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PERSISTENCE AND EXTINCTION OF A NON-AUTONOMOUS LOGISTIC EQUATION WITH RANDOM PERTURBATION

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Abstract. Persistence and extinction of a randomized non-autonomous logistic equation is studied. Sufficient conditions for extinction, non-persistence in the mean, weak persistence and stochastic permanence are established. The critical number between weak persistence and extinction is obtained.

1. Introduction

Logistic system is the most important model in both ecology and mathematical ecology. Persistence and extinction of this model is an interesting and important topic owing to its theoretical and practical significance. The deterministic Logistic equation is usually denoted by:

$$dx(t)/dt = x(t)[r - ax(t)] \tag{1.1}$$

for $t \geq 0$ with initial value $x(0) = x_0 > 0$, and $x(t)$ is the population density at time t. r stands for the growth rate and a denotes the intraspecific competition coefficient; i.e., r/a is the carrying capacity. We refer the reader to May [26] for a detailed model construction. Model (1.1) describes a single species whose members compete among themselves for a limited amount of food and living space.

Owing to its theoretical and practical significance, system (1.1) and its generalization form have been extensively studied and many important results on the global dynamics of solutions have been founded, see e.g. Freedman and Wu [4], Golpalsamy [8] and Lisena [22] and the references therein. Particularly, the book by Golpalsamy [8] is a very good reference in this area.

On the other hand, in the real world, population dynamics is inevitably affected by environmental noise which is an important component in an ecosystem (see e.g. Gard [5, 6, 7]). May [26] pointed out that due to environmental noise, the birth rate in the population system should be stochastic. Therefore lots of authors introduce stochastic perturbation into deterministic models to reveal the effect of environmental variability on the population dynamics in mathematical ecology (see e.g. [1, 2, 3], [11]-[23], [27]-[30].

Especially, under the assumption that the the growth rate r in (1.1) is stochastically perturbed, with

$$r \to r + \sigma x^{\theta} \dot{B}(t),$$

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Ji, Jiang, Shi and O'Regan [11] studied the stochastic Logistic equation

$$dx(t) = x(t)[(r - ax(t))dt + \sigma x^\theta dB(t)], \tag{1.2}$$

where $\dot{B}(t)$ represents the white noise, namely $B(t)$ is a Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$; σ^2 denotes the intensity of the white noise; $\theta \in (0, 0.5)$. Ji et al. [11] showed that following lemma.

Lemma 1.1. *If $r > 0$, $a > 0$, $\theta \in (0, 0.5)$ and $\sigma > 0$, then*

(i) *Equation (1.2) has a unique and positive solution on $t \geq 0$ almost surely (a.s.) with any given initial value $x_0 > 0$.*

(ii) *The solution of (1.2) is stochastically persistent; i.e., for any given $\varepsilon \in (0, 1)$, there are positive constants H_1 and H_2 such that*

$$\limsup_{t\to+\infty} \mathcal{P}\{x(t) \geq H_1\} \geq 1 - \varepsilon, \quad \limsup_{t\to+\infty} \mathcal{P}\{x(t) \leq H_2\} \geq 1 - \varepsilon.$$

Some important and interesting questions are as follows:

(Q1) Since model (1.2) describes a population dynamics, it is critical to find out when the population is extinctive and when is persistent. Furthermore, can we find out the critical number between survival and extinction for model (1.2)?

(Q2) The results of Lemma 1 are based on the conditions $\theta \in (0, 0.5)$, $r > 0$ and $\sigma > 0$. What happens if these conditions are not satisfied?

(Q3) Model (1.2) assumes that the growth rate r and the carrying capacity r/a are independent of time t. However, the natural growth rates of many populations vary with t in real situation, for example, due to the seasonality. Thus, do the conclusions of Lemma 1.1 still hold if all the coefficients vary with t?

In this paper, we shall study the stochastic non-autonomous Logistic model

$$dx(t) = x(t)[r(t) - a(t)x(t)]dt + \sigma(t)x^{1+\theta}(t)dB(t), \tag{1.3}$$

where $\theta \in (0, 1)$; $r(t)$, $a(t)$ and $\sigma(t)$ are continuous and boundedness functions on $R_+ := [0, +\infty)$ and $\min_{t\in R_+} a(t) > 0$. We have the following results:

(R1) Equation (1.3) has a unique and positive solution on $t \geq 0$ a.s. with any given initial value $x_0 > 0$.

(R2) Define $\langle r\rangle^* = \limsup_{t\to+\infty} t^{-1} \int_0^t r(s)ds$.

(R21) If $\langle r\rangle^* < 0$, then the species, $x(t)$, represented by model (1.3) goes to extinction a.s., i.e. $\lim_{t\to+\infty} x(t) = 0$, a.s..

(R22) If $\langle r\rangle^* = 0$, then $x(t)$ is nonpersistent in the mean a.s., i.e.

$$\lim_{t\to+\infty} \langle x(t)\rangle = \lim_{t\to+\infty} t^{-1} \int_0^t x(s)ds = 0, \quad \text{a.s..}$$

(R23) If $\langle r\rangle^* > 0$, then $x(t)$ is weakly persistent (see e.g. [1, 9]) a.s.; i.e., $x^* = \limsup_{t\to+\infty} x(t) > 0$, a.s.

(R3) Define $r_* = \liminf_{t\to+\infty} r(s)$. If $r_* > 0$, then $x(t)$ is stochastically permanent; i.e., for any given $\varepsilon \in (0, 1)$, there are positive constants H_1 and H_2 such that

$$\liminf_{t\to+\infty} \mathcal{P}\{x(t) \geq H_1\} \geq 1 - \varepsilon, \quad \liminf_{t\to+\infty} \mathcal{P}\{x(t) \leq H_2\} \geq 1 - \varepsilon.$$

The important contributions of this paper is therefore clear.

Remark 1.2. It is useful to point out that our definition of stochastic permanence is different from the definition of stochastic persistence given in [11]. It is easy to see that if $x(t)$ is stochastically permanent, then it is stochastically persistent. But the converse is not true.

The rest of the paper is organized as follows. In Section 2, we give the proofs of our main results. In Section 3, we work out some figures to illustrate our main theorems. The last section gives the conclusions.

2. Proofs

For the sake of convenience, we define the following symbols:

$$\langle f(t)\rangle = t^{-1}\int_0^t f(s)ds, \quad f^* = \limsup_{t\to+\infty} f(t), \quad f_* = \liminf_{t\to+\infty} f(t),$$
$$\hat{\nu} = \max_{t\in R_+}\nu(t), \quad \check{\nu} = \min_{t\in R_+}\nu(t).$$

Theorem 2.1. *Equation* (1.3) *has a unique and positive solution on* $t \geq 0$ *with any given initial value* $x_0 > 0$.

Proof. Our proof is motivated by the works of Mao, Marion and Renshaw [25]. Since the coefficients of Eq. (1.3) are locally Lipschitz continuous, then for any given initial value $x(0) \in R_+$, there is a unique maximal local solution $x(t)$ on $t \in [0, \tau_e]$, where τ_e is the explosion time (see e.g. [24]). To show this solution is global, we only need to show that $\tau_e = \infty$. For this end, let $n_0 > 0$ be so large that x_0 lying within the interval $[1/n_0, n_0]$. For each integer $n > n_0$, define the stopping times

$$\tau_n = \inf\{t \in [0, \tau_e] : x(t) \notin (1/n, n)\}.$$

Clearly, τ_n is increasing as $n \to \infty$. Let $\tau_\infty = \lim_{n\to+\infty}\tau_n$, whence $\tau_\infty \leq \tau_e$ *a.s.* Now, we only need to show $\tau_\infty = \infty$. If this statement is false, there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$\mathcal{P}\{\tau_\infty < \infty\} > \varepsilon$$

Consequently, there exists an integer $n_1 \geq n_0$ such that

$$\mathcal{P}\{\tau_n < T\} > \varepsilon, n > n_1. \tag{2.1}$$

Define

$$V(x) = \sqrt{x} - 1 - 0.5\ln x.$$

If $x(t) \in R_+$, in view of Itô's formula (see e.g. [24]), we have

$$\begin{aligned} dV(x) &= V_x dx + 0.5V_{xx}(dx)^2 \\ &= 0.5x^{-0.5}(1 - x^{-0.5})\big[x(r(t) - a(t)x)dt + \sigma(t)x^{1+\theta}dB(t)\big] \\ &\quad + 0.5(-0.25x^{-1.5} + 0.5x^{-2})\sigma^2(t)x^{2+2\theta}dt \\ &= \Big[-0.125\sigma^2(t)x^{0.5+2\theta} + 0.25\sigma^2(t)x^{2\theta} - 0.5a(t)x^{1.5} + 0.5a(t)x \\ &\quad + 0.5r(t)x^{0.5} - 0.5r(t)\Big]dt + 0.5\sigma(t)x^\theta(x^{0.5} - 1)dB(t). \end{aligned} \tag{2.2}$$

Note that $\min_{t\in R_+} a(t) > 0$, then there is clearly a constant $G_1 > 0$ such that

$$-0.25\sigma^2(t)x^{2\theta}(0.5x^{0.5} - 1) - 0.5a(t)x^{1.5} + 0.5a(t)x + 0.5r(t)x^{0.5} - 0.5r(t) < G_1.$$

Substituting this inequality into (2.2), we see that

$$dV(x(t)) \le G_1 dt + 0.5\sigma(t)x^\theta(x^{0.5}-1)dB(t),$$

which implies that

$$\int_0^{\tau_n\wedge T} dV(x(t)) \le \int_0^{\tau_n\wedge T} G_1 dt + \int_0^{\tau_n\wedge T} 0.5\sigma(s)x^\theta(s)(x^{0.5}(s)-1)dB(s),$$

where $\rho\wedge\varrho = \min\{\rho,\varrho\}$. Taking expectation on both sides of the above inequality, we can derive that

$$EV(x(\tau_n\wedge T)) \le V(x_0) + G_1E(\tau_n\wedge T) \le V(x_0) + G_1T \tag{2.3}$$

Set $\Omega_n = \{\tau_n \le T\}$, then by inequality (2.1) we have $\mathcal{P}(\Omega_n) \ge \varepsilon$. Note that for every $\omega\in\Omega_n$, $x(\tau_n,\omega)$ equals either n or $1/n$, hence $V(x(\tau_n,\omega))$ is no less than

$$\min\{\sqrt{n}-1-0.5\ln n, 1/\sqrt{n}-1+0.5\ln n\}.$$

It then follows from (2.3) that

$$V(x_0)+G_1T \ge E[1_{\Omega_n}(\omega)V(x(\tau_n))] \ge \varepsilon\min\{\sqrt{n}-1-0.5\ln n, 1/\sqrt{n}-1+0.5\ln n\}$$

where 1_{Ω_n} is the indicator function of Ω_n. Letting $n\to\infty$ leads to the contradiction

$$\infty > V(x_0)+G_1T = \infty,$$

which completes the proof. □

Theorem 2.2. *If $\langle r\rangle^* < 0$, then the species, $x(t)$, represented by model* (1.3) *goes to extinction a.s.*

Proof. Applying Itô's formula to (1.3), it gives

$$d\ln x = \frac{dx}{x} - \frac{(dx)^2}{2x^2} = [r(t)-a(t)x-0.5\sigma^2(t)x^{2\theta}]dt + \sigma(t)x^\theta dB(t).$$

In other words,

$$\ln x(t) - \ln x_0 = \int_0^t [r(s)-a(s)x(s)-0.5\sigma^2(s)x^{2\theta}(s)]ds + M(t), \tag{2.4}$$

where $M(t) = \int_0^t \sigma(s)x^\theta(s)dB(s)$ is a local martingale, whose quadratic variation is

$$\langle M(t), M(t)\rangle = \int_0^t \sigma^2(s)x^{2\theta}(s)ds.$$

In view of the exponential martingale inequality (see e.g. [24]), for any positive constants T,α and β, we have

$$\mathcal{P}\Big\{\sup_{0\le t\le T}\big[M(t) - \frac{\alpha}{2}\langle M(t),M(t)\rangle\big] > \beta\Big\} \le \exp\{-\alpha\beta\}. \tag{2.5}$$

Choose $T=n$, $\alpha=1$, $\beta=2\ln n$, then it follows that

$$\mathcal{P}\Big\{\sup_{0\le t\le n}\big[M(t) - \frac{1}{2}\langle M(t),M(t)\rangle\big] > 2\ln n\Big\} \le 1/n^2.$$

Using Borel-Cantalli Lemma [24] leads to that for almost all $\omega\in\Omega$, there is a random integer $n_0 = n_0(\omega)$ such that for $n\ge n_0$,

$$\sup_{0\le t\le n}\big[M(t) - \frac{1}{2}\langle M(t),M(t)\rangle\big] \le 2\ln n.$$

That is to say

$$M(t) \leq 2\ln n + \frac{1}{2}\langle M(t), M(t)\rangle = 2\ln n + 0.5\int_0^t \sigma^2(s)x^{2\theta}(s)ds$$

for all $0 \leq t \leq n$, $n \geq n_0$ almost surely. Substituting the above inequality into (2.4), it results in

$$\ln x(t) - \ln x_0 \leq \int_0^t r(s)ds - \int_0^t a(s)x(s)ds + 2\ln n \leq \int_0^t r(s)ds + 2\ln n \tag{2.6}$$

for all $0 \leq t \leq n,\ n \geq n_0$ almost surely. In other words, we have shown that for $0 < n-1 \leq t \leq n$,

$$t^{-1}\{\ln x(t) - \ln x_0\} \leq \langle r(t)\rangle + \frac{2\ln n}{n-1},$$

which means that $[t^{-1}\ln x(t)]^* \leq \langle r\rangle^*$. That is to say, if $\langle r\rangle^* < 0$, one can see that $\lim_{t\to+\infty} x(t) = 0$. $\square$

Theorem 2.3. *If $\langle r\rangle^* = 0$, then $x(t)$ is nonpersistent in the mean a.s.*

Proof. For any given $\varepsilon > 0$, there exists a T_1 such that

$$t^{-1}\int_0^t r(s)ds \leq \langle r\rangle^* + \varepsilon/2 = \varepsilon/2, \quad t \geq T_1.$$

Substituting this inequality into (2.6), one can see that

$$\ln x(t) - \ln x_0 \leq \int_0^t r(s)ds - \int_0^t a(s)x(s)ds + 2\ln n \leq \varepsilon t/2 - \check{a}\int_0^t x(s)ds + 2\ln n$$

for all $T_1 \leq t \leq n$, $n \geq n_0$ almost surely. Note that there exists a $T > T_1$ such that for all $T \leq n-1 \leq t \leq n$ and $n \geq n_0$ we have $(\ln n)/t \leq \varepsilon/4$. In other words, we have already shown that

$$\ln x(t) - \ln x_0 \leq \varepsilon t - \check{a}\int_0^t x(s)ds$$

for sufficiently large $t > T$. Let $g(t) = \int_0^t x(s)ds$, then we obtain

$$\ln(dg/dt) < \varepsilon t - \check{a}g(t) + \ln\ x_0, \quad t > T,$$

which means that

$$\exp(\check{a}g(t))(dg/dt) < x_0\exp(\varepsilon t), \quad t > T.$$

Integrating this inequality from T to t gives

$$\check{a}^{-1}\big[\exp(\check{a}g(t)) - \exp(\check{a}g(T))\big] < x_0\varepsilon^{-1}\big[\exp(\varepsilon t) - \exp(\varepsilon T)\big].$$

Rewriting this inequality one then sees that

$$\exp(\check{a}g(t)) < \exp(\check{a}g(T)) + x_0\check{a}\varepsilon^{-1}\exp(\varepsilon t) - x_0\check{a}\varepsilon^{-1}\exp(\varepsilon T).$$

Taking the logarithm of both sides leads to

$$g(t) < \check{a}^{-1}\ln\Big\{x_0\check{a}\varepsilon^{-1}\exp(\varepsilon t) + \exp(\check{a}g(T)) - x_0\check{a}\varepsilon^{-1}\exp(\varepsilon T)\Big\}.$$

In other words, we have already shown that

$$\Big\{t^{-1}\int_0^t x(s)ds\Big\}^* \leq \check{a}^{-1}\Big\{t^{-1}\ln\big[x_0\check{a}\varepsilon^{-1}\exp(\varepsilon t) + \exp(\check{a}g(T)) - x_0\check{a}\varepsilon^{-1}\exp(\varepsilon T)\big]\Big\}^*.$$

An application of the L'Hopital's rule, one can derive

$$\langle x\rangle^* \le \check{a}^{-1}\Big\{t^{-1}\ln\big[x_0\check{a}\varepsilon^{-1}\exp(\varepsilon t)\big]\Big\}^* = \varepsilon/\check{a}.$$

Since ε is arbitrary, we get $\langle x\rangle^* \le 0$, which is the required assertion. □

Theorem 2.4. *If $\langle r\rangle^* > 0$, then $x(t)$ is weakly persistent a.s.*

Proof. First, let us show that

$$[t^{-1}\ln x(t)]^* \le 0 \quad \text{a.s.} \tag{2.7}$$

In fact, applying Itô's formula to (1.3), it results in

$$\begin{aligned} d(\exp(t)\ln x) &= \exp(t)\ln x dt + \exp(t) d\ln x \\ &= \exp(t)[\ln x + r(t) - a(t)x - 0.5\sigma^2(t)x^{2\theta}]dt + \exp(t)\sigma(t)x^\theta dB(t). \end{aligned}$$

Thus, we have shown that

$$\exp(t)\ln x(t) - \ln x_0 = \int_0^t \exp(s)[\ln x(s) + r(s) - a(s)x(s) - 0.5\sigma^2(s)x^{2\theta}(s)]ds + N(t), \tag{2.8}$$

where $N(t) = \int_0^t \exp(s)\sigma(s)x^\theta(s)dB(s)$ is a martingale with the quadratic form

$$\langle N(t), N(t)\rangle = \int_0^t \exp(2s)\sigma^2(s)x^{2\theta}(s)ds.$$

It then follows from the exponential martingale inequality (2.5), by choosing $T = \gamma k$, $\alpha = \exp(-\gamma k)$ and $\beta = \theta\exp(\gamma k)\ln k$, that

$$\mathcal{P}\Big\{\sup_{0\le t\le\gamma k}\big[N(t) - 0.5\exp(-\gamma k)\langle N(t), N(t)\rangle\big] > \theta\exp(\gamma k)\ln k\Big\} \le k^{-\theta},$$

where $\theta > 1$ and $\gamma > 1$. By virtue of the famous Borel-Cantelli lemma, for almost all $\omega\in\Omega$, there exists $k_0(\omega)$ such that for every $k \ge k_0(\omega)$,

$$N(t) \le 0.5\exp(-\gamma k)\langle N(t), N(t)\rangle + \theta\exp(\gamma k)\ln k, \quad 0 \le t \le \gamma k.$$

Substituting the above inequality into (2.8) yields

$$\begin{aligned} &\exp(t)\ln x(t) - \ln x_0 \\ &\le \int_0^t \exp(s)\big[\ln x(s) + r(s) - a(s)x(s) - 0.5\sigma^2(s)x^{2\theta}(s)\big]ds \\ &\quad + 0.5\exp(-\gamma k)\int_0^t \exp(2s)\sigma^2(s)x^{2\theta}(s)ds + \theta\exp(\gamma k)\ln k \\ &= \int_0^t \exp(s)\big[\ln x(s) + r(s) - a(s)x(s) \\ &\quad - 0.5\sigma^2(s)x^{2\theta}(s)[1-\exp(s-\gamma k)]\big]ds + \theta\exp(\gamma k)\ln k. \end{aligned}$$

It is easy to see that for any $0 \le s \le \gamma k$ and $x > 0$, since $\min_{t\in R_+} a(t) > 0$, then there exists a constant C independent of k such that

$$\ln x + r(s) - a(s)x - 0.5\sigma^2(s)x^{2\theta}[1-\exp(s-\gamma k)] \le C.$$

In other words, for any $0 \le t \le \gamma k$, we have

$$\exp(t)\ln x(t) - \ln x_0 \le C[\exp(t) - 1] + \theta\exp(\gamma k)\ln k.$$

That is to say

$$\ln x(t) \le \exp(-t)\ln x_0 + C[1-\exp(-t)] + \theta \exp(-t)\exp(\gamma k)\ln k.$$

If $\gamma(k-1) \le t \le \gamma k$ and $k \ge k_0(\omega)$, we have

$$\ln x(t)/t \le \exp(-t)\ln x_0/t + C[1-\exp(-t)]/t + \theta \exp(-\gamma(k-1))\exp(\gamma k)\ln k/t,$$

which becomes the desired assertion (2.7) by letting $t \to +\infty$.

Now suppose that $\langle r\rangle^* > 0$, we prove that $x^* > 0$ a.s.. If this assertion is not true, let S be the set $S = \{x^* = 0\}$, then $\mathcal{P}(S) > 0$. It follows from (2.4) that

$$t^{-1}[\ln x(t) - \ln x(0)] = \langle r(t)\rangle - \langle a(t)x(t)\rangle - 0.5\langle \sigma^2(t)x^{2\theta}(t)\rangle + M(t)/t. \tag{2.9}$$

On the other hand, for for all $\omega \in S$, we have $\lim_{t\to+\infty} x(t,\omega) = 0$, then the law of large numbers for local martingales (see e.g. [24]) implies that $\lim_{t\to+\infty} M(t)/t = 0$. Substituting the above inequality into (2.9) gives

$$[\ln x(t,\omega)/t]^* = \langle r(t)\rangle^* > 0.$$

Then $\mathcal{P}\Big([\ln x(t)/t]^* > 0\Big) > 0$, this contradicts (2.7). □

Theorem 2.5. *If $r_* > 0$, then species $x(t)$ represents by model* (1.3) *will be stochastically permanent.*

Proof. First we demonstrate that for any given $0 < \varepsilon < 1$, there exists constant $H_1 > 0$ such that $\mathcal{P}_*\{x(t) \ge H_1\} \ge 1 - \varepsilon$. Define

$$V_1(x) = 1/x^{1+\theta}$$

for $x \in R_+$. Applying Itô's formula to equation (1.3) we can obtain

$$\begin{aligned} dV_1(x(t)) &= -(1+\theta)x^{-2-\theta}dx + 0.5(1+\theta)(2+\theta)x^{-3-\theta}(dx)^2 \\ &= (1+\theta)V_1(x)[a(t)x - r(t)]dt + 0.5(1+\theta)(2+\theta)\sigma^2(t)x^{\theta-1}dt \\ &\quad - (1+\theta)\sigma(t)x^{-1}dB(t). \end{aligned}$$

Define

$$V_2(x) = (1 + V_1(x))^\kappa,$$

where $0 < \kappa < 1$. Applying Itô's formula again leads to

$$\begin{aligned} dV_2(x(t)) &= \kappa(1+V_1(x(t)))^{\kappa-1}dV_1 + 0.5\kappa(\kappa-1)(1+V_1(x(t)))^{\kappa-2}(dV_1)^2 \\ &= \kappa(1+V_1(x))^{\kappa-2}\Big\{(1+V_1(x))\Big[(1+\theta)V_1(x)[a(t)x - r(t)] \\ &\quad + 0.5(1+\theta)(2+\theta)\sigma^2(t)x^{\theta-1}\Big] + 0.5(\kappa-1)(1+\theta)^2\sigma^2(t)x^{-2}\Big\}dt \\ &\quad - \kappa(1+V_1(x))^{\kappa-1}(1+\theta)\sigma(t)x^{-1}dB(t) \\ &= \kappa(1+\theta)(1+V_1(x))^{\kappa-2}\Big\{-r(t)V_1^2(x) - r(t)V_1(x) + a(t)V_1(x)x^{-\theta} \\ &\quad + a(t)x^{-\theta} + 0.5(2+\theta)\sigma^2(t)x^{\theta-1} \\ &\quad + 0.5(2+\theta)\sigma^2(t)x^{-2} + 0.5(\kappa-1)(1+\theta)\sigma^2(t)x^{-2}\Big\}dt \\ &\quad - \kappa(1+V_1(x))^{\kappa-1}(1+\theta)\sigma(t)x^{-1}dB(t) \\ &\le \kappa(1+\theta)(1+V_1(x))^{\kappa-2}\Big\{-\big(r_* - \varepsilon\big)V_1^2(x) \\ &\quad + \hat{a}V_1(x)x^{-\theta} + \hat{a}x^{-\theta} + 1.5\hat{\sigma^2}x^{\theta-1} + 1.5\hat{\sigma^2}x^{-2}\Big\}dt \end{aligned}$$

$$-\kappa(1+V_1(x))^{\kappa-1}(1+\theta)\sigma(t)x^{-1}dB(t)$$

for sufficiently large t. In the last inequality, we have used the facts that $r_*>0$, $\theta<1$ and $\kappa<1$. Now, choose $\eta>0$ sufficiently small to satisfy

$$0<\frac{\eta}{\kappa(1+\theta)}<r_*-\varepsilon.$$

Define $V_3(x)=\exp\{\eta t\}V_2(x)$. By Itô's formula,

$$\begin{aligned}
&dV_3(x(t))\\
&=\eta\exp\{\eta t\}V_2(x)dt+\exp\{\eta t\}dV_2(x)\\
&\le(1+\theta)\kappa\exp\{\eta t\}(1+V_1(x))^{\kappa-2}\Big\{\frac{\eta(1+V_1(x))^2}{\kappa(1+\theta)}\\
&\quad-(r_*-\varepsilon)V_1^2(x)+\hat{a}V_1(x)x^{-\theta}+\hat{a}x^{-\theta}+1.5\hat{\sigma^2}x^{\theta-1}+1.5\hat{\sigma^2}x^{-2}\Big\}dt\\
&\quad-\exp\{\eta t\}\kappa(1+V_1(x))^{\kappa-1}(1+\theta)\sigma(t)x^{-1}dB(t)\\
&=(1+\theta)\kappa\exp\{\eta t\}(1+V_1(x))^{\kappa-2}\Big\{-\Big(r_*-\varepsilon-\frac{\eta}{\kappa(1+\theta)}\Big)V_1^2(x)\\
&\quad+\frac{2\eta}{\kappa(1+\theta)}V_1(x)+\frac{\eta}{\kappa(1+\theta)}+\hat{a}V_1(x)x^{-\theta}+\hat{a}x^{-\theta}+1.5\hat{\sigma^2}x^{\theta-1}+1.5\hat{\sigma^2}x^{-2}\Big\}dt\\
&\quad-\exp\{\eta t\}\kappa(1+V_1(x))^{\kappa-1}(1+\theta)\sigma(t)x^{-1}dB(t)\\
&=\exp\{\eta t\}J(x)dt-\exp\{\eta t\}\kappa(1+V_1(x))^{\kappa-1}(1+\theta)\sigma(t)x^{-1}dB(t)
\end{aligned}$$

for sufficiently large t, where

$$\begin{aligned}
J(x)&=(1+\theta)\kappa(1+V_1(x))^{\kappa-2}\Big\{-\Big(r_*-\varepsilon-\frac{\eta}{\kappa(1+\theta)}\Big)V_1^2(x)+\frac{2\eta}{\kappa(1+\theta)}V_1(x)\\
&\quad+\frac{\eta}{\kappa(1+\theta)}+\hat{a}V_1(x)x^{-\theta}+\hat{a}x^{-\theta}+1.5\hat{\sigma^2}x^{\theta-1}+1.5\hat{\sigma^2}x^{-2}\Big\}.
\end{aligned}\tag{2.10}$$

Now, let us show that $J(x)$ is upper bounded in R_+. To prove this, without loss of generality, let us suppose that $\hat{\sigma^2}>0$. Set

$$K=\min\Big\{1,\Big(\frac{r_*-\varepsilon-\eta/[\kappa(1+\theta)]}{3\hat{\sigma^2}}\Big)^{-2\theta}\Big\}.$$

(a) If $x\ge K$, then it follows from the definition of $V_1(x)$ that $J(x)$ is upper bounded, namely, there exists a positive number J_1 such that $\sup_{x\ge K}J(x)<J_1$.

(b) If $x<K$, then making use of $x<1$ and $0<\theta<1$ lead to that

$$x^{-\theta}\le x^{-0.5-0.5\theta}=V_1^{0.5}(x),\quad x^{\theta-1}=x^{2\theta}x^{-\theta-1}\le V_1(x).\tag{2.11}$$

At the same time, it follows from $x<\Big(\frac{r_*-\varepsilon-\eta/[\kappa(1+\theta)]}{3\hat{\sigma^2}}\Big)^{-2\theta}$ that

$$-0.5\Big(r_*-\varepsilon-\frac{\eta}{\kappa(1+\theta)}\Big)V_1^2(x)+1.5\hat{\sigma^2}x^{-2}<0.\tag{2.12}$$

Substituting (2.11) and (2.12) into (2.10) gives

$$\begin{aligned}
J(x)&\le(1+\theta)\kappa(1+V_1(x))^{\kappa-2}\Big\{-0.5\Big(r_*-\varepsilon-\frac{\eta}{\kappa(1+\theta)}\Big)V_1^2(x)+\frac{2\eta}{\kappa(1+\theta)}V_1(x)\\
&\quad+\frac{\eta}{\kappa(1+\theta)}+\hat{a}V_1^{1.5}(x)+\hat{a}V_1^{0.5}(x)+1.5\hat{\sigma^2}V_1(x)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= (1+\theta)\kappa(1+V_1(x))^{\kappa-2}\Big\{ -0.5\Big(r_* - \varepsilon - \frac{\eta}{\kappa(1+\theta)}\Big)V_1^2(x) + \hat{a}V_1^{1.5}(x) \\
&\quad + [\frac{2\eta}{\kappa(1+\theta)} + 1.5\hat{\sigma^2}]V_1(x) + \hat{a}V_1^{0.5}(x) + \frac{\eta}{\kappa(1+\theta)}\Big\} \\
&=: (1+\theta)\kappa(1+V_1(x))^{\kappa-2}H(x).
\end{aligned}
$$

Note that $r_* - \varepsilon - \frac{\eta}{\kappa(1+\theta)} > 0$, then there is a positive constant $x_0 \le K$ such that if $x \le x_0$, then $H(x) \le 0$. Therefore if $0 < x \le x_0$, then $J(x) \le 0$. On the other hand, if $x_0 \le x \le K$, by the continuity of $(1+\theta)\kappa(1+V_1(x))^{\kappa-2}H(x)$, there is a positive number J_2 such that $\sup_{x_0 \le x \le K} J(x) < J_2$. In other words, we have shown that if $x \le K$, then $\sup_{x \le K} J(x) < J_2$. Consequently, $J(x)$ is upper bounded in R_+, namely $J_3 := \sup_{x \in R_+} J(x) < +\infty$. Therefore,

$$
dV_3(x(t)) \le J_3 \exp\{\eta t\}dt - \exp\{\eta t\}\kappa(1+V_1(x))^{\kappa-1}(1+\theta)\sigma(t)x^{-1}dB(t)
$$

for sufficiently large t. Integrating both sides of the above inequality and then taking expectations give

$$
E\Big[\exp\{\eta t\}\Big(1+V_1(x(t))\Big)^{\kappa}\Big] \le \Big(1+V_1(x(T))\Big)^{\kappa} + J_3(\exp\{\eta t\} - \exp\{\eta T\})/\eta.
$$

That is to say

$$
\limsup_{t\to+\infty} E[V_1^{\kappa}(x(t))] \le \limsup_{t\to+\infty} E[(1+V_1(x(t)))^{\kappa}] \le J_3/\eta.
$$

In other words, we have already shown that

$$
\limsup_{t\to+\infty} E[x^{-\kappa(1+\theta)}(t)] \le J_3/\eta =: J_4.
$$

Thus for any given $\varepsilon > 0$, let $H_1 = \varepsilon^{-\kappa(1+\theta)}/J_4^{-\kappa(1+\theta)}$, by Chebyshev's inequality, we can derive that

$$
\mathcal{P}\{x(t) < H_1\} = \mathcal{P}\{x^{-\kappa(1+\theta)}(t) > H_1^{-\kappa(1+\theta)}\} \le H_1^{\kappa(1+\theta)}E[x^{-\kappa(1+\theta)}(t)],
$$

that is to say $\limsup_{t\to+\infty} \mathcal{P}\{x(t) < H_1\} \le H_1^{\kappa(1+\theta)}J_4 = \varepsilon$. Consequently

$$
\liminf_{t\to+\infty} \mathcal{P}\{x(t) \ge H_1\} \ge 1 - \varepsilon.
$$

Next we show that for arbitrary fixed $\varepsilon > 0$, there exists $H_2 > 0$ such that $\mathcal{P}_*(x(t) \le H_2) \ge 1 - \varepsilon$. The following proof is motivated by the works of Luo and Mao [23]. Define

$$
V(x) = x^q
$$

for $x \in R_+$, where $0 < q < 1$. Then it follows from Itô's formula that

$$
\begin{aligned}
dV(x) &= qx^{q-1}dx + \frac{q(q-1)}{2}x^{q-2}(dx)^2 \\
&= qx^{q-1}\Big\{x[r(t) - a(t)x] + \sigma(t)x^{1+\theta}dB(t)\Big\} + \frac{q-1}{2}x^{q-2}\sigma^2(t)x^{2+2\theta}dt \\
&= qx^q\big[r(t) - a(t)x - \frac{1-q}{2}\sigma^2(t)x^{2\theta}\big]dt + q\sigma(t)x^{q+\theta}dB(t).
\end{aligned}
$$

Let $k_0 > 0$ be so large that x_0 lying within the interval $[1/k_0, k_0]$. For each integer $k \ge k_0$, define the stopping time

$$
\tau_k = \inf\{t \ge 0 : x(t) \notin (1/k, k)\}.
$$

Clearly $\tau_k \to \infty$ almost surely as $k \to \infty$. Applying Itô's formula again to $\exp\{t\}V(x)$ gives

$$\begin{aligned}
&d(\exp\{t\}V(x)) = \exp\{t\}V(x)dt + \exp\{t\}dV(x)\\
&= \exp\{t\}\big[x^q + qx^q(r(t) - a(t)x - \frac{1-q}{2}\sigma^2(t)x^{2\theta})\big]dt + \exp\{t\}q\sigma(t)x^{q+\theta}dB(t)\\
&\le \exp\{t\}[x^q + qx^q(r(t) - a(t)x)]dt + \exp\{t\}q\sigma(t)x^{q+\theta}dB(t)\\
&\le \exp\{t\}M_5 + \exp\{t\}q\sigma(t)x^{q+\theta}dB(t),
\end{aligned}$$

where M_5 is a positive constant. Integrating this inequality and then taking expectations on both sides, one can see that

$$E\big[\exp\{t\wedge\tau_k\}x^q(t\wedge\tau_k)\big] - x_0^q \le E\int_0^{t\wedge\tau_k}\exp\{s\}M_5ds \le M_5(\exp\{t\}-1),$$

Letting $k \to \infty$ yields

$$\exp\{t\}E[x^q(t)] \le x_0^q + M_5(\exp\{t\}-1),$$

which indicates that

$$\limsup_{t\to+\infty} E[x^q(t)] \le M_5.$$

Then the desired assertion follows from Chebyshev's inequality. □

3. Numerical simulations

In this section we shall use the Milstein method mentioned in Higham [10] to illustrate the analytical results. Consider the discretization equation

$$\begin{aligned}
x_{k+1} = x_k &+ x_k[r(k\ \Delta t) - a(k\ \Delta t)x_k]\Delta t + \sigma(k\Delta t)x_k^{1+\theta}\sqrt{\Delta t}\xi_k\\
&+ 0.5\sigma^2(k\ \Delta t)x_k^{2+2\theta}(\xi_k^2\Delta t - \Delta t),
\end{aligned}$$

where ξ_k, $k = 1, 2, \dots, n$ are Gaussian random variables.

In Figure 1, we choose $\theta = 0.8$, $a(t) = 0.3 + 0.1\sin(2t)$ and $\sigma^2(t) = 8$. The only difference between conditions of Figure 1(a), Figure 1(b), Figure 1(c) and Figure 1(d) is that the representation of $r(t)$ is different. In Figure 1(a), we choose $r(t) = -0.001 + 0.2\sin t$. Then we have $\langle r(t)\rangle^* < 0$. In view of Theorem 2, x goes to extinction. Figure 1(a) confirms this. In Figure 1(b), we choose $r(t) = 0.2\sin t$. Then it is easy to obtain $\langle r(t)\rangle^* = 0$. It follows from Theorem 3 that x is non-persistence in the mean. See Figure 1(b). In Figure 1(c), we choose $r(t) = 0.001 + 0.2\sin t$. Then $\langle r(t)\rangle^* > 0$. By virtue of Theorem 4, one can obtain that x is weakly persistent. This can be seen from Figure 1(c). In Figure 1(d), we choose $r(t) = 0.12 + 0.02\sin t$. Then $\liminf_{t\to+\infty} r(t) > 0$. By Theorem 5, x is stochastically permanent. Figure 1(d) confirms this.

4. Concluding remarks

For a stochastic non-autonomous Logistic equation we obtained sufficient conditions for extinction, non-persistence in the mean, weak persistence and stochastic permanence. The critical number between weak persistence and extinction was obtained initially. The behavior of the model for several coefficient cases was studied. More precisely,

(I) If $\langle r\rangle^* < 0$, then $x(t)$ is extinctive with probability one.
(II) If $\langle r\rangle^* = 0$, then $x(t)$ is non-persistence in the mean with probability one.

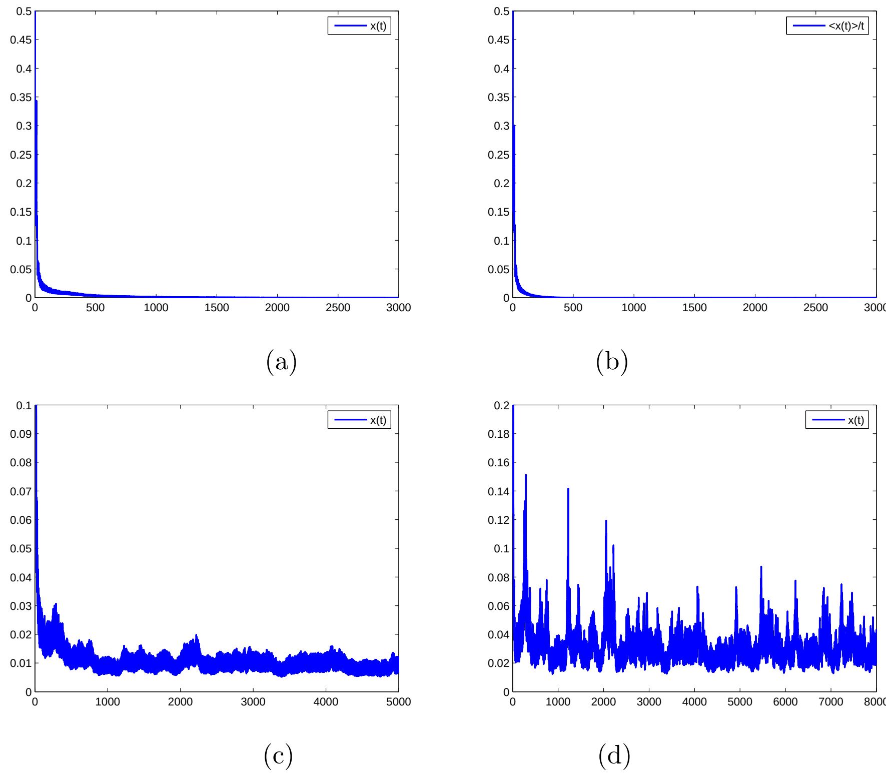

FIGURE 1. Solutions of system (1.3) for $\theta = 0.8$, $a(t) = 0.3 + 0.1\sin(2t)$, $\sigma^2(t) = 8$, $x(0) = 0.4$, step size $\Delta t = 0.001$. The horizontal axis represents the time t. (a) with $r(t) = -0.001 + 0.2\sin t$; (b) with $r(t) = 0.2\sin t$; (c) with $r(t) = 0.001 + 0.2\sin t$; (d) with $r(t) = 0.12 + 0.2\sin t$

(III) If $\langle r\rangle^* > 0$, then $x(t)$ is be weakly persistent with probability one.
(IV) If $r_* > 0$, then $x(t)$ is stochastically permanent.

Our key contributions in this article are:

(A) We obtained the critical number between weak persistence and extinction for the first time, which is neglected by all the existing papers.
(B) Our conditions of Theorem 5 are much weaker than (ii) in Lemma 1.1. And our results are stronger than (ii) in Lemma 1.1 (see Remark 1 above).
(C) This article deals with the *non-autonomous* stochastic logistic model, while [11] considered the *autonomous* case.

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