mathbb{R}^n} u f(u)\mathrm{d}x < 0$ and $\varphi(\lambda) = -\frac{1}{\lambda}\int_{\mathbb{R}^n} u f(\lambda u)\mathrm{d}x$, then $I(\lambda u) > 0$ for $\forall \lambda > 0$.*
(iii) $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$.
(iv) *$\varphi(\lambda)$ is increasing on $0 < \lambda < \infty$.*
(v) $\lim_{\lambda\to 0} \varphi(\lambda) = 0$, $\lim_{\lambda\to+\infty} \varphi(\lambda) = +\infty$.
(vi) *In the interval $0 < \lambda < \infty$, there exists a unique $\lambda^* = \lambda^*(u)$ such that*
$$\frac{\mathrm{d}}{\mathrm{d}\lambda} J(\lambda u)\Big|_{\lambda=\lambda^*} = 0.$$
(vii) *$J(\lambda u)$ is increasing on $0 < \lambda \le \lambda^*$, decreasing on $\lambda^* \le \lambda < \infty$ and takes the maximum at $\lambda = \lambda^*$.*
(viii) *$I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < \infty$ and $I(\lambda^* u) = 0$.*

Proof. Parts (i)–(iii) are obvious. Part (iv) and Part (v) follow from

$$\varphi(\lambda) = -\frac{1}{\lambda}\int_{\mathbb{R}^n} u f(\lambda u)\mathrm{d}x = -\lambda^{p-1}\int_{\mathbb{R}^n} u f(u)\mathrm{d}x.$$

Note that $\int_{\mathbb{R}^n} u f(u)\mathrm{d}x \neq 0$ implies $\|u\| \neq 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} J(\lambda u) = \lambda\left(\|u\|^2 - \varphi(\lambda)\right), \tag{2.10}$$

which together with Part (iv) and Part (v) give Part (vi) and Part (vii).

Part (viii) follows from Part (ii) and (2.10). □

Lemma 2.4. *Let $f(u)$ satisfy* (A1) *and $u \in H^1$. We obtain*

(i) *If $0 < \|u\| < r_0$, then $I(u) > 0$;*
(ii) *If $I(u) < 0$, then $\|u\| > r_0$;*
(iii) *If $I(u) = 0$ and $\|u\| \neq 0$, i.e. $u \in \mathcal{N}$, then $\|u\| \geq r_0$, where*

$$r_0 = \Big(\frac{1}{aC_*^{p+1}}\Big)^{\frac{1}{p-1}}, \quad C_* = \sup_{u\in H^1, u\neq 0} \frac{\|u\|_{p+1}}{\|u\|}.$$

Proof. (i) If $0 < \|u\| < r_0$, then $I(u) > 0$ follows from

$$\begin{aligned} -\int_{\mathbb{R}^n} uf(u)dx &\leq \int_{\mathbb{R}^n} |uf(u)|dx = a\|u\|_{p+1}^{p+1} \leq aC_*^{p+1}\|u\|^{p+1} \\ &= aC_*^{p+1}\|u\|^{p-1}\|u\|^2 < \|u\|^2. \end{aligned}$$

(ii) If $I(u) < 0$, then $\|u\| > r_0$ follows from

$$\|u\|^2 < -\int_{\mathbb{R}^n} uf(u)dx \leq aC_*^{p+1}\|u\|^{p-1}\|u\|^2.$$

(iii) If $I(u) = 0$ and $\|u\| \neq 0$, then we have

$$\|u\|^2 = -\int_{\mathbb{R}^n} uf(u)dx \leq aC_*^{p+1}\|u\|^{p-1}\|u\|^2,$$

which together with $\|u\| \neq 0$ gives $\|u\| \geq r_0$. □

Lemma 2.5. *Let $f(u)$ satisfy* (A1), *we have*

(i)

$$d \geq d_0 = \frac{p-1}{2(p+1)}\Big(\frac{1}{aC_*^{p+1}}\Big)^{\frac{2}{p-1}}. \tag{2.11}$$

(ii) *If $u \in H^1$ and $I(u) < 0$, then*

$$I(u) < (p+1)\left(J(u) - d\right). \tag{2.12}$$

Proof. (i) For any $u \in \mathcal{N}$, by Lemma 2.4 we have $\|u\| \geq r_0$ and

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^n} F(u)dx = \frac{1}{2}\|u\|^2 + \frac{1}{p+1}\int_{\mathbb{R}^n} uf(u)dx \\ &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|u\|^2 + \frac{1}{p+1}I(u) = \frac{p-1}{2(p+1)}\|u\|^2 \geq \frac{p-1}{2(p+1)}r_0^2, \end{aligned}$$

which gives (2.11).

(ii) Let $u \in H^1$ and $I(u) < 0$, then from Lemma 2.3 it follows that there exists a λ^* such that $0 < \lambda^* < 1$ and $I(\lambda^* u) = 0$. From the definition of d we have

$$\begin{aligned} d \le J(\lambda^* u) &= \frac{1}{2}\|\lambda^* u\|^2 + \int_{\mathbb{R}^n} F(\lambda^* u)dx \\ &= \frac{1}{2}\|\lambda^* u\|^2 + \frac{1}{p+1}\int_{\mathbb{R}^n} \lambda^* u f(\lambda^* u)dx \\ &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|\lambda^* u\|^2 + \frac{1}{p+1} I(\lambda^* u) \\ &= \frac{p-1}{2(p+1)}\|\lambda^* u\|^2 = \lambda^{*2}\frac{p-1}{2(p+1)}\|u\|^2 \\ &< \frac{p-1}{2(p+1)}\|u\|^2. \end{aligned} \tag{2.13}$$

From (2.13) and

$$J(u) = \frac{p-1}{2(p+1)}\|u\|^2 + \frac{1}{p+1} I(u),$$

we obtain

$$d < \frac{p-1}{2(p+1)}\|u\|^2 = J(u) - \frac{1}{p+1} I(u),$$

which gives (2.12). □

Lemma 2.6. *Let $f(u)$ satisfy* (A1), *$u_0 \in H^1$ and $u_1 \in H$. We conclude that $F(u_0) \in L^1$. And for the approximate solutions u_m defined by* (2.4)–(2.7), *there holds $E_m(0) \to E(0)$ as $m \to \infty$, where*

$$E(0) = \frac{1}{2}\left(\|u_1\|_H^2 + \|u_0\|^2\right) + \int_{\mathbb{R}^n} F(u_0)\mathrm{d}x.$$

Proof. First from the assumptions we have

$$|F(u)| \le \frac{a}{p+1}|u|^{p+1}, \quad \forall u \in \mathbb{R},$$

where $\frac{2n+2}{n} \le p+1 < \frac{2n}{n-2}$ for $n \ge 3$ or $2 < p+1 < \infty$ for $n = 1, 2$. It is obvious that $F(u_0) \in L^1$.

From (2.6) and (2.7) we obtain that as $m \to \infty$

$$\begin{aligned} &\|(-\Delta)^{-1/2} u_{mt}(0)\|^2 + \|u_m(0)\|^2 + \|u_{mt}(0)\|^2 + \|\nabla u_{mt}(0)\|^2 \\ &\to \|(-\Delta)^{-1/2} u_1\|^2 + \|u_0\|^2 + \|u_1\|^2 + \|\nabla u_1\|^2 = \|u_1\|_H^2. \end{aligned}$$

Next we prove that

$$\int_{\mathbb{R}^n} F(u_m(0))\mathrm{d}x \to \int_{\mathbb{R}^n} F(u_0)\mathrm{d}x \text{ as } m \to \infty.$$

In fact we have

$$\begin{aligned} \Big|\int_{\mathbb{R}^n} F(u_m(0))\mathrm{d}x - \int_{\mathbb{R}^n} F(u_0)\mathrm{d}x\Big| &\le \int_{\mathbb{R}^n} |f(\varphi_m)||u_m(0) - u_0|\mathrm{d}x \\ &\le \|f(\varphi_m)\|_r \|u_m(0) - u_0\|_q, \end{aligned}$$

where $1 < q$, $r < \infty$, $\frac{1}{q} + \frac{1}{r} = 1$, $\varphi_m = u_0 + \theta\,(u_m(0) - u_0)$, $0 < \theta < 1$.

(i) If $n \ge 3$. Choose $q = \frac{2n}{n-2}$, $r = \frac{2n}{n+2}$. We have

$$\|u_m(0) - u_0\|_q \le C\|u_m(0) - u_0\| \to 0 \quad \text{as } m \to \infty,$$

$$\|f(\varphi_m)\|_r^r = \int_{\mathbb{R}^n} (a|\varphi_m|^p)^r \,\mathrm{d}x = A\|\varphi_m\|_{pr}^{pr}.$$

From the conditions we have $2 \leq pr \leq \frac{2n}{n-2}$, hence $\|f(\varphi_m)\|_r \leq C$.

(ii) If $n = 1, 2$. Choose $q = r = 2$, then we have

$$\|u_m(0) - u_0\|_q \leq \|u_m(0) - u_0\| \to 0 \quad \text{as } m \to \infty.$$
$$\|f(\varphi_m)\|_r^r = \|f(\varphi_m)\|^2 \leq A\|\varphi_m\|_{2p}^{2p}.$$

Since $2 < 2p < \infty$, we obtain $\|f(\varphi_m)\|_r < C$.

Thus for two cases above we always have

$$\int_{\mathbb{R}^n} F(u_m(0))\mathrm{d}x \to \int_{\mathbb{R}^n} F(u_0)\mathrm{d}x \quad \text{as } m \to \infty$$

and $E_m(0) \to E(0)$ as $m \to \infty$. □

Lemma 2.7. *Let $f(u)$ satisfy* (A), *$u_0 \in H^1$ and $u_1 \in H$, $E(0) < d$. Assume that $I(u_0) > 0$ or $\|u_0\| = 0$, i.e $u_0 \in W'$. Then for the approximate solutions u_m defined by* (2.4)–(2.7) *there holds $u_m \in W'$ for $0 \leq t < \infty$ and sufficiently large m. Furthermore we have*

$$\|u_m\|^2 \leq \frac{2(p+1)}{p-1}d, \quad \|u_{mt}\|_H^2 < 2d, \quad 0 \leq t < \infty, \tag{2.14}$$

for sufficiently large m.

Proof. Arguing by contradiction, we assume that there exists a $\bar{t} > 0$ such that $u_m(\bar{t}) \notin W'$ for some sufficiently large m. Then by the continuity of $I(u_m)$ with respect to t it follows that there exists a $t_0 > 0$ such that $u_m(t_0) \in \partial W'$. On the other hand, from the definition of W' we have $0 \notin \partial W'$. Hence $I(u_m(t_0)) = 0$ and $\|u_m(t_0)\| \neq 0$ for some sufficiently large m. From the definition of d we obtain $J(u_m(t_0)) \geq d$, which contradicts (by (2.8))

$$E_m(t) = \frac{1}{2}\|u_{mt}\|_H^2 + J(u_m) \leq E_m(0) < d, \quad 0 \leq t < \infty \tag{2.15}$$

for sufficiently large m.

On the other hand, from (2.15) we obtain that for sufficiently large m there holds

$$\frac{1}{2}\|u_{mt}\|_H^2 + \frac{p-1}{2(p+1)}\|u_m\|^2 + \frac{1}{p+1}I(u_m) \leq E_m(0) < d, \quad 0 \leq t < \infty,$$

which together with $u_m(t) \in W'$ gives (2.14). □

3. Existence and nonexistence of global solutions

We first give the invariance of both subsets W and V of $H^1(\mathbb{R}^n)$ under the flow of (1.1), (1.2).

Theorem 3.1 (Invariant sets)**.** *Let $f(u)$ satisfy* (A1), *$u_0 \in H^1$ and $u_1 \in H$. Assume that $E(0) < d$. Then both sets W' and V' are invariant under the flow of problem* (1.1)-(1.2). *Furthermore*

(i) *All weak solutions of problem* (1.1), (1.2) *belong to W provided $u_0 \in W'$.*
(ii) *All weak solutions of problem* (1.1), (1.2) *belong to V provided $u_0 \in V'$.*

Proof. We only prove the invariance of W', the proof for the invariance of V' is similar. Let $u(t)$ be any weak solution of (1.1), (1.2) with $u_0 \in W'$, T be the maximal existence time of $u(t)$. Next we prove that $u(t) \in W'$ for $0 < t < T$. Arguing by contradiction we assume there is a $\bar{t} \in (0,T)$ such that $u(\bar{t}) \notin W'$. According to the continuity of $I(u(t))$ with respect to t, there is a $t_0 \in (0,T)$ such that $u(t_0) \in \partial W'$. From the definition of W' and (i) of Lemma 2.4 we have $B_{r_0} \subset W'$, $B_{r_0} = \{u \in H^1 : \|u\| < r_0\}$. Hence we know $0 \notin \partial W'$. So $u(t_0) \in \partial W$ reads $I(u(t_0)) = 0$ with $\|u(t_0)\| \neq 0$. The definition of d tells $J(u(t_0)) \geq d$, which contradicts

$$\frac{1}{2}\|u_{mt}\|_H^2 + k\int_0^t \|u_\tau\|^2 \mathrm{d}\tau + J(u) \leq E(0) < d, \quad 0 \leq t < T. \tag{3.1}$$

So the prove can be completed. □

Next we show the existence of global solution for (1.1), (1.2). And we give some sufficient conditions for global well-posedness and finite time blow up. What is more, these results are independent of the local existence theory, so they are not restricted by the conditions for the local solution.

Theorem 3.2. *Let $f(u)$ satisfy* (A1), *$u_0 \in H^1$, $u_1 \in H$ and $E(0) < d$. Then problem* (1.1), (1.2) *admits a global weak solution $u(t) \in L^\infty(0,\infty;H^1)$ with $u_t(t) \in L^\infty(0,\infty;H)$ and $u(t) \in W$ for $0 \leq t < \infty$ provided $u_0 \in W'$.*

Proof. First let us turn to the existence of global solution for problem (1.1), (1.2).

For problem (1.1), (1.2), construct the approximate solutions $u_m(x,t)$ by (2.4)-(2.7). From Lemma 2.7, it follows that $\{u_m\}$ in $L^\infty(0,\infty;H^1)$ and $\{\nabla u_{mt}\}$ in $L^\infty(0,\infty;H)$ are bounded respectively. Moreover by an argument similar to that in the proof of Lemma 2.6 we can get $\{f(u_m)\}$ are bounded in $L^\infty(0,\infty;L^r)$, where r is defined in the proof of Lemma 2.6. Hence there exists a u and a subsequence $\{u_\nu\}$ of $\{u_m\}$ such that as $\nu \to \infty$; $u_\nu \to u$ in $L^\infty\left(0,\infty;H^1\right)$ weakly star and a.e. in $Q = \mathbb{R}^n \times [0,\infty)$; $\nabla u_{\nu t} \to \nabla u_t$ in $L^\infty(0,\infty;H)$ weakly star; $f(u_\nu) \to \chi = f(u)$ in $L^\infty(0,\infty;L^r)$ weakly star.

Integrating (2.5) with respect to t from 0 to t we obtain

$$\begin{aligned}
&\Big((-\Delta)^{-1/2}u_{mt}, (-\Delta)^{-1/2}w_s\Big) + (\nabla u_{mt}, \nabla w_s) + (u_{mt}, w_s)\\
&+ k(u_m, w_s) + \int_0^t ((u_m, w_s) + (f(u_m), w_s))\,\mathrm{d}\tau\\
&= \Big((-\Delta)^{-1/2}u_{mt}(0), (-\Delta)^{-1/2}w_s\Big) + (\nabla u_{mt}(0), \nabla w_s)\\
&\quad + (u_{mt}(0), w_s) + k(u_m(0), w_s).
\end{aligned} \tag{3.2}$$

Let $m = \nu \to \infty$ in (3.2) we obtain

$$\begin{aligned}
&\Big((-\Delta)^{-1/2}u_t, (-\Delta)^{-1/2}w_s\Big) + (\nabla u_t, \nabla w_s) + (u_t, w_s)\\
&+ k(u, w_s) + \int_0^t ((u, w_s) + (f(u), w_s))\,\mathrm{d}\tau\\
&= \Big((-\Delta)^{-1/2}u_1, (-\Delta)^{-1/2}w_s\Big) + (\nabla u_1, \nabla w_s) + (u_1, w_s) + k(u_0, w_s), \ \forall s
\end{aligned}$$

and

$$\Big((-\Delta)^{-1/2}u_t, (-\Delta)^{-1/2}v\Big) + (\nabla u_t, \nabla v) + (u_t, v)$$

$$
\begin{aligned}
&+ k(u,v) + \int_0^t ((u,v) + (f(u),v))\,\mathrm{d}\tau \\
&= \Big((-\Delta)^{-1/2}u_1, (-\Delta)^{-1/2}v\Big) + (\nabla u_1, \nabla v) + (u_1, v) + k(u_0, v),
\end{aligned}
$$

for all $v \in H$ and all $t \in [0,\infty)$. On the other hand, from (2.6), (2.7) we obtain

$$
\begin{aligned}
u(x,0) &= u_0(x) \quad \text{in } H^1, \\
u_t(x,0) &= u_1(x) \quad \text{in } H.
\end{aligned}
$$

Next we prove that above u satisfies (2.3). Note that the embedding $H^1 \hookrightarrow L^{p+1}$ is compact under the condition $2\left(1+\frac{1}{n}\right) \le p+1 < \frac{2n}{n-2}$ for $n \ge 3$ or $2 < p+1 < \infty$ for $n = 1, 2$. Thus from $\{u_m\}$ is bounded in $L^\infty(0,\infty; H^1)$ it follows that there exists a subsequence $\{u_\nu\}$ of $\{u_m\}$ such that as $\nu \to \infty$ $u_\nu \to u$ in L^{p+1} strongly for each $t > 0$. Hence

$$
\Big|\int_{\mathbb{R}^n} F(u_\nu)\mathrm{d}x - \int_{\mathbb{R}^n} F(u)\mathrm{d}x\Big| \le \int_{\mathbb{R}^n} |f(v_\nu)||u_\nu - u|\mathrm{d}x \le \|f(v_\nu)\|_{\bar r}\|u_\nu - u\|_{\bar q},
$$

where $\bar q = p+1$, $\bar r = \frac{p+1}{p}$, $u_\nu = u + \theta(u_\nu - u)$, $0 < \theta < 1$. From $\|u_\nu - u\|_{\bar q} \to 0$ as $\nu \to \infty$ and

$$
\|f(v_\nu)\|_{\bar r}^{\bar r} = \int_{\mathbb{R}^n} (a|v_\nu|^p)^{\bar r}\,\mathrm{d}x = a^{\frac{p+1}{p}}\|v_\nu\|_{p+1}^{p+1} \le C,
$$

we obtain

$$
\int_{\mathbb{R}^n} F(u_\nu)\mathrm{d}x \to \int_{\mathbb{R}^n} F(u)\mathrm{d}x \quad \text{as } \nu \to \infty.
$$

Hence from (2.8) we obtain

$$
\begin{aligned}
&\frac{1}{2}\Big(\|(-\Delta)^{-1/2}u_t\|^2 + \|\nabla u_t\|^2 + \|u_t\|^2 + \|u\|^2\Big) + k\int_0^t \|u_\tau\|^2\mathrm{d}\tau \\
&\le \frac{1}{2}\Big(\liminf_{\nu\to\infty}\|(-\Delta)^{-1/2}u_{\nu t}\|^2 + \liminf_{\nu\to\infty}\|\nabla u_{\nu t}\|^2 + \liminf_{\nu\to\infty}\|u_{\nu t}\|^2 + \liminf_{\nu\to\infty}\|u_\nu\|^2\Big) \\
&\quad + k\liminf_{\nu\to\infty}\int_0^t \|u_\tau\|^2\mathrm{d}\tau \\
&\le \liminf_{\nu\to\infty}\Big(\frac{1}{2}\|(-\Delta)^{-1/2}u_{\nu t}\|^2 + \frac{1}{2}\|\nabla u_{\nu t}\|^2 + \frac{1}{2}\|u_{\nu t}\|^2 + \frac{1}{2}\|u_\nu\|^2 + k\int_0^t \|u_\tau\|^2\mathrm{d}\tau\Big) \\
&= \liminf_{\nu\to\infty}\Big(E_\nu(0) - \int_{\mathbb{R}^n} F(u_\nu)\mathrm{d}x\Big) \\
&= \lim_{\nu\to\infty}\Big(E_\nu(0) - \int_{\mathbb{R}^n} F(u_\nu)\mathrm{d}x\Big) \\
&= E(0) - \int_{\mathbb{R}^n} F(u)\mathrm{d}x,
\end{aligned}
$$

which gives $E(t) \le E(0)$ for $0 \le t < \infty$. Therefore $u(x)$ is a global weak solution of problem (1.1), (1.2). Finally from Theorem 3.1 we obtain $u(t) \in W$ for $0 \le t < \infty$. ∎

Theorem 3.3. *Let $f(u)$ satisfy* (A1), *$u_0 \in H^1$, $u_1 \in H$, $(-\Delta)^{-1/2}u_0 \in L^2$ and $E(0) < d$. Then the solutionu(t) of* (1.1), (1.2) *belongs to $L^\infty(0,\infty; H^1)$, with*

$u_t(t) \in L^\infty(0,\infty;H)$ *blows up in finite time when* $I(u_0) < 0$, *and* k *satisfies*

$$
\begin{gathered}
0 < k < p-1, \quad \text{if } E(0) \le 0; \\
0 < k < (p-1)\sqrt{1-\frac{E(0)}{d_0}}, \quad \text{if } 0 < E(0) < d_0,
\end{gathered} \tag{3.3}
$$

where

$$
d_0 = \frac{p-1}{2(p+1)}\Big(\frac{1}{aC_*^{p+1}}\Big)^{\frac{2}{p-1}}, \quad C_* = \sup_{u\in H^1,\ u\neq 0} \frac{\|u\|_{p+1}}{\|u\|}.
$$

Proof. Let $u(t) \in L^\infty(0,\infty;H^1)$ with $u_t(t) \in L^\infty(0,\infty;H)$ be any weak solution of (1.1), (1.2), T be the maximal existence time of $u(t)$. Now we need to show $T < \infty$. Arguing by contradiction, we suppose that $T = +\infty$. Let $\phi(t) = \|u\|_H^2$, then

$$
\dot\phi(t) = 2((-\Delta)^{-1/2}u_t, (-\Delta)^{-1/2}u) + 2(\nabla u_t, \nabla u) + 2(u_t, u).
$$

From Schwartz inequality we obtain

$$
\begin{aligned}
&\Big(((-\Delta)^{-1/2}u_t, (-\Delta)^{-1/2}u) + (\nabla u_t, \nabla u) + (u_t, u)\Big)^2 \\
&= ((-\Delta)^{-1/2}u_t, (-\Delta)^{-1/2}u)^2 + (\nabla u_t, \nabla u)^2 + (u_t, u)^2 \\
&\quad + 2((-\Delta)^{-1/2}u_t, (-\Delta)^{-1/2}u)(\nabla u_t, \nabla u) + 2(\nabla u, \nabla u_t)(u, u_t) \\
&\quad + ((-\Delta)^{-1/2}u_t, (-\Delta)^{-1/2}u)(u_t, u) \\
&\le \|(-\Delta)^{-1/2}u_t\|^2\|(-\Delta)^{-1/2}u\|^2 + \|\nabla u_t\|^2\|\nabla u\|^2 + \|u\|^2\|u_t\|^2 \\
&\quad + \|(-\Delta)^{-1/2}u_t\|^2\|\nabla u\|^2 + \|(-\Delta)^{-1/2}u\|^2\|\nabla u_t\|^2 + \|\nabla u\|^2\|u_t\|^2 \\
&\quad + \|\nabla u_t\|^2\|u\|^2 + \|(-\Delta)^{-1/2}u_t\|^2\|u\|^2 + \|(-\Delta)^{-1/2}u\|^2\|u_t\|^2 \\
&= \|(-\Delta)^{-1/2}u\|^2\|u_t\|_H^2 + \|\nabla u\|^2\|u_t\|_H^2 + \|u\|^2\|u_t\|_H^2.
\end{aligned}
$$

we have

$$
\big(\dot\phi(t)\big)^2 \le 4\phi(t)\|u_t\|_H^2, \tag{3.4}
$$

and

$$
\begin{aligned}
\ddot\phi(t) &= 2\|(-\Delta)^{-1/2}u_t\|^2 + 2\|\nabla u_t\|^2 + 2\|u_t\|^2 \\
&\quad + 2((-\Delta)^{-1/2}u_{tt}, (-\Delta)^{-1/2}u) + 2(\nabla u_{tt}, \nabla u) + 2(u, u_{tt}) \\
&= 2\|u_t\|_H^2 - 2((-\Delta)^{-1}u_{tt}, u) - 2(\Delta u_{tt}, u) + 2(u_{tt}, u) \\
&= 2\|u_t\|_H^2 - 2(u, u) - 2(f(u), u) - 2k(u_t, u) \\
&= 2\|u_t\|_H^2 - 2(u, u) - 2(f(u), u) - 2k(u_t, u) \\
&= 2\|u_t\|_H^2 - 2\Big(\|u\|^2 + \int_{\mathbb{R}^n} uf(u)\mathrm{d}x\Big) - 2k(u_t, u) \\
&= 2\|u_t\|_H^2 - 2I(u) - 2k(u_t, u).
\end{aligned} \tag{3.5}
$$

From (2.3) we have

$$
\begin{gathered}
\frac{1}{2}\|u_t\|_H^2 + \frac{p-1}{2(p+1)}\|u\|^2 + \frac{1}{p+1}I(u) \le \frac{1}{2}\|u_t\|_H^2 + J(u) \le E(0) < d, \\
-2I(u) \ge (p+1)\|u_t\|_H^2 + (p-1)\|u\|^2 - 2(p+1)E(0).
\end{gathered}
$$

Hence we have

$$
\ddot\phi(t) \ge (p+3)\|u_t\|_H^2 + (p-1)\|u\|^2 - 2k(u_t, u) - 2(p+1)E(0). \tag{3.6}
$$

Next we consider the following three cases:

(i) We consider $E(0) < 0$. In this case from $0 < k < p-1$ it follows that there exists a ε such that $0 < \varepsilon < p-1$ and $k^2 < (p-1-\varepsilon)(p-1)$. And (3.6) gives

$$\begin{aligned} \ddot{\phi}(t) \geq (4+\varepsilon)\|u_t\|_H^2 &+ (p-1-\varepsilon)\left(\|(-\Delta)^{-1/2}u_t\|^2 + \|\nabla u_t\|^2\right) \\ &+ (p-1-\varepsilon)\|u_t\|^2 + (p-1)\|u\|^2 - 2k(u_t,u) - 2(p+1)E(0). \end{aligned} \tag{3.7}$$

And from

$$\begin{aligned} 2k|(u_t,u)| &\leq (p-1-\varepsilon)\|u_t\|^2 + \frac{k^2}{p-1-\varepsilon}\|u\|^2 \\ &\leq (p-1-\varepsilon)\|u_t\|^2 + (p-1)\|u\|^2, \end{aligned}$$

we have

$$\ddot{\phi}(t) \geq (4+\varepsilon)\|u_t\|_H^2 - 2(p+1)E(0). \tag{3.8}$$

(ii) Suppose $E(0) = 0$. In this case from $0 < k < p-1$ it follows that there exists a ε such that $0 < \varepsilon < p-1$ and $k < p-1-\varepsilon$. And (3.6) gives

$$\begin{aligned} \ddot{\phi}(t) \geq (4+\varepsilon)\|u_t\|_H^2 &+ (p-1-\varepsilon)\left(\|(-\Delta)^{-1/2}u_t\|^2 + \|\nabla u_t\|^2\right) \\ &+ (p-1-\varepsilon)\|u_t\|^2 + (p-1-\varepsilon)\|u\|^2 + \varepsilon\|u\|^2 - 2k(u_t,u), \end{aligned} \tag{3.9}$$

and from

$$\begin{aligned} 2k|(u_t,u)| &\leq (p-1-\varepsilon)\|u_t\|^2 + \frac{k^2}{p-1-\varepsilon}\|u\|^2 \\ &\leq (p-1-\varepsilon)\|u_t\|^2 + (p-1-\varepsilon)\|u\|^2, \end{aligned}$$

we have

$$\ddot{\phi}(t) \geq (4+\varepsilon)\|u_t\|_H^2 + \varepsilon\|u\|^2 \geq (4+\varepsilon)\|u_t\|_H^2 + \varepsilon r_0^2. \tag{3.10}$$

(iii) We consider $0 < E(0) < d_0$. In this case from $0 < k < (p-1)\sqrt{1-\frac{E(0)}{d_0}}$, it follows that there exists a ε such that $0 < \varepsilon < (p-1)\big(1-\frac{E(0)}{d_0}\big)$ and $k^2 < (p-1-\varepsilon)\big((p-1)\big(1-\frac{E(0)}{d_0}\big)-\varepsilon\big)$. And (3.6) gives

$$\begin{aligned} \ddot{\phi}(t) \geq (4+\varepsilon)\|u_t\|_H^2 &+ (p-1-\varepsilon)\Big(\|(-\Delta)^{-1/2}u_t\|^2 + \|\nabla u_t\|^2 + \|u_t\|^2\Big) \\ &+ \Big((p-1)\big(1-\frac{E(0)}{d_0}\big)-\varepsilon\Big)\|u\|^2 + \varepsilon\|u\|^2 \\ &+ (p-1)\Big(\frac{E(0)}{d_0}\Big)\|u\|^2 - 2k(u_t,u) - 2(p+1)E(0). \end{aligned} \tag{3.11}$$

And from Theorem 3.1 we have $u(t) \in V$ for $0 \leq t < \infty$. By Lemma 2.4, we obtain $\|u\| > r_0$. From $d_0 = \frac{p-1}{2(p+1)}r_0^2$, we obtain

$$(p-1)\Big(\frac{E(0)}{d_0}\Big)\|u\|^2 \geq (p-1)\Big(\frac{E(0)}{d_0}\Big)r_0^2 = 2(p+1)E(0). \tag{3.12}$$

We can derive

$$\ddot{\phi}(t) \geq (4+\varepsilon)\|u_t\|^2 + (p-1-\varepsilon)\left(\|(-\Delta)^{-1/2}u_t\|^2 + \|\nabla u_t\|^2\right)$$

$$+ (p-1-\varepsilon)\|u_t\|^2 + \Big((p-1)\Big(1-\frac{E(0)}{d_0}\Big)-\varepsilon\Big)\|u\|^2 + \varepsilon\|u\|^2$$
$$+ (p-1)\Big(\frac{E(0)}{d_0}\Big)\|u\|^2 - 2k(u_t,u) - 2(p+1)E(0).$$

On the other hand,

$$\begin{aligned} 2k|(u_t,u)| &\le (p-1-\varepsilon)\|u_t\|^2 + \frac{k^2}{(p-1-\varepsilon)}\|u\|^2 \\ &\le (p-1-\varepsilon)\|u_t\|^2 + \Big((p-1)\Big(1-\frac{E(0)}{d_0}\Big)-\varepsilon\Big)\|u\|^2. \end{aligned}$$

Hence we have

$$\ddot{\phi}(t) \ge (4+\varepsilon)\|u_t\|_H^2 + \varepsilon\|u\|^2 > (4+\varepsilon)\|u_t\|_H^2 + \varepsilon r_0^2. \tag{3.13}$$

From (3.8), (3.10) and (3.13), it follows that there exists a δ_0 such that for all above cases there holds

$$\ddot{\phi}(t) \ge (4+\varepsilon)\|u_t\|_H^2 + \delta_0. \tag{3.14}$$

Hence

$$\phi(t)\ddot{\phi}(t) - \frac{\varepsilon+4}{4}\big(\dot{\phi}(t)\big)^2 \ge \delta_0\|u_t\|_H^2 \ge 0, \tag{3.15}$$

and

$$\big(\phi^{-\alpha}(t)\big)'' = \frac{-\alpha}{\phi(t)^{\alpha+2}}\Big(\phi(t)\ddot{\phi}(t) - (\alpha+1)\big(\dot{\phi}(t)\big)^2\Big) \le 0, \tag{3.16}$$
$$\alpha = \frac{\varepsilon}{4}, \quad 0 < t < \infty.$$

On the other hand, from (3.14) we obtain

$$\dot{\phi}(t) \ge \delta_0 t + \dot{\phi}(0), \quad 0 < t < \infty.$$

Hence there exists a $t_0 \ge 0$ such that $\dot{\phi}(t) > \dot{\phi}(t_0) > 0$ for $t > t_0$ and

$$\phi(t) > \dot{\phi}(t_0)(t-t_0) + \phi(t_0) \ge \dot{\phi}(t_0)(t-t_0),\ t_0 < t < \infty.$$

Therefore there exits a $t_1 > 0$ such that $\phi(t_1) > 0$ and $\dot{\phi}(t_1) > 0$. From this and (3.16) it follows that there exists a $T_1 > 0$ such that

$$\lim_{t\to T_1} \phi^{-\alpha}(t) = 0,$$

and

$$\lim_{t\to T_1} \phi(t) = +\infty, \tag{3.17}$$

which contradicts $T = +\infty$. So we prove the nonexistence of global weak solutions. □

4. Remarks

In this section, we give some remarks on the main results of this paper. First Theorem 3.3 can be written as follows

Theorem 4.1. *Let $f(u)$, u_0 and u_1 be same as those in Theorem 3.2 and 3.3. Assume that $E(0) < d_0$, where d_0 is defined in Lemma 2.5, i.e.*

$$d_0 = \frac{p-1}{2(p+1)}\Big(\frac{1}{aC_*^{p+1}}\Big)^{\frac{2}{p-1}}.$$

Then when $\|u_0\| < r_0$ *problem* (1.1), (1.2) *admits a global weak solution; and when* $\|u_0\| \geq r_0$*, and* k *satisfies*

$$\begin{gathered} 0 < k < p-1, \quad \text{if } E(0) \leq 0; \\ 0 < k < (p-1)\sqrt{1 - \frac{E(0)}{d_0}}, \quad \text{if } 0 < E(0) < d_0. \end{gathered} \tag{4.1}$$

Then problem (1.1), (1.2) *does not admit any global weak solution, where* r_0 *is defined in Lemma 2.4; i.e.,*

$$r_0 = \Big(\frac{1}{aC_*^{p+1}}\Big)^{\frac{1}{p-1}}, \quad C_* = \sup_{u \in H^1, u \neq 0} \frac{\|u\|_{p+1}}{\|u\|}.$$

Proof. We will complete this proof by considering case $\|u_0\| < r_0$ and case $\|u_0\| \geq r_0$ separately.

(i) Since $\|u_0\| < r_0$ implies $0 < \|u_0\| < r_0$ or $\|u_0\| = 0$. If $0 < \|u_0\| < r_0$, from Lemma 2.4 we can derive $I(u_0) > 0$. Hence the weak solution exists globally.

(ii) If $\|u_0\| \geq r_0$, then from

$$\begin{aligned} &\frac{1}{2}\|u_1\|_H^2 + k\int_0^t \|u_1\|^2 \mathrm{d}\tau + \frac{p-1}{2(p+1)}\|u_0\|^2 + \frac{1}{p+1}I(u_0) \\ &\leq E(0) < d_0 = \frac{p-1}{2(p+1)}\Big(\frac{1}{aC_*^{p+1}}\Big)^{\frac{2}{p-1}} = \frac{p-1}{2(p+1)}r_0^2, \end{aligned}$$

we obtain $I(u_0) < 0$. Then Theorem 3.3 gives that there is no global weak solution for problem (1.1), (1.2). □

So the results of Theorem 4.1 show that the space $H = \{u \in H^1 \mid (-\Delta)^{-1/2}u \in L^2\}$ is divided into two subspaces: $\|u\| < r_0$ and $\|u\| > r_0$ by the surface $\|u\| = r_0$. Furthermore, we have all weak solutions $u(t)$ of problem (1.1), (1.2) with $E(0) < d_0$ belong to $B_{r_0} = \{u \in H \mid \|u\| < r_0\}$, and problem (1.1), (1.2) does not admit any global weak solutions if $u_0 \in \bar{B}_{r_0}^c = \{u \in H \mid \|u\| \geq r_0\}$ and k satisfies (4.1).

In the case $E(0) \leq 0$, which is a special case of the energy restriction $E(0) < d$. We have the following result.

Theorem 4.2. *Let* $f(u)$ *satisfy* (A1) *and* $u_0, u_1 \in H$*. Assume that* $E(0) < 0$ *or* $E(0) = 0$*,* $\|u_0\| \neq 0$*. Then all weak solutions of problem* (1.1)-(1.2) *belong to* V*.*

Proof. Let $u(t)$ be any weak solution of problem (1.1)-(1.2) with $E(0) < 0$ or $E(0) = 0$, $\|u_0\| \neq 0$, T be the maximal existence time of $u(t)$. From

$$\frac{1}{2}\|u_1\|_H^2 + k\int_0^t \|u_1\|^2 d\tau + \frac{p-1}{2(p+1)}\|u_0\|^2 + \frac{1}{p+1}I(u_0) \leq E(0),$$

we see that if $E(0) < 0$ or $E(0) = 0$ with $\|u_0\| \neq 0$, then $I(u_0) < 0$. Hence from Theorem 3.1 we obtain $u(t) \in V$ for $0 \leq t < T$. □

Furthermore from Theorem 3.3 and Theorem 4.2 we can conclude the following corollary.

Corollary 4.3. *Let* $f(u)$ *satisfy* (A1) *and* $u_0, u_1 \in H$*. Assume that* $E(0) < 0$ *or* $E(0) = 0$*,* $\|u_0\| \neq 0$ *and* k *satisfies*

$$0 < k < p-1, \quad \text{if } E(0) \leq 0;$$

$$0 < k < (p-1)\sqrt{1 - \frac{E(0)}{d_0}}, \quad \textit{if } 0 < E(0) < d_0.$$

Then problem (1.1), (1.2) *does not admit any global weak solution.*

Corollary 4.4. *Under the conditions of Theorem 3.3, for the global weak solution of problem* (1.1)-(1.2) *given in Theorem 3.3 we further have*

$$u(t) \in L^\infty(0,T;H), \quad \forall T > 0.$$

Proof. From

$$(-\Delta)^{-1/2}u = \int_0^t (-\Delta)^{-1/2}u_\tau \mathrm{d}\tau + (-\Delta)^{-1/2}u_0, \quad 0 \le t < \infty,$$

we obtain

$$\begin{aligned}
\|(-\Delta)^{-1/2}u\| &\le \int_0^t \|(-\Delta)^{-1/2}u_\tau\|\mathrm{d}\tau + \|(-\Delta)^{-1/2}u_0\| \\
&\le T \max_{0\le t\le T} \Big(\|(-\Delta)^{-1/2}u_t\|\Big) + \|(-\Delta)^{-1/2}u_0\|, \quad 0 \le t \le T,
\end{aligned}$$

which gives

$$\begin{gathered}
(-\Delta)^{-1/2}u \in L^\infty(0,T;L^2), \quad \forall T > 0, \\
u(t) \in L^\infty(0,T;H), \quad \forall T > 0.
\end{gathered}$$

□

Acknowledgements. This work was supported by the National Natural Science Foundation of China (11471087, 41306086).

References

[1] Boussinesq, J.; *Theorie des ondes et de remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contene dans ce canal des vitesses sensiblement pareilles de la surface au foud.* J. Math Pures Appl., 1872, **217**: 55-108.

[2] Liu, Y. C.; Xu, R. Z.; *Global existence and blow up of solutions for Cauchy problem of generalized Boussinesq equation.* Phys. D, 2008, **237**(6): 721-731.

[3] Makhankov, V. G.; *Dynamics of classical solitons (in nonintegrable systems).* Phys. Rep., 1978, **35**(1): 1-128.

[4] Polat, N.; Ertaş, A.; *Existence and blow-up of solution of Cauchy problem for the generalized damped multidimensional Boussinesq equation.* J. Math. Anal. Appl., 2009, **349**(1): 10-20.

[5] Polat, N.; Pişkin, E.; *Asymptotic behavior of a solution of the Cauchy problem for the generalized damped multidimensional Boussinesq equation.* Applied Mathematics Letters, 2012, **25**: 1871-1874.

[6] Schneider, G.; Eugene, C. W.; *Kawahara dynamics in dispersive media.* Phys. D, 2001, **152/153**: 108-110.

[7] Wang, S. B.; Chen, G. W.; *Small amplitude solutions of the generalized IMBq equation.* J. Math. Anal. Appl, 2002, **274**(2): 846-866.

[8] Wang, S. B.; Chen, G. W.; *The Cauchy problem for the generalized IMBq equation in $W^{s,p}(\mathbb{R}^n)$.* J. Math. Anal. Appl., 2002, **262**(1): 38-54.

[9] Wang, Y. P.; , Guo, B. L.; *The Cauchy problem for a generalized Boussinesq type equation.* Chinese Ann. Math. Ser. A, 2008, **29**(2): 185-194.

[10] Wang, Y., Mu, C. L.; *Blow-up and scattering of solution for a generalized Boussinesq equation.* Appl. Math. Comput., 2007, **188**(2): 1131-1141.

[11] Xia, S. X.; Yuan, J.; *Existence and scattering of small solutions to a Boussinesq type equation of sixth order.* Nonlinear Anal., 2010, **73**(4): 1015-1027.

[12] Xu, R. Z.; *Cauchy problem of generalized Boussinesq equation with combined power-type nonlinearities.* Mathematical Methods in the Applied Sciences, 2011, **34**(18): 2318-2328.

[13] Xu, R. Z.; Ding, Y. H.; *Global solutions and finite time blow up for damped Klein-Gordon equation*. Acta Mathematica Scientia, 2013, **33**(3): 643-652.
[14] Xu, R. Z.; Liu, Y. C.; *Global existence and blow-up of solutions for generalized Pochhammer-Chree equations*. Acta Mathematica Scientia, 2010, **30**(5): 1793-1807.
[15] Xu, R. Z.; Liu, Y. C.; Liu, B. W.; *The Cauchy problem for a class of the multidimensional Boussinesq-type equation*. Nonlinear Anal., 2011, **74**(6): 2425-2437.
[16] Xu, R. Z.; Liu, Y. C.; *Global existence and nonexistence of solution for Cauchy problem of multidimensional double dispersion equations*. J. Math. Anal. Appl., 2009, **359**(2): 739-751.

Yi Niu
College of Automation, Harbin Engineering University, 150001, China
E-mail address: `niuyipde@163.com`

Xiuyan Peng
College of Automation, Harbin Engineering University, 150001, China
E-mail address: `pxygll@163.com`

Mingyou Zhang
College of Automation, Harbin Engineering University, 150001, China
E-mail address: `zmy1985624@163.com`