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# OSCILLATION CRITERIA FOR FOURTH-ORDER NONLINEAR DELAY DYNAMIC EQUATIONS

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Abstract. We obtain criteria for the oscillation of all solutions to a fourth-order nonlinear delay dynamic equation on a time scale that is unbounded from above. The results obtained are illustrated with examples

## 1. Introduction

This article concerns the oscillation of all solutions to the fourth-order nonlinear delay dynamic equation

$$x^{\Delta^4}(t) + p(t)x^{\gamma}(\tau(t)) = 0 \tag{1.1}$$

on a time scale $\mathbb{T}$, where $\gamma$ is the ratio of positive odd integers, $p$ is a positive real-valued rd-continuous function defined on $\mathbb{T}$, $\tau \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{T})$, $\tau(t) \le t$, and $\lim_{t\to\infty} \tau(t) = \infty$. As we are interested in oscillatory behavior, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above and is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$ with $t_0 \in \mathbb{T}$.

By a solution to (1.1) we mean a nontrivial real-valued function $x \in \mathrm{C}^4_{\mathrm{rd}}[T_x, \infty)_{\mathbb{T}}$, $T_x \in [t_0, \infty)_{\mathbb{T}}$ which satisfies (1.1) on $[T_x, \infty)_{\mathbb{T}}$. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. A solution $x$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger [20] in his PhD thesis in 1988 in order to unify continuous and discrete analysis. The study of the oscillation of dynamic equations on time scales is a new area of applied mathematics, and work in this topic is rapidly growing. Recently, there has been an increasing interest in obtaining sufficient conditions for oscillation and nonoscillation of solutions of various equations on time scales, we refer the reader to the books [3, 4, 7, 8, 29] and the articles [1, 2, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 31], and the references cited therein. Regarding the oscillation of first-order and second-order dynamic equations, Agarwal and Bohner [1], Bohner et al. [6], Braverman and

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Karpuz [9], Şahiner and Stavroulakis [26] examined a first-order delay dynamic equation

$$x^{\Delta}(t) + p(t)x(\tau(t)) = 0.$$

Agarwal et al. [2], Erbe et al. [13], Şahiner [27], Zhang and Zhu [30] considered a second-order delay dynamic equation

$$x^{\Delta^2}(t) + p(t)x(\tau(t)) = 0.$$

Akın-Bohner et al. [5] investigated a second-order Emden–Fowler dynamic equation

$$x^{\Delta^2}(t) + p(t)x^{\gamma}(\sigma(t)) = 0.$$

Saker [28] studied a second-order dynamic equation

$$(rx^{\Delta})^{\Delta}(t) + p(t)f(x(\sigma(t))) = 0.$$

For the oscillation of higher-order dynamic equations on time scales, Erbe et al. [14] investigated a third-order dynamic equation

$$x^{\Delta^3}(t) + p(t)x(t) = 0.$$

Hassan [19] and Li et al. [21] considered a third-order nonlinear delay dynamic equation

$$\left(a((rx^{\Delta})^{\Delta})^{\gamma}\right)^{\Delta}(t) + f(t, x(\tau(t))) = 0.$$

Grace et al. [16] studied a fourth-order dynamic equation

$$x^{\Delta^4}(t) + p(t)x^{\gamma}(\sigma(t)) = 0.$$

Grace et al. [18] examined a fourth-order dynamic equation

$$x^{\Delta^4}(t) + p(t)x^{\gamma}(t) = 0.$$

Zhang et al. [31] investigated a fourth-order dynamic equation

$$(rx^{\Delta^3})^{\Delta}(t) + p(t)f(x(\sigma(t))) = 0.$$

Erbe et al. [12] considered a higher-order neutral delay dynamic equation

$$(x(t) + A(t)x(\alpha(t)))^{\Delta^n} + B(t)x(\beta(t)) = 0.$$

Karpuz [23, 24] studied a higher-order neutral delay dynamic equation

$$(x(t) + A(t)x(\alpha(t)))^{\Delta^n} + B(t)F(x(\beta(t))) = \varphi(t).$$

The Riccati transformation technique plays an important role in obtaining sufficient conditions for oscillation of dynamic equations. For instance, Erbe et al. [13], Şahiner [27], and Saker [28] applied the Riccati substitution as

$$\omega := \delta \frac{x^{\Delta}}{x}$$

to the second-order dynamic equations, where $x > 0$, $x^{\Delta} > 0$, and $\delta$ is an optional function. Hassan [19] used the Riccati transformation

$$\omega := \delta \frac{a((rx^{\Delta})^{\Delta})^{\gamma}}{(x \circ \tau)^{\gamma}},$$

where $x \circ \tau > 0$, $(rx^{\Delta})^{\Delta} > 0$, and $\delta$ is an optional function. Erbe et al. [14] utilized the Riccati substitution

$$\omega := \delta \frac{x^{\Delta^2}}{x^{\Delta}},$$

where $x^{\Delta} > 0$, $x^{\Delta^2} > 0$, and $\delta$ is an optional function.

The aim of this paper is to give some new oscillation theorems for (1.1). This article is organized as follows: In the next section, we present the basic definitions and the theory of calculus on time scales. In the section 3, we will establish some oscillation results for (1.1) by employing some different Riccati substitutions. In the section 4, we shall give two examples to illustrate our main results.

## 2. Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above and is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$. On any time scale we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} | s < t\},$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, right-dense if $\sigma(t) = t$ and $t < \sup \mathbb{T}$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess $\mu$ of the time scale is defined by $\mu(t) := \sigma(t) - t$.

For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of $f$ may actually be replaced by any Banach space), the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if $f$ is continuous at $t$ and $t$ is right-scattered. If $t$ is not right-scattered then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t^+} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{s \to t^+} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.

A function $f$ is said to be differentiable if its derivative exists. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous function is denoted by $\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.

The derivative and the shift operator $\sigma$ are related by the formula

$$f^{\sigma}(t) := f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Let $f$ be a real-valued function defined on an interval $[a, b]_{\mathbb{T}}$. We say that $f$ is increasing, decreasing, nondecreasing, and non-increasing on $[a, b]_{\mathbb{T}}$ if $t_1,\ t_2 \in [a, b]_{\mathbb{T}}$ and $t_2 > t_1$ imply $f(t_2) > f(t_1)$, $f(t_2) < f(t_1)$, $f(t_2) \geq f(t_1)$ and $f(t_2) \leq f(t_1)$, respectively. Let $f$ be a differentiable function on $[a, b]_{\mathbb{T}}$. Then $f$ is increasing, decreasing, nondecreasing, and non-increasing on $[a, b]_{\mathbb{T}}$ if $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geq 0$, and $f^{\Delta}(t) \leq 0$ for all $t \in [a, b)_{\mathbb{T}}$, respectively.

We will use the following product and quotient rules for the derivative of the product $fg$ and the quotient $f/g$ (where $g(t)g(\sigma(t)) \neq 0$) of two differentiable functions $f$ and $g$,

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)),$$

$$\Big(\frac{f}{g}\Big)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

For $a, b \in \mathbb{T}$ and a differentiable function $f$, the Cauchy integral of $f^{\Delta}$ is defined by

$$\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a).$$

The integration by parts formula reads

$$\int_a^b f^{\Delta}(t)g(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t,$$

and infinite integrals are defined as

$$\int_a^{\infty} f(s)\Delta s = \lim_{t\to\infty}\int_a^t f(s)\Delta s.$$

## 3. Main results

Below, all occurring functional inequalities are assumed to hold for all sufficiently large $t$. We begin with the following lemma.

**Lemma 3.1.** *Assume that there exists $T \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$y(t) > 0, \quad y^{\Delta}(t) > 0, \quad y^{\Delta^2}(t) < 0, \quad t \in [T, \infty)_{\mathbb{T}}.$$

*Then, for each $k \in (0,1)$, there exists a constant $T_k \in [T, \infty)_{\mathbb{T}}$ such that*

$$\frac{y(\tau(t))}{y(\sigma(t))} \geq \frac{\tau(t) - T}{\sigma(t) - T} \geq k\frac{\tau(t)}{\sigma(t)} \quad \text{and} \quad \frac{y(\tau(t))}{y(t)} \geq \frac{\tau(t) - T}{t - T} \geq k\frac{\tau(t)}{t}$$

*for $t \in [T_k, \infty)_{\mathbb{T}}$.*

*Proof.* The proof is similar to that of [11, Lemma 2.4], and so is omitted. □

The Taylor monomials (See [7, Section 1.6]) $\{h_n(t,s)\}_{n=0}^{\infty}$ are defined recursively by

$$h_0(t,s) = 1, \quad h_{n+1}(t,s) = \int_s^t h_n(\tau, s)\Delta\tau, \quad t,\ s \in \mathbb{T},\ n \geq 0.$$

For any time scale, $h_1(t,s) = t - s$, but simple formulas in general do not hold for $n \geq 2$.

**Lemma 3.2** (See [14, Lemma 4])**.** *Assume that $y$ satisfies*

$$y(t) > 0, \quad y^{\Delta}(t) > 0, \quad y^{\Delta^2}(t) > 0, \quad y^{\Delta^3}(t) \leq 0$$

*for $t \in [t_1, \infty)_{\mathbb{T}}$. Then*

$$\liminf_{t\to\infty} \frac{ty(t)}{h_2(t,t_0)y^{\Delta}(t)} \geq 1.$$

**Lemma 3.3.** *Assume that $x$ is an eventually positive solution of* (1.1)*. Then there are only the following two cases eventually:*

$$(1)\ x > 0, \quad x^{\Delta} > 0, \quad x^{\Delta^2} > 0, \quad x^{\Delta^3} > 0, \quad x^{\Delta^4} < 0,$$

*or*

$$(2)\ x > 0, \quad x^{\Delta} > 0, \quad x^{\Delta^2} < 0, \quad x^{\Delta^3} > 0, \quad x^{\Delta^4} < 0.$$

*Proof.* Let $x$ be an eventually positive solution of (1.1). Then there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. From (1.1), we have

$$x^{\Delta^4}(t) = -p(t)x^{\gamma}(\tau(t)) < 0, \quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{3.1}$$

Thus $x^{\Delta}$, $x^{\Delta^2}$, $x^{\Delta^3}$ each is of constant sign eventually. We claim that $x^{\Delta^3}(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. If not, then there exist a constant $c < 0$ and $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that

$$x^{\Delta^3}(t) \le c < 0, \quad t \in [t_2,\infty)_{\mathbb{T}}.$$

Integrating the above inequality from $t_2$ to $t$, we obtain

$$x^{\Delta^2}(t) - x^{\Delta^2}(t_2) \le c(t - t_2),$$

which implies that

$$\lim_{t\to\infty} x^{\Delta^2}(t) = -\infty,$$

and so there exist a constant $c_1 < 0$ and $t_3 \in [t_2,\infty)_{\mathbb{T}}$ such that

$$x^{\Delta^2}(t) \le c_1 < 0, \quad t \in [t_3,\infty)_{\mathbb{T}}.$$

Integrating the above inequality from $t_3$ to $t$, we obtain

$$x^{\Delta}(t) - x^{\Delta}(t_3) \le c(t - t_3).$$

This gives

$$\lim_{t\to\infty} x^{\Delta}(t) = -\infty,$$

which yields $\lim_{t\to\infty} x(t) = -\infty$ from $x^{\Delta} < 0$ and $x^{\Delta^2} < 0$. This is a contradiction. Hence

$$x^{\Delta^3}(t) > 0, \quad t \in [t_1,\infty)_{\mathbb{T}}.$$

If

$$x^{\Delta^2} > 0,$$

then $x^{\Delta} > 0$ due to $x^{\Delta^3} > 0$. If

$$x^{\Delta^2} < 0,$$

then $x^{\Delta} > 0$ due to $x > 0$. The proof is complete. □

**Lemma 3.4.** *Assume that $x$ is an eventually positive bounded solution of* (1.1). *Then $x$ only satisfies Case* (2) *of Lemma 3.3.*

*Proof.* Suppose that $x$ is an eventually positive solution of (1.1). Proceeding as in the proof of Lemma 3.3, $x$ satisfies Case (1) or Case (2). It is easy to see that $\lim_{t\to\infty} x(t) = \infty$ when Case (1) holds. Thus, $x$ only satisfies Case (2) of Lemma 3.3. The proof is complete. □

Next we give the main results. For simplification, we let $d_+^{\Delta}(t) := \max\{0, d^{\Delta}(t)\}$.

**Theorem 3.5.** *Let $\gamma \ge 1$. Assume that there exist positive functions $\alpha,\beta \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that, for some $k \in (0,1)$, for all constants $M, P \in (0,\infty)$ and sufficiently large $t_1$, for $t_2 > t_1$, and for $t_3 > t_2$, one has $\tau(t) > t_2$ for $t \ge t_3$,*

$$\begin{aligned}&\limsup_{t\to\infty} \int_{t_3}^{t} \Big[\alpha^{\sigma}(s)p(s)\Big(kh_2(\tau(s),t_2)\frac{t_2 - t_1}{\tau(s) - t_1}\frac{\tau(s)}{\sigma(s)}\Big)^{\gamma} \\ &- \frac{(\alpha_+^{\Delta}(s))^2}{4\gamma M^{\gamma-1}\alpha^{\sigma}(s)}\Big(\frac{\sigma(s)}{ks}\Big)^{\gamma}\Big]\Delta s = \infty,\end{aligned} \tag{3.2}$$

*and*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\Big[k^{2\gamma}\beta^{\sigma}(\xi)\Big(\frac{\xi}{\sigma(\xi)}\Big)^{\gamma}f(\xi)-\frac{\sigma^{\gamma}(\xi)(\beta_{+}^{\Delta}(\xi))^2}{4\gamma k^{\gamma}P^{\gamma-1}\beta^{\sigma}(\xi)\xi^{\gamma}}\Big]\Delta\xi=\infty, \tag{3.3}$$

*where*

$$f(\xi)=\int_{\xi}^{\infty}\int_{s}^{\infty}p(v)\Big(\frac{\tau(v)}{v}\Big)^{\gamma}\Delta v\Delta s$$

*is well defined. Then* (1.1) *is oscillatory.*

*Proof.* Suppose that (1.1) has a nonoscillatory solution $x$ on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that there exists a $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$ and $x(\tau(t))>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. Proceeding as in the proof of Lemma 3.3, we get (3.1) and then $x$ satisfies either Case (1) or Case (2).

Assume Case (1) holds. Define

$$\omega(t):=\alpha(t)\frac{x^{\Delta^3}(t)}{(x^{\Delta^2}(t))^{\gamma}},\quad t\in[t_1,\infty)_{\mathbb{T}}. \tag{3.4}$$

Then $\omega(t)>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$, and

$$\omega^{\Delta}(t)=\alpha^{\Delta}(t)\frac{x^{\Delta^3}(t)}{(x^{\Delta^2}(t))^{\gamma}}+\alpha^{\sigma}(t)\Big(\frac{x^{\Delta^3}(t)}{(x^{\Delta^2}(t))^{\gamma}}\Big)^{\Delta},$$

which implies that

$$\omega^{\Delta}(t)=\alpha^{\Delta}(t)\frac{x^{\Delta^3}(t)}{(x^{\Delta^2}(t))^{\gamma}}+\alpha^{\sigma}(t)\frac{x^{\Delta^4}(t)}{(x^{\Delta^2}(\sigma(t)))^{\gamma}}-\alpha^{\sigma}(t)\frac{x^{\Delta^3}(t)((x^{\Delta^2})^{\gamma})^{\Delta}(t)}{(x^{\Delta^2}(t))^{\gamma}(x^{\Delta^2}(\sigma(t)))^{\gamma}}. \tag{3.5}$$

By Pötzsche chain rule [7, Theorem 1.90], we see that

$$\begin{aligned}((x^{\Delta\Delta})^{\gamma})^{\Delta}(t)&=\gamma x^{\Delta^3}(t)\int_0^1\Big[hx^{\Delta^2}(\sigma(t))+(1-h)x^{\Delta^2}(t)\Big]^{\gamma-1}\mathrm{d}h\\&\geq\gamma(x^{\Delta^2}(t))^{\gamma-1}x^{\Delta^3}(t).\end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.5), we have

$$\begin{aligned}\omega^{\Delta}(t)&\leq\alpha^{\Delta}(t)\frac{x^{\Delta^3}(t)}{(x^{\Delta^2}(t))^{\gamma}}+\alpha^{\sigma}(t)\frac{x^{\Delta^4}(t)}{(x^{\Delta^2}(\sigma(t)))^{\gamma}}\\&\quad-\gamma\alpha^{\sigma}(t)\frac{(x^{\Delta^3}(t))^2}{(x^{\Delta^2}(t))^{2\gamma}}\Big(\frac{x^{\Delta^2}(t)}{x^{\Delta^2}(\sigma(t))}\Big)^{\gamma}(x^{\Delta^2}(t))^{\gamma-1}.\end{aligned}$$

In view of the above inequality, (3.1), and (3.4), we obtain

$$\begin{aligned}\omega^{\Delta}(t)&\leq\frac{\alpha^{\Delta}(t)}{\alpha(t)}\omega(t)-\alpha^{\sigma}(t)p(t)\frac{x^{\gamma}(\tau(t))}{(x^{\Delta^2}(\sigma(t)))^{\gamma}}\\&\quad-\gamma\alpha^{\sigma}(t)\frac{\omega^2(t)}{\alpha^2(t)}\Big(\frac{x^{\Delta^2}(t)}{x^{\Delta^2}(\sigma(t))}\Big)^{\gamma}(x^{\Delta^2}(t))^{\gamma-1}.\end{aligned} \tag{3.7}$$

Note that

$$\frac{x^{\gamma}(\tau(t))}{(x^{\Delta^2}(\sigma(t)))^{\gamma}}=\Big(\frac{x(\tau(t))}{x^{\Delta^2}(\tau(t))}\frac{x^{\Delta^2}(\tau(t))}{x^{\Delta^2}(\sigma(t))}\Big)^{\gamma}. \tag{3.8}$$

From $x^{\Delta^2}(t_1)>0$ and $x^{\Delta^4}(t)<0$, we have

$$x^{\Delta^2}(t)>\int_{t_1}^{t}x^{\Delta^3}(s)\Delta s\geq(t-t_1)x^{\Delta^3}(t).$$

Then

$$\Big(\frac{x^{\Delta^2}}{h_1(\cdot,t_1)}\Big)^{\Delta}(t) = \frac{(t-t_1)x^{\Delta^3}(t) - x^{\Delta^2}(t)}{(t-t_1)(\sigma(t)-t_1)} < 0,$$

which implies that $x^{\Delta^2}/h_1(\cdot,t_1)$ is decreasing. Using Taylor's formula [7, Theorem 1.113] and choosing any $t_2 \in (t_1,\infty)_{\mathbb{T}}$, we have

$$x(t) = \sum_{k=0}^{n-1} h_k(t,t_2)x^{\Delta^k}(t_2) + \int_{t_2}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\eta))x^{\Delta^n}(\eta)\Delta\eta.$$

Substituting $n=3$ into the above equality and using $x^{\Delta^i} > 0$, $i = 0,1,2,3$, we obtain

$$x(t) \ge h_2(t,t_2)x^{\Delta^2}(t_2) \ge h_2(t,t_2)\frac{t_2-t_1}{t-t_1}x^{\Delta^2}(t).$$

Hence

$$\frac{x(\tau(t))}{x^{\Delta^2}(\tau(t))} \ge h_2(\tau(t),t_2)\frac{t_2-t_1}{\tau(t)-t_1}. \tag{3.9}$$

Letting $y := x^{\Delta^2}$, we have

$$y>0, \quad y^{\Delta}>0, \quad y^{\Delta^2}<0.$$

Then from Lemma 3.1, for each $k \in (0,1)$,

$$\frac{y(\tau(t))}{y(\sigma(t))} \ge k\frac{\tau(t)}{\sigma(t)} \quad \text{and} \quad \frac{y(t)}{y(\sigma(t))} \ge k\frac{t}{\sigma(t)}.$$

That is,

$$\frac{x^{\Delta^2}(\tau(t))}{x^{\Delta^2}(\sigma(t))} \ge k\frac{\tau(t)}{\sigma(t)} \quad \text{and} \quad \frac{x^{\Delta^2}(t)}{x^{\Delta^2}(\sigma(t))} \ge k\frac{t}{\sigma(t)}. \tag{3.10}$$

It follows from (3.8), (3.9), and (3.10) that

$$\frac{x^{\gamma}(\tau(t))}{(x^{\Delta^2}(\sigma(t)))^{\gamma}} \ge \Big(kh_2(\tau(t),t_2)\frac{t_2-t_1}{\tau(t)-t_1}\frac{\tau(t)}{\sigma(t)}\Big)^{\gamma} \tag{3.11}$$

for each $k \in (0,1)$. On the other hand, there exists a constant $M>0$ such that

$$(x^{\Delta^2}(t))^{\gamma-1} \ge M^{\gamma-1} \tag{3.12}$$

due to $x^{\Delta^3} > 0$ and $\gamma \ge 1$. From (3.7), (3.10), (3.11), and (3.12), we obtain

$$\begin{aligned}\omega^{\Delta}(t) \le{}& -\alpha^{\sigma}(t)p(t)\Big(kh_2(\tau(t),t_2)\frac{t_2-t_1}{\tau(t)-t_1}\frac{\tau(t)}{\sigma(t)}\Big)^{\gamma} + \frac{\alpha_+^{\Delta}(t)}{\alpha(t)}\omega(t)\\ &- \gamma M^{\gamma-1}\frac{\alpha^{\sigma}(t)}{\alpha^2(t)}\Big(k\frac{t}{\sigma(t)}\Big)^{\gamma}\omega^2(t).\end{aligned}$$

Thus

$$\begin{aligned}\omega^{\Delta}(t) \le{}& -\alpha^{\sigma}(t)p(t)\Big(kh_2(\tau(t),t_2)\frac{t_2-t_1}{\tau(t)-t_1}\frac{\tau(t)}{\sigma(t)}\Big)^{\gamma}\\ &+ \frac{(\alpha_+^{\Delta}(t))^2}{4\gamma M^{\gamma-1}\alpha^{\sigma}(t)}\Big(\frac{\sigma(t)}{kt}\Big)^{\gamma}.\end{aligned}$$

Integrating the above inequality from $t_3$ ($\tau(t) > t_2$ when $t \ge t_3$) to $t$, we obtain

$$\begin{aligned}&\int_{t_3}^{t}\Big[\alpha^{\sigma}(s)p(s)\Big(kh_2(\tau(s),t_2)\frac{t_2-t_1}{\tau(s)-t_1}\frac{\tau(s)}{\sigma(s)}\Big)^{\gamma} - \frac{(\alpha_+^{\Delta}(s))^2}{4\gamma M^{\gamma-1}\alpha^{\sigma}(s)}\Big(\frac{\sigma(s)}{ks}\Big)^{\gamma}\Big]\Delta s\\ &\le \omega(t_3) - \omega(t) \le \omega(t_3),\end{aligned}$$

which contradicts (3.2).

Assume Case (2) holds. Define the function

$$u(t) := \beta(t)\frac{x^{\Delta}(t)}{x^{\gamma}(t)}, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.13}$$

Then $u(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$, and

$$u^{\Delta}(t) = \beta^{\Delta}(t)\frac{x^{\Delta}(t)}{x^{\gamma}(t)} + \beta^{\sigma}(t)\frac{x^{\Delta^2}(t)}{x^{\gamma}(\sigma(t))} - \beta^{\sigma}(t)\frac{x^{\Delta}(t)(x^{\gamma})^{\Delta}(t)}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}. \tag{3.14}$$

It follows from Pötzsche chain rule [7, Thm. 1.90] that $(x^{\gamma})^{\Delta}(t) \geq \gamma x^{\gamma-1}(t)x^{\Delta}(t)$. Hence by (3.13) and (3.14), we have

$$u^{\Delta}(t) \leq \frac{\beta^{\Delta}(t)}{\beta(t)}u(t) + \beta^{\sigma}(t)\frac{x^{\Delta^2}(t)}{x^{\gamma}(\sigma(t))} - \gamma\frac{\beta^{\sigma}(t)}{\beta^2(t)}\Big(\frac{x(t)}{x(\sigma(t))}\Big)^{\gamma}x^{\gamma-1}(t)u^2(t). \tag{3.15}$$

Since $x > 0$, $x^{\Delta} > 0$, and $x^{\Delta^2} < 0$, we obtain

$$\frac{x(t)}{x(\sigma(t))} \geq k\frac{t}{\sigma(t)} \quad \text{for each } k \in (0,1) \tag{3.16}$$

due to Lemma 3.1. From $x^{\Delta} > 0$, there exists a constant $P > 0$ such that $x^{\gamma-1}(t) \geq P^{\gamma-1}$. Thus, by (3.15), we see that

$$u^{\Delta}(t) \leq \beta^{\sigma}(t)\frac{x^{\Delta^2}(t)}{x^{\gamma}(\sigma(t))} + \frac{\beta_+^{\Delta}(t)}{\beta(t)}u(t) - \gamma k^{\gamma}P^{\gamma-1}\frac{\beta^{\sigma}(t)}{\beta^2(t)}\Big((\frac{t}{\sigma(t)}\Big)^{\gamma}u^2(t). \tag{3.17}$$

On the other hand, by (1.1), we calculate

$$x^{\Delta^3}(z) - x^{\Delta^3}(t) + \int_t^z p(s)x^{\gamma}(\tau(s))\Delta s = 0.$$

Let $y := x$. By Lemma 3.1, we have

$$\frac{x(\tau(t))}{x(t)} \geq k\frac{\tau(t)}{t}$$

for any $k \in (0,1)$. Thus, from $x^{\Delta} > 0$, we have

$$x^{\Delta^3}(z) - x^{\Delta^3}(t) + k^{\gamma}x^{\gamma}(t)\int_t^z p(s)\Big(\frac{\tau(s)}{s}\Big)^{\gamma}\Delta s \leq 0.$$

Letting $z \to \infty$ in the above inequality, we obtain

$$-x^{\Delta^3}(t) + k^{\gamma}x^{\gamma}(t)\int_t^{\infty} p(s)\left(\frac{\tau(s)}{s}\right)^{\gamma}\Delta s \leq 0$$

due to $\lim_{z\to\infty} x^{\Delta^3}(z) \geq l \geq 0$. Therefore,

$$-x^{\Delta^2}(z) + x^{\Delta^2}(t) + k^{\gamma}x^{\gamma}(t)\int_t^z\int_s^{\infty} p(v)\Big(\frac{\tau(v)}{v}\Big)^{\gamma}\Delta v\Delta s \leq 0.$$

Letting $z \to \infty$ in the last inequality and using $\lim_{z\to\infty}(-x^{\Delta^2}(z)) \geq l_1 \geq 0$, we have

$$x^{\Delta^2}(t) + k^{\gamma}x^{\gamma}(t)\int_t^{\infty}\int_s^{\infty} p(v)\Big(\frac{\tau(v)}{v}\Big)^{\gamma}\Delta v\Delta s \leq 0.$$

Thus by (3.16), we have

$$\begin{aligned}\frac{x^{\Delta^2}(t)}{x^\gamma(\sigma(t))} &\le -k^\gamma \frac{x^\gamma(t)}{x^\gamma(\sigma(t))}\int_t^\infty\int_s^\infty p(v)\Big(\frac{\tau(v)}{v}\Big)^\gamma \Delta v\Delta s\\ &\le -k^{2\gamma}\Big(\frac{t}{\sigma(t)}\Big)^\gamma\int_t^\infty\int_s^\infty p(v)\Big(\frac{\tau(v)}{v}\Big)^\gamma \Delta v\Delta s.\end{aligned}\tag{3.18}$$

Substituting (3.18) into (3.17), we obtain

$$\begin{aligned}u^\Delta(t) &\le -k^{2\gamma}\beta^\sigma(t)\Big(\frac{t}{\sigma(t)}\Big)^\gamma\int_t^\infty\int_s^\infty p(v)\Big(\frac{\tau(v)}{v}\Big)^\gamma \Delta v\Delta s\\ &\quad + \frac{\beta_+^\Delta(t)}{\beta(t)}u(t) - \gamma k^\gamma P^{\gamma-1}\frac{\beta^\sigma(t)}{\beta^2(t)}\Big(\frac{t}{\sigma(t)}\Big)^\gamma u^2(t),\end{aligned}$$

which implies that

$$\begin{aligned}u^\Delta(t) &\le -k^{2\gamma}\beta^\sigma(t)\Big(\frac{t}{\sigma(t)}\Big)^\gamma\int_t^\infty\int_s^\infty p(v)\Big(\frac{\tau(v)}{v}\Big)^\gamma \Delta v\Delta s\\ &\quad + \frac{\sigma^\gamma(t)(\beta_+^\Delta(t))^2}{4\gamma k^\gamma P^{\gamma-1}\beta^\sigma(t)t^\gamma}.\end{aligned}$$

Integrating the last inequality from $t_1$ to $t$, we have

$$\begin{aligned}&\int_{t_1}^t\Big[k^{2\gamma}\beta^\sigma(\xi)\Big(\frac{\xi}{\sigma(\xi)}\Big)^\gamma\Big[\int_\xi^\infty\int_s^\infty p(v)\Big(\frac{\tau(v)}{v}\Big)^\gamma\Delta v\Delta s\Big] - \frac{\sigma^\gamma(\xi)(\beta_+^\Delta(\xi))^2}{4\gamma k^\gamma P^{\gamma-1}\beta^\sigma(\xi)\xi^\gamma}\Big]\Delta\xi\\ &\le u(t_1) - u(t) \le u(t_1),\end{aligned}$$

which contradicts (3.3). The proof is complete. □

Combining Theorem 3.5 with Lemma 3.4, we obtain the following criterion for oscillation of all bounded solutions of (1.1).

**Corollary 3.6.** *Let $\gamma \ge 1$. Assume that there exists a positive function $\beta \in C^1_{\rm rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that, for some $k\in(0,1)$, for all constants $P\in(0,\infty)$ and sufficiently large $t_1$, one has* (3.3). *Then every bounded solution of* (1.1) *is oscillatory.*

Next, we establish another oscillation result for (1.1) under the case when $\gamma > 1$.

**Theorem 3.7.** *Let $\gamma > 1$. If for all sufficiently large $t_1$, for $t_2 > t_1$, and for $t_3 > t_2$, one has $\tau(t) > t_2$ for $t \ge t_3$,*

$$\int_{t_3}^\infty \sigma(t)p(t)\Big(h_2(\tau(t),t_2)\frac{t_2-t_1}{\tau(t)-t_1}\frac{\tau(t)}{\sigma(t)}\Big)^\gamma\Delta t = \infty,\tag{3.19}$$

*and*

$$\int_{t_1}^\infty \sigma(\xi)\Big(\frac{\xi}{\sigma(\xi)}\Big)^\gamma\Big[\int_\xi^\infty\int_s^\infty p(v)\Big(\frac{\tau(v)}{v}\Big)^\gamma\Delta v\Delta s\Big]\Delta\xi = \infty,\tag{3.20}$$

*where*

$$\int_\xi^\infty\int_s^\infty p(v)\Big(\frac{\tau(v)}{v}\Big)^\gamma\Delta v\Delta s$$

*is well defined, then* (1.1) *is oscillatory.*

*Proof.* Let $x$ be a non-oscillatory solution of (1.1). Without loss of generality, we may assume that there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$ and $x(\tau(t))>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. Proceeding as in the proof of Lemma 3.3, we obtain (3.1) and then $x$ satisfies either Case (1) or Case (2).

Suppose that Case (1) holds. We define the function $\omega$ by

$$\omega(t) := \frac{tx^{\Delta^3}(t)}{(x^{\Delta^2}(t))^{\gamma}}, \quad t\in[t_1,\infty)_{\mathbb{T}}. \tag{3.21}$$

Then $\omega(t)>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$, and

$$\begin{aligned}\omega^{\Delta}(t) &= \big(x^{\Delta^3}(t)+\sigma(t)x^{\Delta^4}(t)\big)\big(x^{\Delta^2}(\sigma(t))\big)^{-\gamma} + tx^{\Delta^3}(t)((x^{\Delta^2})^{-\gamma})^{\Delta}(t)\\ &\le x^{\Delta^3}(t)(x^{\Delta^2}(\sigma(t)))^{-\gamma} - \sigma(t)p(t)\Big(\frac{x(\tau(t))}{x^{\Delta^2}(\sigma(t))}\Big)^{\gamma}\end{aligned} \tag{3.22}$$

due to (3.1) and $((x^{\Delta^2})^{-\gamma})^{\Delta}\le 0$ (see Pötzsche chain rule [7, Theorem 1.90]). On the other hand, by Pötzsche chain rule [7, Theorem 1.90], we have

$$\begin{aligned}((x^{\Delta^2})^{1-\gamma})^{\Delta}(t) &= (1-\gamma)x^{\Delta^3}(t)\int_0^1 \big[hx^{\Delta^2}(\sigma(t))+(1-h)x^{\Delta^2}(t)\big]^{-\gamma}\mathrm{d}h\\ &\le (1-\gamma)x^{\Delta^3}(t)\int_0^1 \big[hx^{\Delta^2}(\sigma(t))+(1-h)x^{\Delta^2}(\sigma(t))\big]^{-\gamma}\mathrm{d}h\\ &= (1-\gamma)x^{\Delta^3}(t)(x^{\Delta^2}(\sigma(t)))^{-\gamma}.\end{aligned}$$

Then by (3.22), we see that

$$\omega^{\Delta}(t) \le \frac{((x^{\Delta^2})^{1-\gamma})^{\Delta}(t)}{1-\gamma} - \sigma(t)p(t)\left(\frac{x(\tau(t))}{x^{\Delta^2}(\sigma(t))}\right)^{\gamma}. \tag{3.23}$$

Similar as in the proof of Theorem 3.5, we have (3.11). Hence from (3.23), we obtain

$$\omega^{\Delta}(t) \le \frac{((x^{\Delta^2})^{1-\gamma})^{\Delta}(t)}{1-\gamma} - \sigma(t)p(t)\left(kh_2(\tau(t),t_2)\frac{t_2-t_1}{\tau(t)-t_1}\frac{\tau(t)}{\sigma(t)}\right)^{\gamma}$$

for each $k\in(0,1)$ and $t_2\in(t_1,\infty)_{\mathbb{T}}$. Integrating the last inequality from $t_3$ ($\tau(t)>t_2$ when $t\ge t_3$) to $t$, we get

$$\begin{aligned}&\int_{t_3}^{t}\sigma(s)p(s)\Big(kh_2(\tau(s),t_2)\frac{t_2-t_1}{\tau(s)-t_1}\frac{\tau(s)}{\sigma(s)}\Big)^{\gamma}\Delta s\\ &\le -\int_{t_3}^{t}\Big(\omega^{\Delta}(s)-\frac{((x^{\Delta^2})^{1-\gamma})^{\Delta}(s)}{1-\gamma}\Big)\Delta s \le \omega(t_3)+\frac{(x^{\Delta^2})^{1-\gamma}(t_3)}{\gamma-1},\end{aligned}$$

which contradicts (3.19).

Assume Case (2) holds. We define the function $u$ by

$$u(t) := \frac{tx^{\Delta}(t)}{x^{\gamma}(t)}, \quad t\in[t_1,\infty)_{\mathbb{T}}. \tag{3.24}$$

Then $u(t)>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$, and

$$\begin{aligned}u^{\Delta}(t) &= (x^{\Delta}(t)+\sigma(t)x^{\Delta^2}(t))x^{-\gamma}(\sigma(t)) + tx^{\Delta}(t)(x^{-\gamma})^{\Delta}(t)\\ &\le x^{\Delta}(t)x^{-\gamma}(\sigma(t)) + \sigma(t)\frac{x^{\Delta^2}(t)}{x^{\gamma}(\sigma(t))}\end{aligned} \tag{3.25}$$

due to $(x^{-\gamma})^{\Delta} \le 0$ (see Pötzsche chain rule [7, Theorem 1.90]). On the other hand, by Pötzsche chain rule [7, Theorem 1.90], we get

$$\begin{aligned}
(x^{1-\gamma})^{\Delta}(t) &= (1-\gamma)x^{\Delta}(t)\int_0^1 \big[hx(\sigma(t)) + (1-h)x(t)\big]^{-\gamma}\mathrm{d}h \\
&\le (1-\gamma)x^{\Delta}(t)\int_0^1 \big[hx(\sigma(t)) + (1-h)x(\sigma(t))\big]^{-\gamma}\mathrm{d}h \\
&= (1-\gamma)x^{\Delta}(t)(x(\sigma(t)))^{-\gamma}.
\end{aligned}$$

Then from (3.25), we obtain

$$u^{\Delta}(t) \le \frac{(x^{1-\gamma})^{\Delta}(t)}{1-\gamma} + \sigma(t)\frac{x^{\Delta^2}(t)}{x^{\gamma}(\sigma(t))}. \tag{3.26}$$

As in the proof of Theorem 3.5, we obtain (3.18). It follows from (3.18) and (3.26) that

$$u^{\Delta}(t) \le \frac{(x^{1-\gamma})^{\Delta}(t)}{1-\gamma} - k^{2\gamma}\sigma(t)\Big(\frac{t}{\sigma(t)}\Big)^{\gamma}\int_t^{\infty}\int_s^{\infty} p(v)\Big(\frac{\tau(v)}{v}\Big)^{\gamma}\Delta v\Delta s$$

for each $k \in (0,1)$. Integrating the last inequality from $t_1$ to $t$, we have

$$\begin{aligned}
&\int_{t_1}^{t} k^{2\gamma}\sigma(\xi)\Big(\frac{\xi}{\sigma(\xi)}\Big)^{\gamma}\Big[\int_{\xi}^{\infty}\int_s^{\infty} p(v)\Big(\frac{\tau(v)}{v}\Big)^{\gamma}\Delta v\Delta s\Big]\Delta\xi \\
&\le -\int_{t_1}^{t}\Big(u^{\Delta}(s) - \frac{(x^{1-\gamma})^{\Delta}(s)}{1-\gamma}\Big)\Delta s \le u(t_1) + \frac{x^{1-\gamma}(t_1)}{\gamma-1},
\end{aligned}$$

which contradicts (3.20). This completes the proof. □

Combining Theorem 3.7 with Lemma 3.4, we obtain the following result for oscillation of all bounded solutions of (1.1).

**Corollary 3.8.** *Let $\gamma > 1$. Suppose that* (3.20) *holds for all sufficiently large $t_1$. Then every bounded solution of* (1.1) *is oscillatory.*

Next, we give a new oscillation criterion for (1.1) by using a different class of Riccati substitution.

**Theorem 3.9.** *Let $\gamma \ge 1$. Suppose that $\tau \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$, $\tau^{\Delta} > 0$, and $\tau([t_0,\infty)_{\mathbb{T}}) := [\tau(t_0),\infty)_{\mathbb{T}}$. Assume also that there exist positive functions $\alpha,\beta \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that, for some $k \in (0,1)$, for all constants $M,P \in (0,\infty)$ and sufficiently large $t_1$, for $t_2 > t_1$, one has* (3.3) *and*

$$\limsup_{t\to\infty}\int_{t_2}^{t}\Big[\alpha(s)p(s) - \frac{\sigma(s)(\alpha_+^{\Delta}(s))^2}{4k^2\gamma M^{\gamma-1}\tau^{\Delta}(s)\alpha(s)h_2(\tau(s),t_0)\tau(s)}\Big]\Delta s = \infty. \tag{3.27}$$

*Then* (1.1) *is oscillatory.*

*Proof.* Suppose that (1.1) has a nonoscillatory solution $x$ on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Proceeding as in the proof of Lemma 3.3, we get (3.1) and then $x$ satisfies either Case (1) or Case (2).

Assume Case (1) holds. Define the function

$$\omega(t) := \frac{\alpha(t)}{x^{\gamma}(\tau(t))}x^{\Delta^3}(t), \quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{3.28}$$

Clearly, $\omega(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$, and

$$\omega^{\Delta}(t) = \Big(\frac{\alpha(t)}{x^{\gamma}(\tau(t))}\Big)^{\Delta} x^{\Delta^3}(\sigma(t)) + \frac{\alpha(t)}{x^{\gamma}(\tau(t))} x^{\Delta^4}(t),$$

which yields

$$\begin{aligned} \omega^{\Delta}(t) = \frac{\alpha(t)}{x^{\gamma}(\tau(t))} x^{\Delta^4}(t) + \frac{\alpha^{\Delta}(t) x^{\Delta^3}(\sigma(t))}{x^{\gamma}(\tau(\sigma(t)))} \\ - \alpha(t) \frac{x^{\Delta^3}(\sigma(t))(x^{\gamma}(\tau(t)))^{\Delta}}{x^{\gamma}(\tau(t)) x^{\gamma}(\tau(\sigma(t)))}. \end{aligned} \tag{3.29}$$

From chain rules [7, Theorem 1.90 and Theorem 1.93], we have

$$\begin{aligned} (x^{\gamma}(\tau(t)))^{\Delta} &= \gamma x^{\Delta}(\tau(t)) \tau^{\Delta}(t) \int_0^1 \big[h x(\tau(\sigma(t))) + (1-h) x(\tau(t))\big]^{\gamma-1} \mathrm{d}h \\ &\geq \gamma (x(\tau(t)))^{\gamma-1} x^{\Delta}(\tau(t)) \tau^{\Delta}(t). \end{aligned} \tag{3.30}$$

Substituting (3.30) into (3.29), we find that

$$\omega^{\Delta}(t) \leq \frac{\alpha(t)}{x^{\gamma}(\tau(t))} x^{\Delta^4}(t) + \frac{\alpha^{\Delta}(t) x^{\Delta^3}(\sigma(t))}{x^{\gamma}(\tau(\sigma(t)))} - \gamma \alpha(t) \frac{x^{\Delta^3}(\sigma(t)) x^{\Delta}(\tau(t)) \tau^{\Delta}(t)}{x(\tau(t)) x^{\gamma}(\tau(\sigma(t)))}.$$

In view of (3.1), (3.28), and the above inequality, we obtain

$$\begin{aligned} \omega^{\Delta}(t) \leq -\alpha(t) p(t) + \frac{\alpha^{\Delta}(t)}{\alpha^{\sigma}(t)} \omega^{\sigma}(t) \\ - \gamma \tau^{\Delta}(t) \frac{\alpha(t)}{(\alpha^{\sigma}(t))^2} \frac{x^{\gamma}(\tau(\sigma(t)))}{x(\tau(t))} \frac{x^{\Delta}(\tau(t))}{x^{\Delta^3}(\sigma(t))} (\omega^{\sigma}(t))^2. \end{aligned} \tag{3.31}$$

Let $y := x^{\Delta}$. Then from Lemma 3.2, we see that

$$\frac{x^{\Delta}(t)}{x^{\Delta^2}(t)} \geq k \frac{h_2(t, t_0)}{t}$$

for each $k \in (0, 1)$. Since

$$x^{\Delta^2}(t) > 0, \quad x^{\Delta^3}(t) > 0, \quad x^{\Delta^4}(t) < 0, \quad t \in [t_1, \infty)_{\mathbb{T}},$$

we have

$$x^{\Delta^2}(t) > (t - t_1) x^{\Delta^3}(t) \geq k t x^{\Delta^3}(t).$$

Thus

$$\frac{x^{\Delta}(t)}{x^{\Delta^3}(t)} = \frac{x^{\Delta}(t)}{x^{\Delta^2}(t)} \frac{x^{\Delta^2}(t)}{x^{\Delta^3}(t)} \geq k^2 h_2(t, t_0).$$

Then

$$\frac{x^{\Delta}(\tau(t))}{x^{\Delta^3}(\sigma(t))} = \frac{x^{\Delta}(\tau(t))}{x^{\Delta^3}(\tau(t))} \frac{x^{\Delta^3}(\tau(t))}{x^{\Delta^3}(\sigma(t))} \geq k^2 \frac{h_2(\tau(t), t_0) \tau(t)}{\sigma(t)} \tag{3.32}$$

due to

$$\Big(\frac{x^{\Delta^3}(t)}{t}\Big)^{\Delta} = \frac{t x^{\Delta^4}(t) - x^{\Delta^3}(t)}{t \sigma(t)} < 0.$$

On the other hand, from $x^{\Delta} > 0$ and $\tau^{\Delta} > 0$, we have

$$\frac{x(\tau(\sigma(t)))}{x(\tau(t))} \geq 1, \tag{3.33}$$

and there exists a constant $M > 0$ such that

$$x^{\gamma-1}(\tau(\sigma(t))) \geq M^{\gamma-1}. \tag{3.34}$$

Substituting (3.32), (3.33), and (3.34) into (3.31), we obtain

$$\omega^\Delta(t) \leq -\alpha(t)p(t) + \frac{\alpha_+^\Delta(t)}{\alpha^\sigma(t)}\omega^\sigma(t) - k^2\gamma M^{\gamma-1}\tau^\Delta(t)\frac{\alpha(t)}{(\alpha^\sigma(t))^2}\frac{h_2(\tau(t),t_0)\tau(t)}{\sigma(t)}(\omega^\sigma(t))^2.$$

Therefore,

$$\omega^\Delta(t) \leq -\alpha(t)p(t) + \frac{\sigma(t)(\alpha_+^\Delta(t))^2}{4k^2\gamma M^{\gamma-1}\tau^\Delta(t)\alpha(t)h_2(\tau(t),t_0)\tau(t)}.$$

Integrating the above inequality from $t_2$ ($t_2 > t_1$) to $t$, we have

$$\int_{t_2}^t \Big[\alpha(s)p(s) - \frac{\sigma(s)(\alpha_+^\Delta(s))^2}{4k^2\gamma M^{\gamma-1}\tau^\Delta(s)\alpha(s)h_2(\tau(s),t_0)\tau(s)}\Big]\Delta s \leq \omega(t_2) - \omega(t) \leq \omega(t_2),$$

which contradicts (3.27). The proof of Case (2) is the same as that of Case (2) in Theorem 3.5, and so is omitted. This finishes the proof. □

In the following, we will establish some oscillation results for (1.1) in the case when $\gamma \leq 1$.

**Theorem 3.10.** *Let $\gamma \leq 1$. Assume that there exist positive functions $\alpha, \beta \in \mathrm{C}^1_{\rm rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ such that, for some $k \in (0,1)$, for all constants $M, P \in (0,\infty)$ and sufficiently large $t_1$, for $t_2 > t_1$, and for $t_3 > t_2$, one has $\tau(t) > t_2$ for $t \geq t_3$,*

$$\begin{aligned}\limsup_{t\to\infty}\int_{t_3}^t \Big[&\alpha^\sigma(s)p(s)\Big(kh_2(\tau(s),t_2)\frac{t_2-t_1}{\tau(s)-t_1}\frac{\tau(s)}{\sigma(s)}\Big)^\gamma \\ &- \frac{(\alpha_+^\Delta(s))^2}{4\gamma(M\sigma(s))^{\gamma-1}\alpha^\sigma(s)}\Big(\frac{\sigma(s)}{ks}\Big)^\gamma\Big]\Delta s = \infty,\end{aligned} \tag{3.35}$$

*and*

$$\limsup_{t\to\infty}\int_{t_1}^t \Big[k^{2\gamma}\beta^\sigma(\xi)\Big(\frac{\xi}{\sigma(\xi)}\Big)^\gamma f(\xi) - \frac{\sigma^\gamma(\xi)(\beta_+^\Delta(\xi))^2}{4\gamma k^\gamma(P\sigma(s))^{\gamma-1}\beta^\sigma(\xi)\xi^\gamma}\Big]\Delta\xi = \infty, \tag{3.36}$$

*where*

$$f(\xi) = \int_\xi^\infty\int_s^\infty p(v)\Big(\frac{\tau(v)}{v}\Big)^\gamma\Delta v\Delta s$$

*is well defined. Then (1.1) is oscillatory.*

*Proof.* Suppose that (1.1) has a nonoscillatory solution $x$ on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Proceeding as in the proof of Lemma 3.3, we obtain (3.1) and then $x$ satisfies either Case (1) or Case (2).

Assume Case (1) holds. Define a Riccati substitution as in (3.4). Then we have (3.5). From Pötzsche chain rule [7, Theorem 1.90], we find that

$$\begin{aligned}((x^{\Delta^2})^\gamma)^\Delta(t) &= \gamma x^{\Delta^3}(t)\int_0^1 \big[hx^{\Delta^2}(\sigma(t)) + (1-h)x^{\Delta^2}(t)\big]^{\gamma-1}\mathrm{d}h \\ &\geq \gamma(x^{\Delta^2}(\sigma(t)))^{\gamma-1}x^{\Delta^3}(t).\end{aligned} \tag{3.37}$$

Substituting (3.37) into (3.5), we have

$$\begin{aligned}\omega^{\Delta}(t) \le \alpha^{\Delta}(t)\frac{x^{\Delta^3}(t)}{(x^{\Delta^2}(t))^{\gamma}} + \alpha^{\sigma}(t)\frac{x^{\Delta^4}(t)}{(x^{\Delta^2}(\sigma(t)))^{\gamma}} \\ - \gamma\alpha^{\sigma}(t)\frac{(x^{\Delta^3}(t))^2}{(x^{\Delta^2}(t))^{2\gamma}}\Big(\frac{x^{\Delta^2}(t)}{x^{\Delta^2}(\sigma(t))}\Big)^{\gamma}(x^{\Delta^2}(\sigma(t)))^{\gamma-1}.\end{aligned}$$

By (3.1), (3.4), and the above inequality, we obtain

$$\begin{aligned}\omega^{\Delta}(t) \le \frac{\alpha^{\Delta}(t)}{\alpha(t)}\omega(t) - \alpha^{\sigma}(t)p(t)\frac{x^{\gamma}(\tau(t))}{(x^{\Delta^2}(\sigma(t)))^{\gamma}} \\ - \gamma\alpha^{\sigma}(t)\frac{\omega^2(t)}{\alpha^2(t)}\Big(\frac{x^{\Delta^2}(t)}{x^{\Delta^2}(\sigma(t))}\Big)^{\gamma}(x^{\Delta^2}(\sigma(t)))^{\gamma-1}.\end{aligned} \tag{3.38}$$

As in the proof of Theorem 3.5, we have (3.10) and (3.11) for each $k \in (0,1)$. On the other hand, there exists a constant $M > 0$ such that

$$x^{\Delta^2}(t) = x^{\Delta^2}(t_1) + \int_{t_1}^{t} x^{\Delta^3}(s)\Delta s \le Mt. \tag{3.39}$$

It follows from (3.10), (3.11), (3.38), and (3.39) that

$$\begin{aligned}\omega^{\Delta}(t) \le -\alpha^{\sigma}(t)p(t)\left(kh_2(\tau(t),t_2)\frac{t_2-t_1}{\tau(t)-t_1}\frac{\tau(t)}{\sigma(t)}\right)^{\gamma} + \frac{\alpha_+^{\Delta}(t)}{\alpha(t)}\omega(t) \\ - \gamma(M\sigma(t))^{\gamma-1}\frac{\alpha^{\sigma}(t)}{\alpha^2(t)}\Big(k\frac{t}{\sigma(t)}\Big)^{\gamma}\omega^2(t).\end{aligned}$$

Then

$$\begin{aligned}\omega^{\Delta}(t) \le -\alpha^{\sigma}(t)p(t)\Big(kh_2(\tau(t),t_2)\frac{t_2-t_1}{\tau(t)-t_1}\frac{\tau(t)}{\sigma(t)}\Big)^{\gamma} \\ + \frac{(\alpha_+^{\Delta}(t))^2}{4\gamma\Big(M\sigma(t)\Big)^{\gamma-1}\alpha^{\sigma}(t)}\Big(\frac{\sigma(t)}{kt}\Big)^{\gamma}.\end{aligned}$$

Integrating the last inequality from $t_3$ ($\tau(t) > t_2$ when $t \ge t_3$) to $t$, we obtain

$$\begin{aligned}&\int_{t_3}^{t}\Big[\alpha^{\sigma}(s)p(s)\Big(kh_2(\tau(s),t_2)\frac{t_2-t_1}{\tau(s)-t_1}\frac{\tau(s)}{\sigma(s)}\Big)^{\gamma} - \frac{(\alpha_+^{\Delta}(s))^2}{4\gamma(M\sigma(s))^{\gamma-1}\alpha^{\sigma}(s)}\Big(\frac{\sigma(s)}{ks}\Big)^{\gamma}\Big]\Delta s \\ &\le \omega(t_3) - \omega(t) \le \omega(t_3),\end{aligned}$$

which contradicts (3.35).

If Case (2) holds, we define the function $u$ by (3.13). Then, we have (3.14). By Pötzsche chain rule [7, Theorem 1.90], $(x^{\gamma})^{\Delta}(t) \ge \gamma x^{\gamma-1}(\sigma(t))x^{\Delta}(t)$. Hence from (3.13) and (3.14), we have

$$u^{\Delta}(t) \le \frac{\beta^{\Delta}(t)}{\beta(t)}u(t) + \beta^{\sigma}(t)\frac{x^{\Delta^2}(t)}{x^{\gamma}(\sigma(t))} - \gamma\frac{\beta^{\sigma}(t)}{\beta^2(t)}\Big(\frac{x(t)}{x(\sigma(t))}\Big)^{\gamma}x^{\gamma-1}(\sigma(t))u^2(t). \tag{3.40}$$

Note that there exists a constant $P > 0$ such that

$$x(t) = x(t_1) + \int_{t_1}^{t} x^{\Delta}(s)\Delta s \le Pt. \tag{3.41}$$

Thus, by (3.40) and (3.41), we see that

$$u^{\Delta}(t) \le \beta^{\sigma}(t)\frac{x^{\Delta^2}(t)}{x^{\gamma}(\sigma(t))} + \frac{\beta_+^{\Delta}(t)}{\beta(t)}u(t) - \gamma(P\sigma(t))^{\gamma-1}\frac{\beta^{\sigma}(t)}{\beta^2(t)}\Big(\frac{x(t)}{x(\sigma(t))}\Big)^{\gamma}u^2(t). \quad (3.42)$$

The rest of the proof is similar to that of Case (2) in Theorem 3.5, and we can obtain a contradiction to (3.36). This completes the proof. □

Combining Theorem 3.10 with Lemma 3.4, we give the following criterion for oscillation of all bounded solutions of (1.1).

**Corollary 3.11.** *Let $\gamma \le 1$. Assume that there exists a positive function $\beta \in C^1_{\rm rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that, for some $k \in (0,1)$, for all constants $P \in (0,\infty)$ and sufficiently large $t_1$, one has* (3.36). *Then every bounded solution of* (1.1) *is oscillatory.*

## 4. Examples

In this section, we shall give two examples to illustrate the main results. Here we set $\mathbb{T} := \overline{2^{\mathbb{Z}}} := 2^{\mathbb{Z}} \cup \{0\} := \{2^k : k \in \mathbb{Z}\} \cup \{0\}$. To get the conditions for oscillation, we will use the following facts; see [7, Example 1.104])

$$h_2(t,s) = \frac{(t-s)(t-2s)}{3} \quad \text{and} \quad h_3(t,s) = \frac{(t-s)(t-2s)(t-4s)}{21}.$$

**Example 4.1.** Consider a fourth-order super-linear delay dynamic equation

$$x^{\Delta^4}(t) + \frac{\lambda}{h_3(t,t_0)}x^{\gamma}(2^{-k_1}t) = 0, \quad t \in [t_0,\infty)_{\overline{2^{\mathbb{Z}}}}, \quad (4.1)$$

where $t_0 > 0$, $\gamma > 1$, $\lambda > 0$, and $k_1$ is a positive integer. Let $p(t) = \lambda/h_3(t,t_0)$ and $\tau(t) = 2^{-k_1}t$. Then

$$\begin{aligned}
\int_s^{\infty} p(v)\Big(\frac{\tau(v)}{v}\Big)^{\gamma}\Delta v &= 2^{-k_1\gamma}\lambda\int_s^{\infty}\frac{1}{h_3(v,t_0)}\Delta v \\
&\ge 21\lambda \times 2^{-k_1\gamma}\int_s^{\infty}\frac{1}{(v-t_0)^3}\Delta v \\
&\ge \frac{21\lambda \times 2^{-(k_1\gamma+1)}}{(s-t_0)^2}
\end{aligned}$$

and

$$\int_{\xi}^{\infty}\int_s^{\infty} p(v)\Big(\frac{\tau(v)}{v}\Big)^{\gamma}\Delta v\Delta s \ge \frac{21\lambda \times 2^{-(k_1\gamma+1)}}{\xi - t_0}.$$

It is easy to see that all assumptions of Theorem 3.7 hold. Thus equation (4.1) is oscillatory.

**Example 4.2.** Consider a fourth-order linear delay dynamic equation

$$x^{\Delta^4}(t) + \frac{\lambda h_2(t,t_0)}{h_3(t,t_0)h_3(2t,t_0)}x(2^{-k_1}t) = 0, \quad t \in [t_0,\infty)_{\overline{2^{\mathbb{Z}}}}, \quad (4.2)$$

where $t_0 > 0$, $\lambda > 0$, and $k_1$ is a positive integer. We now let

$$p(t) = \lambda h_2(t,t_0)/(h_3(t,t_0)h_3(2t,t_0))$$

and $\tau(t) = 2^{-k_1}t$. Then

$$\int_s^{\infty} p(v)\Big(\frac{\tau(v)}{v}\Big)^{\gamma}\Delta v = 2^{-k_1}\lambda\int_s^{\infty}\frac{h_2(v,t_0)}{h_3(v,t_0)h_3(2v,t_0)}\Delta v$$

$$
\begin{aligned}
&= 2^{-k_1}\lambda \int_s^\infty \Big(-\frac{1}{h_3(v,t_0)}\Big)^\Delta \Delta v \\
&= \frac{2^{-k_1}\lambda}{h_3(s,t_0)}
\end{aligned}
$$

and

$$
\int_\xi^\infty \int_s^\infty p(v)\Big(\frac{\tau(v)}{v}\Big)^\gamma \Delta v \Delta s \geq \frac{21\lambda \times 2^{-(k_1+1)}}{(\xi - t_0)^2}.
$$

Note that

$$
p(t) = \lambda \frac{h_2(t,t_0)}{(h_3(t,t_0)h_3(2t,t_0))} \geq \frac{147\lambda}{8t^4}.
$$

Let $\gamma = 1$, $\alpha(t) = t^3$, and $\beta(t) = t$. If $\lambda > 2^{(3+4k_1)}/(7k^2)$ for some $k \in (0,1)$, then

$$
\limsup_{t\to\infty} \int_{t_2}^t \Big[\alpha(s)p(s) - \frac{\sigma(s)((\alpha^\Delta(s))_+)^2}{4k^2\gamma M^{\gamma-1}\tau^\Delta(s)\alpha(s)h_2(\tau(s),t_0)\tau(s)}\Big]\Delta s = \infty.
$$

If $\lambda > 2^{(k_1-1)}/(21k^3)$ for some $k \in (0,1)$, then (3.3) holds. Hence by Theorem 3.9, equation (4.2) oscillates if $\lambda > \max\{2^{(3+4k_1)}/(7k^2), 2^{(k_1-1)}/(21k^3)\}$ for some $k \in (0,1)$.

The results obtained can be extended to a fourth-order neutral delay dynamic equation

$$
\big[x(t) + p(t)x(\delta(t))\big]^{\Delta^4}(t) + q(t)x^\gamma(\tau(t)) = 0.
$$

Moreover, similar methods can be applied to a fourth-order quasi-linear neutral delay dynamic equation

$$
\Big[\Big((x(t) + p(t)x(\delta(t)))^{\Delta^3}\Big)^\gamma\Big]^\Delta(t) + q(t)x^\gamma(\tau(t)) = 0.
$$

The details are left to the reader.

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