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# STEPANOV-LIKE PSEUDO-ALMOST AUTOMORPHIC FUNCTIONS IN LEBESGUE SPACES WITH VARIABLE EXPONENTS $L^{p(x)}$

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Abstract. In this article we introduce and study a new class of functions called Stepanov-like pseudo-almost automorphic functions with variable exponents, which generalizes in a natural way classical Stepanov-like pseudo-almost automorphic spaces. Basic properties of these new spaces are investigated. The existence of pseudo-almost automorphic solutions to some first-order differential equations with $S^{p,q(x)}$-pseudo-almost automorphic coefficients will also be studied.

## 1. Introduction

The impetus of this article comes from three main sources. The first one is a series of papers by Liang et al [16, 22, 23] in which the concept of pseudo-almost automorphy was introduced and intensively studied. Pseudo-almost automorphic functions are natural generalizations to various classes of functions including almost periodic functions, almost automorphic functions, and pseudo-almost periodic functions.

The second source is a paper by Diagana [7] in which the concept of $S^p$-pseudo-almost automorphy ($p \geq 1$ being a constant) was introduced and studied. Note that $S^p$-pseudo-almost automorphic functions (or Stepanov-like pseudo-almost automorphic functions) are natural generalizations of pseudo-almost automorphic functions. The spaces of Stepanov-like pseudo-almost automorphic functions are now fairly well-understood as most of their fundamental properties have recently been established through the combined efforts of several mathematicians. Some of the recent developments on these functions can be found in [6, 9, 12, 13, 15].

The third and last source is a paper by Diagana and Zitane [11] in which the class of $S^{p,q(x)}$-pseudo-almost periodic functions was introduced and studied, where $q : \mathbb{R} \mapsto \mathbb{R}$ is a measurable function satisfying some additional conditions. The construction of these new spaces makes extensive use of basic properties of the Lebesgue spaces with variable exponents $L^{q(x)}$ (see [5, 14, 21]).

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In this article we extend $S^p$-pseudo-almost automorphic spaces by introducing $S^{p,q(x)}$-pseudo-almost automorphic spaces (or Stepanov-like pseudo-almost automorphic spaces with variable exponents). Basic properties as well as some composition results for these new spaces are established (see Theorems 4.18 and 4.20).

To illustrate our above-mentioned findings, we will make extensive use of the newly-introduced functions to investigate the existence of pseudo-almost automorphic solutions to the first-order differential equations

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, \tag{1.1}$$

and

$$u'(t) = A(t)u(t) + F(t, Bu(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

where $A(t) : D(A(t)) \subset \mathbb{X} \mapsto \mathbb{X}$ is a family of closed linear operators on a Banach space $\mathbb{X}$, satisfying the well-known Acquistapace–Terreni conditions, the forcing terms $f : \mathbb{R} \to \mathbb{X}$ is an $S^{p,q(x)}$-pseudo-almost automorphic function and $F : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is an $S^{p,q}$-pseudo-almost automorphic function, satisfying some additional conditions, and $B : \mathbb{X} \mapsto \mathbb{X}$ is a bounded linear operator. Such result (Theorems 5.3 and5.4) generalize most of the known results encountered in the literature on the existence and uniqueness of pseudo-almost automorphic solutions to Equations (1.1)-(1.2).

## 2. Preliminaries

Let $(\mathbb{X}, \|\cdot\|), (\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces. Let $BC(\mathbb{R}, \mathbb{X})$ (respectively, $BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denote the collection of all bounded continuous functions from $\mathbb{R}$ into $\mathbb{X}$ (respectively, the class of jointly bounded continuous functions $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$). The space $BC(\mathbb{R}, \mathbb{X})$ equipped with the sup norm $\|\cdot\|_\infty$ is a Banach space. Furthermore, $C(\mathbb{R}, \mathbb{Y})$ (respectively, $C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denotes the class of continuous functions from $\mathbb{R}$ into $\mathbb{Y}$ (respectively, the class of jointly continuous functions $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$). Let $B(\mathbb{X}, \mathbb{Y})$ stand for the Banach space of bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ equipped with its natural operator topology $\|\cdot\|_{B(\mathbb{X},\mathbb{Y})}$ with $B(\mathbb{X}, \mathbb{X}) := B(\mathbb{X})$.

### 2.1. **Pseudo-almost automorphic functions.**

**Definition 2.1** ([4, 6, 20])**.** A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$f(t) = \lim_{n\to\infty} g(t - s_n)$$

for each $t \in \mathbb{R}$.

The collection of all such functions will be denoted by $AA(\mathbb{X})$, which turns out to be a Banach space when it is equipped with the sup-norm.

**Definition 2.2** ([6, 16])**.** A function $F \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is said to be almost automorphic if $F(t, u)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all $u \in K$, where $K \subset \mathbb{Y}$ is an arbitrary bounded subset. The collection of all such functions will be denoted by $AA(\mathbb{R} \times \mathbb{X})$.

**Definition 2.3** ([15])**.** A function $L \in C(\mathbb{R}\times\mathbb{R},\mathbb{X})$ is called bi-almost automorphic if for every sequence of real numbers $(s'_n)_n$ we can extract a subsequence $(s_n)_n$ such that

$$H(t,s) := \lim_{n\to\infty} L(t+s_n, s+s_n)$$

is well defined for each $t, s \in \mathbb{R}$, and

$$L(t,s) = \lim_{n\to\infty} H(t-s_n, s-s_n)$$

for each $t, s \in \mathbb{R}$. The collection of all such functions will be denoted by $bAA(\mathbb{R}\times\mathbb{R},\mathbb{X})$.

**Proposition 2.4** ([20])**.** *Assume $f, g : \mathbb{R} \to \mathbb{X}$ are almost automorphic and $\lambda$ is any scalar. Then the following hold*

(a) $f+g, \lambda f, f_\tau(t) := f(t+\tau)$ *and* $\widehat{f}(t) := f(-t)$ *are almost automorphic;*
(b) *The range $R_f$ of $f$ is precompact, so $f$ is bounded;*
(c) *If $\{f_n\}$ is a sequence of almost automorphic functions and $f_n \to f$ uniformly on $\mathbb{R}$, then $f$ is almost automorphic.*

Define

$$PAA_0(\mathbb{X}) := \Big\{ f \in BC(\mathbb{R},\mathbb{X}) : \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|f(\sigma)\| d\sigma = 0 \Big\}.$$

Similarly, define $PAA_0(\mathbb{R}\times\mathbb{X})$ as the collection of jointly continuous functions $F : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$ such that $F(\cdot, y)$ is bounded for each $y \in \mathbb{Y}$ and

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|F(s,y)\| ds = 0$$

uniformly in $y \in \mathbb{Y}$.

**Definition 2.5** ([4])**.** A function $f \in BC(\mathbb{R},\mathbb{X})$ is said to be pseudo-almost automorphic if it can be decomposed as $f = g+\varphi$ where $g \in AA(\mathbb{X})$ and $\varphi \in PAA_0(\mathbb{X})$. The set of all such functions will be denoted by $PAA(\mathbb{X})$.

**Definition 2.6** ([16])**.** A function $F \in C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ is said to be pseudo-almost automorphic if it can be decomposed as $f = G + \Phi$ where $G \in AA(\mathbb{R}\times\mathbb{X})$ and $\Phi \in AA_0(\mathbb{R}\times\mathbb{X})$. The collection of such functions will be denoted by $PAA(\mathbb{R}\times\mathbb{X})$.

**Theorem 2.7** ([22])**.** The space $PAA(\mathbb{X})$ equipped with the sup-norm is a Banach space.

**Theorem 2.8** ([15])**.** If $u \in PAA(\mathbb{X})$ and if $C \in B(\mathbb{X})$, then the function $t \mapsto Cu(t)$ belongs to $PAA(\mathbb{X})$.

**Theorem 2.9** ([7, 15])**.** *Assume $F \in PAA(\mathbb{R}\times\mathbb{X})$. Suppose that $u \mapsto F(t,u)$ is Lipschitz uniformly in $t \in \mathbb{R}$, in the sense that there exists $L > 0$ such that*

$$\|F(t,u) - F(t,v)\| \le L\|u-v\| \quad \textit{for all } t \in \mathbb{R}, u, v \in \mathbb{X} \tag{2.1}$$

*If $\Phi \in PAA(\mathbb{X})$, then $F(.,\Phi(.)) \in PAA(\mathbb{X})$.*

### 2.2. **Evolution family and exponential dichotomy.**

**Definition 2.10** ([6, 18])**.** A family of bounded linear operators $(U(t,s))_{t\geq s}$ on a Banach space $\mathbb{X}$ is called a strongly continuous evolution family if

(i) $U(t,t)=I$ for all $t\in\mathbb{R}$;
(ii) $U(t,s)=U(t,r)U(r,s)$ for all $t\geq r\geq s$ and $t,r,s\in\mathbb{R}$; and
(iii) the map $(t,s)\mapsto U(t,s)x$ is continuous for all $x\in\mathbb{X}, t\geq s$ and $t,s\in\mathbb{R}$.

**Definition 2.11** ([6, 18])**.** *An evolution family $(U(t,s))_{t\geq s}$ on a Banach space $\mathbb{X}$ is called hyperbolic (or has exponential dichotomy) if there exist projections $P(t), t\in\mathbb{R}$, uniformly bounded and strongly continuous in $t$, and constants $M>0,\delta>0$ such that*

(i) *$U(t,s)P(s)=P(t)U(t,s)$ for $t\geq s$ and $t,s\in\mathbb{R}$;*
(ii) *The restriction $U_Q(t,s):Q(s)\mathbb{X}\mapsto Q(t)\mathbb{X}$ of $U(t,s)$ is invertible for $t\geq s$ (and we set $U_Q(s,t):=U(t,s)^{-1}$);*
(iii) *$\|U(t,s)P(s)\|\leq Me^{-\delta(t-s)}$, $\|U_Q(s,t)Q(t)\|\leq Me^{-\delta(t-s)}$ for $t\geq s$ and $t,s\in\mathbb{R}$,*

*where $Q(t):=I-P(t)$ for all $t\in\mathbb{R}$.*

**Definition 2.12** ([18])**.** *Given a hyperbolic evolution family $U(t,s)$, we define its so-called Green's function by*

$$\Gamma(t,s):=\begin{cases}U(t,s)P(s), & \text{for } t\geq s, \quad t,s\in\mathbb{R},\\ U_Q(t,s)Q(s), & \text{for } t<s, \quad t,s\in\mathbb{R}.\end{cases} \tag{2.2}$$

## 3. Lebesgue spaces with variable exponents $L^{p(x)}$

The setting of this section follows that of Diagana and Zitane [11]. This section is mainly devoted to the so-called Lebesgue spaces with variable exponents $L^{p(x)}(\mathbb{R},\mathbb{X})$. Various basic properties of these functions are reviewed. For more on these spaces and related issues we refer to Diening et al [5].

Let $(\mathbb{X},\|\cdot\|)$ be a Banach space and let $\Omega\subseteq\mathbb{R}$ be a subset. Let $M(\Omega,\mathbb{X})$ denote the collection of all measurable functions $f:\Omega\mapsto\mathbb{X}$. Let us recall that two functions $f$ and $g$ of $M(\Omega,\mathbb{X})$ are equal whether they are equal almost everywhere. Set $m(\Omega):=M(\Omega,\mathbb{R})$ and fix $p\in m(\Omega)$. Let $\varphi(x,t)=t^{p(x)}$ for all $x\in\Omega$ and $t\geq0$, and define

$$\rho(u)=\rho_{p(x)}(u)=\int_\Omega\varphi(x,\|u(x)\|)dx=\int_\Omega\|u(x)\|^{p(x)}dx,$$
$$L^{p(x)}(\Omega,\mathbb{X})=\Big\{u\in M(\Omega,\mathbb{X}):\lim_{\lambda\to0^+}\rho(\lambda u)=0\Big\},$$
$$L^{p(x)}_{OC}(\Omega,\mathbb{X})=\Big\{u\in L^{p(x)}(\Omega,\mathbb{X}):\rho(u)<\infty\Big\}, \text{ and}$$
$$E^{p(x)}(\Omega,\mathbb{X})=\Big\{u\in L^{p(x)}(\Omega,\mathbb{X}): \text{ for all } \lambda>0,\ \rho(\lambda u)<\infty\Big\}.$$

Note that the space $L^{p(x)}(\Omega,\mathbb{X})$ defined above is a Musielak-Orlicz type space while $L^{p(x)}_{OC}(\Omega,\mathbb{X})$ is a generalized Orlicz type space. Further, the sets $E^{p(x)}(\Omega,\mathbb{X})$ and $L^{p(x)}(\Omega,\mathbb{X})$ are vector subspaces of $M(\Omega,\mathbb{X})$. In addition, $L^{p(x)}_{OC}(\Omega,\mathbb{X})$ is a convex subset of $L^{p(x)}(\Omega,\mathbb{X})$, and the following inclusions hold

$$E^{p(x)}(\Omega,\mathbb{X})\subset L^{p(x)}_{OC}(\Omega,\mathbb{X})\subset L^{p(x)}(\Omega,\mathbb{X}).$$

**Definition 3.1** ([5]). A convex and left-continuous function $\psi : [0,\infty) \to [0,\infty]$ is called a $\Phi-$function if it satisfies the following conditions:

(a) $\psi(0) = 0$;
(b) $\lim_{t\to 0^+} \psi(t) = 0$; and
(c) $\lim_{t\to\infty} \psi(t) = \infty$.

Moreover, $\psi$ is said to be positive whether $\psi(t) > 0$ for all $t > 0$.

Let us mention that if $\psi$ is a $\Phi$-function, then on the set $\{t > 0 : \psi(t) < \infty\}$, the function $\psi$ is of the form

$$\psi(t) = \int_0^t k(t)dt,$$

where $k(\cdot)$ is the right-derivative of $\psi(t)$. Moreover, $k$ is a non-increasing and right-continuous function. For more on these functions and related issues we refer to [5].

**Example 3.2.** (a) Consider the function $\varphi_p(t) = p^{-1}t^p$ for $1 \le p < \infty$. It can be shown that $\varphi_p$ is a $\Phi$-function. Furthermore, the function $\varphi_p$ is continuous and positive.

(b) It can be shown that the function $\varphi$ defined above; that is, $\varphi(x,t) = t^{p(x)}$ for all $x \in \mathbb{R}$ and $t \ge 0$ is a $\Phi-$function.

For any $p \in m(\Omega)$, we define

$$p^- := \operatorname{ess\,inf}_{x\in\Omega} p(x), \quad p^+ := \operatorname{ess\,sup}_{x\in\Omega} p(x).$$

Define

$$C_+(\Omega) := \Big\{p \in m(\Omega) : 1 < p^- \le p(x) \le p^+ < \infty, \text{ for each } x \in \Omega\Big\}.$$

Let $p \in C_+(\Omega)$. Using similar argument as in [5, Theorem 3.4.1], it can be shown that

$$E^{p(x)}(\Omega,\mathbb{X}) = L^{p(x)}_{OC}(\Omega,\mathbb{X}) = L^{p(x)}(\Omega,\mathbb{X}).$$

In view of the above, we define the Lebesgue space $L^{p(x)}(\Omega,\mathbb{X})$ with variable exponents $p \in C_+(\Omega)$, by

$$L^{p(x)}(\Omega,\mathbb{X}) := \Big\{u \in M(\Omega,\mathbb{X}) : \int_\Omega \|u(x)\|^{p(x)}dx < \infty\Big\}.$$

Define, for each $u \in L^{p(x)}(\Omega,\mathbb{X})$,

$$\|u\|_{p(x)} := \inf\Big\{\lambda > 0 : \int_\Omega \Big\|\frac{u(x)}{\lambda}\Big\|^{p(x)}dx \le 1\Big\}.$$

It can be shown that $\|\cdot\|_{p(x)}$ is a norm upon $L^{p(x)}(\Omega,\mathbb{X})$, which is referred to as the *Luxemburg norm*.

**Remark 3.3.** Let $p \in C_+(\Omega)$. If $p$ is constant, then the space $L^{p(\cdot)}(\Omega,\mathbb{X})$, as defined above, coincides with the usual space $L^p(\Omega,\mathbb{X})$.

We now establish some basic properties for these spaces. For more on these functions and related issues we refer to [5].

**Proposition 3.4** ([11]). *Let $p \in C_+(\Omega)$ and let $u, u_k, v \in M(\Omega,\mathbb{X})$ for $k = 1, 2, \dots$. Then the following statements hold,*

(a) *If $u_k \to u$ a.e., then $\rho_p(u) \le \lim_{k\to\infty} \inf(\rho_p(u_k))$;*

(b) *If* $\|u_k\| \to \|u\|$ a.e., *then* $\rho_p(u) = \lim_{k\to\infty} \rho_p(u_k)$;
(c) *If* $u_k \to u$ a.e., $\|u_k\| \le \|v\|$ *and* $v \in E^{p(x)}(\Omega, \mathbb{X})$, *then* $u_k \to u$ *in the space* $L^{p(x)}(\Omega, \mathbb{X})$.

**Proposition 3.5** ([5, 21])**.** *Let* $p \in C_+(\Omega)$. *If* $u, v \in L^{p(x)}(\Omega, \mathbb{X})$, *then the following properties hold,*

(a) $\|u\|_{p(x)} \ge 0$, *with equality if and only if* $u = 0$;
(b) $\rho_p(u) \le \rho_p(v)$ *and* $\|u\|_{p(x)} \le \|v\|_{p(x)}$ *if* $\|u\| \le \|v\|$;
(c) $\rho_p(u\|u\|^{-1}_{p(x)}) = 1$ *if* $u \ne 0$;
(d) $\rho_p(u) \le 1$ *if and only if* $\|u\|_{p(x)} \le 1$;
(e) *If* $\|u\|_{p(x)} \le 1$, *then*

$$\left[\rho_p(u)\right]^{1/p^-} \le \|u\|_{p(x)} \le \left[\rho_p(u)\right]^{1/p^+}.$$

(f) *If* $\|u\|_{p(x)} \ge 1$, *then*

$$\left[\rho_p(u)\right]^{1/p^+} \le \|u\|_{p(x)} \le \left[\rho_p(u)\right]^{1/p^-}.$$

**Proposition 3.6** ([5])**.** *Let* $p \in C_+(\Omega)$ *and let* $u, u_k, v \in M(\Omega, \mathbb{X})$ *for* $k = 1, 2, \dots$. *Then the following statements hold:*

(a) *If* $u \in L^{p(x)}(\Omega, \mathbb{X})$ *and* $0 \le \|v\| \le \|u\|$, *then* $v \in L^{p(x)}(\Omega, \mathbb{X})$ *and* $\|v\|_{p(x)} \le \|u\|_{p(x)}$.
(b) *If* $u_k \to u$ a.e., *then* $\|u\|_{p(x)} \le \lim_{k\to\infty} \inf(\|u_k\|_{p(x)})$.
(c) *If* $\|u_k\| \to \|u\|$ a.e. *with* $u_k \in L^{p(x)}(\Omega, \mathbb{X})$ *and* $\sup_k \|u_k\|_{p(x)} < \infty$, *then* $u \in L^{p(x)}(\mathbb{R}, \mathbb{X})$ *and* $\|u_k\|_{p(x)} \to \|u\|_{p(x)}$.

Using similar arguments as in Fan et al [14], we obtain the following result.

**Proposition 3.7.** *If* $u, u_n \in L^{p(x)}(\Omega, \mathbb{X})$ *for* $k = 1, 2, \dots$, *then the following statements are equivalent:*

(a) $\lim_{k\to\infty} \|u_k - u\|_{p(x)} = 0$;
(b) $\lim_{k\to\infty} \rho_p(u_k - u) = 0$;
(c) $u_k \to u$ *and* $\lim_{k\to\infty} \rho_p(u_k) = \rho_p(u)$.

**Theorem 3.8** ([5, 14])**.** *Let* $p \in C_+(\Omega)$. *The space* $(L^{p(x)}(\Omega, \mathbb{X}), \|\cdot\|_{p(x)})$ *is a Banach space that is separable and uniform convex. Its topological dual is* $L^{q(x)}(\Omega, \mathbb{X})$, *where* $p^{-1}(x) + q^{-1}(x) = 1$. *Moreover, for any* $u \in L^{p(x)}(\Omega, \mathbb{X})$ *and* $v \in L^{q(x)}(\Omega, \mathbb{R})$, *we have*

$$\|\int_\Omega uv dx\| \le \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big) \|u\|_{p(x)} |v|_{q(x)}. \tag{3.1}$$

Define

$$D_+(\Omega) := \Big\{p \in m(\Omega) : 1 \le p^- \le p(x) \le p^+ < \infty, \text{ for each } x \in \Omega\Big\}.$$

**Corollary 3.9** ([21])**.** *Let* $p, r \in D_+(\Omega)$. *If the function* $q$ *defined by the equation*

$$\frac{1}{q(x)} = \frac{1}{p(x)} + \frac{1}{r(x)}$$

*is in* $D_+(\Omega)$, *then there exists a constant* $C = C(p, r) \in [1, 5]$ *such that*

$$\|uv\|_{q(x)} \le C \|u\|_{p(x)} |v|_{r(x)},$$

*for every* $u \in L^{p(x)}(\Omega, \mathbb{X})$ *and* $v \in L^{r(x)}(\Omega, \mathbb{R})$.

**Corollary 3.10** ([5]). *Let* $\operatorname{meas}(\Omega) < \infty$ *where* $\operatorname{meas}(\cdot)$ *stands for the Lebesgue measure and* $p, q \in D_+(\Omega)$. *If* $q(\cdot) \le p(\cdot)$ *almost everywhere in* $\Omega$, *then the embedding* $L^{p(x)}(\Omega, \mathbb{X}) \hookrightarrow L^{q(x)}(\Omega, \mathbb{X})$ *is continuous whose norm does not exceed* $2(\operatorname{meas}(\Omega) + 1)$.

## 4. Stepanov-like pseudo-almost automorphic functions with variable exponents

**Definition 4.1.** The Bochner transform $f^b(t, s)$, $t \in \mathbb{R}$, $s \in [0, 1]$ of a function $f : \mathbb{R} \to \mathbb{X}$ is defined by $f^b(t, s) := f(t + s)$.

**Remark 4.2.** A function $\varphi(t, s)$, $t \in \mathbb{R}$, $s \in [0, 1]$, is the Bochner transform of a certain function $f$, $\varphi(t, s) = f^b(t, s)$, if and only if $\varphi(t + \tau, s - \tau) = \varphi(s, t)$ for all $t \in \mathbb{R}$, $s \in [0, 1]$ and $\tau \in [s - 1, s]$. Moreover, if $f = h + \varphi$, then $f^b = h^b + \varphi^b$. Moreover, $(\lambda f)^b = \lambda f^b$ for each scalar $\lambda$.

**Definition 4.3.** The Bochner transform $F^b(t, s, u)$, $t \in \mathbb{R}$, $s \in [0, 1]$, $u \in \mathbb{X}$ of a function $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$, is defined by $F^b(t, s, u) := F(t + s, u)$ for each $u \in \mathbb{X}$.

**Definition 4.4.** Let $p \in [1, \infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent $p$, consists of all measurable functions $f$ on $\mathbb{R}$ with values in $\mathbb{X}$ such that $f^b \in L^\infty\big(\mathbb{R}, L^p((0, 1), \mathbb{X})\big)$. This is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t \in \mathbb{R}} \Big( \int_t^{t+1} \|f(\tau)\|^p \, d\tau \Big)^{1/p}.$$

Note that for each $p \ge 1$, we have the following continuous inclusion:

$$(BC(\mathbb{X}), \|\cdot\|_\infty) \hookrightarrow (BS^p(\mathbb{X}), \|\cdot\|_{S^p}).$$

**Definition 4.5** (Diagana and Zitane [11]). Let $p \in C_+(\mathbb{R})$. The space $BS^{p(x)}(\mathbb{X})$ consists of all functions $f \in M(\mathbb{R}, \mathbb{X})$ such that $\|f\|_{S^{p(x)}} < \infty$, where

$$\begin{aligned} \|f\|_{S^{p(x)}} &= \sup_{t \in \mathbb{R}} \Big[ \inf \Big\{ \lambda > 0 : \int_0^1 \Big\| \frac{f(x + t)}{\lambda} \Big\|^{p(x+t)} dx \le 1 \Big\} \Big] \\ &= \sup_{t \in \mathbb{R}} \Big[ \inf \Big\{ \lambda > 0 : \int_t^{t+1} \Big\| \frac{f(x)}{\lambda} \Big\|^{p(x)} dx \le 1 \Big\} \Big]. \end{aligned}$$

Note that the space $\big(BS^{p(x)}(\mathbb{X}), \|\cdot\|_{S^{p(x)}}\big)$ is a Banach space, which, depending on $p(\cdot)$, may or may not be translation-invariant.

**Definition 4.6** (Diagana and Zitane [11]). If $p, q \in C_+(\mathbb{R})$, we then define the space $BS^{p(x),q(x)}(\mathbb{X})$ as follows

$$\begin{aligned} BS^{p(x),q(x)}(\mathbb{X}) &:= BS^{p(x)}(\mathbb{X}) + BS^{q(x)}(\mathbb{X}) \\ &= \Big\{ f = h + \varphi \in M(\mathbb{R}, \mathbb{X}) : h \in BS^{p(x)}(\mathbb{X}) \text{ and } \varphi \in BS^{q(x)}(\mathbb{X}) \Big\}. \end{aligned}$$

We equip $BS^{p(x),q(x)}(\mathbb{X})$ with the norm $\|\cdot\|_{S^{p(x),q(x)}}$ defined by

$$\|f\|_{S^{p(x),q(x)}} := \inf \Big\{ \|h\|_{S^{p(x)}} + \|\varphi\|_{S^{q(x)}} : f = h + \varphi \Big\}.$$

Clearly, $\big(BS^{p(x),q(x)}(\mathbb{X}), \|\cdot\|_{S^{p(x),q(x)}}\big)$ is a Banach space, which, depending on both $p(\cdot)$ and $q(\cdot)$, may or may not be translation-invariant.

**Lemma 4.7** (Diagana and Zitane [11]). Let $p, q \in C_+(\mathbb{R})$. Then the following continuous inclusion holds,

$$\Big(BC(\mathbb{R},\mathbb{X}), \|\cdot\|_\infty\Big) \hookrightarrow \Big(BS^{p(x)}(\mathbb{X}), \|\cdot\|_{S^{p(x)}}\Big) \hookrightarrow \Big(BS^{p(x),q(x)}(\mathbb{X}), \|\cdot\|_{S^{p(x),q(x)}}\Big).$$

*Proof.* The fact that $\big(BS^{p(x)}(\mathbb{X}), \|\cdot\|_{S^{p(x)}}\big) \hookrightarrow \big(BS^{p(x),q(x)}(\mathbb{X}), \|\cdot\|_{S^{p(x),q(x)}}\big)$ is obvious. Thus we will only show that $\big(BC(\mathbb{R},\mathbb{X}), \|\cdot\|_\infty\big) \hookrightarrow \big(BS^{p(x)}(\mathbb{X}), \|\cdot\|_{S^{p(x)}}\big)$. Indeed, let $f \in BC(\mathbb{R},\mathbb{X}) \subset M(\mathbb{R},\mathbb{X})$. If $\|f\|_\infty = 0$, which yields $f = 0$, then there is nothing to prove. Now suppose that $\|f\|_\infty \neq 0$. Using the facts that $0 < \|\frac{f(x)}{\|f\|_\infty}\| \leq 1$ and that $p \in C_+(\mathbb{R})$ it follows that for every $t \in \mathbb{R}$,

$$\int_t^{t+1} \Big\|\frac{f(x)}{\|f\|_\infty}\Big\|^{p(x)} dx \leq \int_t^{t+1} 1^{p(x)} dx = 1,$$

and hence $\|f\|_\infty \in \Big\{\lambda > 0 : \int_t^{t+1} \|\frac{f(x)}{\lambda}\|^{p(x)} dx \leq 1\Big\}$, which yields

$$\inf\Big\{\lambda > 0 : \int_t^{t+1} \|\frac{f(x)}{\lambda}\|^{p(x)} dx \leq 1\Big\} \leq \|f\|_\infty.$$

Therefore, $\|f\|_{S^{p(x)}} \leq \|f\|_\infty < \infty$. This shows that not only $f \in (BS^{p(x)}(\mathbb{X})), \|\cdot\|_{S^{p(x)}})$ but also the injection $(BC(\mathbb{R},\mathbb{X}), \|\cdot\|_\infty) \hookrightarrow (BS^{p(x)}(\mathbb{X}), \|\cdot\|_{S^{p(x)}})$ is continuous. □

**Definition 4.8.** Let $p \geq 1$ be a constant. A function $f \in BS^p(\mathbb{X})$ is said to be $S^p$-almost automorphic (or Stepanov-like almost automorphic function) if $f^b \in AA\big(L^p((0,1),\mathbb{X})\big)$. That is, a function $f \in L^p_{\rm loc}(\mathbb{R},\mathbb{X})$ is said to be Stepanov-like almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p(0,1;\mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $(s'_n)_n$, there exists a subsequence $(s_n)_n$ and a function $g \in L^p_{\rm loc}(\mathbb{R},\mathbb{X})$ such that

$$\Big(\int_0^1 \|f(t+s+s_n)-g(t+s)\|^p ds\Big)^{1/p} \to 0, \quad \Big(\int_0^1 \|g(t+s-s_n)-f(t+s)\|^p ds\Big)^{1/p} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}$. The collection of such functions will be denoted by $S^p_{aa}(\mathbb{X})$.

**Remark 4.9.** There are some difficulties in defining $S^{p(x)}_{aa}(\mathbb{X})$ for a function $p \in C_+(\mathbb{R})$ that is not necessarily constant. This is mainly due to the fact that the space $BS^{p(x)}(\mathbb{X})$ is not always translation-invariant. In other words, the quantities $f^b(t+\tau, s)$ and $f^b(t,s)$ (for $t \in \mathbb{R},\ s \in [0,1]$) that are used in the definition of $S^{p(x)}$-almost automorphy, do not belong to the same space, unless $p$ is constant.

**Remark 4.10.** It is clear that if $1 \leq p < q < \infty$ and $f \in L^q_{\rm loc}(\mathbb{R},\mathbb{X})$ is $S^q$-almost automorphic, then f is $S^p$-almost automorphic. Also if $f \in AA(\mathbb{X})$, then $f$ is $S^p$-almost automorphic for any $1 \leq p < \infty$.

Taking into account Remark 4.9, we introduce the concept of $S^{p,q(x)}$-pseudo-almost automorphy as follows, which obviously generalizes the notion of $S^p$-pseudo-almost automorphy.

**Definition 4.11.** Let $p \geq 1$ be a constant and let $q \in C_+(\mathbb{R})$. A function $f \in BS^{p,q(x)}(\mathbb{X})$ is said to be $S^{p,q(x)}$-pseudo-almost automorphic (or Stepanov-like pseudo-almost automorphic with variable exponents $p, q(x)$) if it can be decomposed as

$$f = h + \varphi,$$

where $h \in S^p_{aa}(\mathbb{X})$ and $\varphi \in S^{q(x)}_{paa_0}(\mathbb{X})$ with $S^{q(x)}_{paa_0}(\mathbb{X})$ being the space of all $\psi \in BS^{q(x)}(\mathbb{X})$ such that

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \inf\Big\{\lambda > 0 : \int_t^{t+1} \|\frac{\psi(x)}{\lambda}\|^{q(x)} dx \le 1\Big\} dt = 0.$$

The collection of $S^{p,q(x)}$-pseudo-almost automorphic functions will be denoted by $S^{p,q(x)}_{paa}(\mathbb{X})$.

**Lemma 4.12.** *Let $r, s \ge 1, p, q \in D_+(\mathbb{R})$. If $s < r, q^+ < p^-$ and $f \in BS^{r,p(x)}(\mathbb{X})$ is $S^{r,p(x)}_{paa}$-pseudo-almost automorphic, then $f$ is $S^{s,q(x)}_{paa}$-pseudo-almost automorphic.*

*Proof.* Suppose that $f \in BS^{r,p(x)}(\mathbb{X})$ is $S^{r,p(x)}$-pseudo-almost automorphic. Thus there exist two functions $h, \varphi : \mathbb{R} \to \mathbb{X}$ such that

$$f = h + \varphi,$$

where $h \in S^r_{aa}(\mathbb{X})$ and $\varphi \in S^{p(x)}_{paa_0}(\mathbb{X})$. From remark 4.10, $h$ is $S^s$-almost automorphic.

In view of $q(\cdot) \le q^+ < p^- \le p(\cdot)$, it follows from Corollary 3.10 that,

$$\begin{aligned} &\Big[\inf\Big\{\lambda > 0 : \int_t^{t+1} \|\frac{\varphi(x)}{\lambda}\|^{q(x)} dx \le 1\Big\}\Big] \\ &\le 4\Big[\inf\Big\{\lambda > 0 : \int_t^{t+1} \|\frac{\varphi(x)}{\lambda}\|^{p(x)} dx \le 1\Big\}\Big]. \end{aligned}$$

Using the fact that $\varphi \in S^{p(x)}_{paa_0}(\mathbb{X})$ and the previous inequality it follows that

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \inf\Big\{\lambda > 0 : \int_t^{t+1} \|\frac{\varphi(x)}{\lambda}\|^{q(x)} dx \le 1\Big\} dt = 0;$$

that is, $\varphi \in S^{q(x)}_{paa_0}(\mathbb{X})$. Therefore, $f \in S^{s,q(x)}_{paa}(\mathbb{X})$. □

**Proposition 4.13.** *Let $p \ge 1$ be a constant and let $q \in C_+(\mathbb{R})$. If $f \in PAA(\mathbb{X})$, then $f$ is $S^{p,q(x)}$-pseudo-almost automorphic.*

*Proof.* Let $f \in PAA(\mathbb{X})$, that is, there exist two functions $h, \varphi : \mathbb{R} \to \mathbb{X}$ such that $f = h + \varphi$ where $h \in AA(\mathbb{X})$ and $\varphi \in PAA_0(\mathbb{X})$. Now from remark 4.10, $h \in AA(\mathbb{X}) \subset S^p_{aa}(\mathbb{X})$. The proof of $\varphi \in S^{q(x)}_{paa_0}(\mathbb{X})$ was given in [11]. However for the sake of clarity, we reproduce it here. Using (e)-(f) of Proposition 3.5 and the usual Hölder inequality, it follows that

$$\begin{aligned} &\int_{-T}^{T} \inf\Big\{\lambda > 0 : \int_0^1 \|\frac{\varphi(x+t)}{\lambda}\|^{q(x+t)} dx \le 1\Big\} dt \\ &\le \int_{-T}^{T} \Big(\int_0^1 \|\varphi(t+x)\|^{q(t+x)}\, dx\Big)^{\gamma} dt \\ &\le (2T)^{1-\gamma}\Big[\int_{-T}^{T} \Big(\int_0^1 \|\varphi(t+x)\|^{q(t+x)}\, dx\Big)\, dt\Big]^{\gamma} \\ &\le (2T)^{1-\gamma}\Big[\int_{-T}^{T} \Big(\int_0^1 \|\varphi(t+x)\| \|\varphi\|_\infty^{q(t+x)-1}\, dx\Big)\, dt\Big]^{\gamma} \\ &\le (2T)^{1-\gamma}\Big(\|\varphi\|_\infty + 1\Big)^{\frac{q^+-1}{\gamma}}\Big[\int_{-T}^{T} \Big(\int_0^1 \|\varphi(t+x)\|\, dx\Big)\, dt\Big]^{\gamma} \end{aligned}$$

$$
\begin{aligned}
&= (2T)^{1-\gamma}\Big(\|\varphi\|_\infty + 1\Big)^{\frac{q^+-1}{\gamma}} \Big[\int_0^1 \Big(\int_{-T}^{T} \|\varphi(t+x)\|\,dt\Big)\,dx\Big]^\gamma \\
&= (2T)\Big(\|\varphi\|_\infty + 1\Big)^{\frac{q^+-1}{\gamma}} \Big[\int_0^1 \Big(\frac{1}{2T}\int_{-T}^{T} \|\varphi(t+x)\|\,dt\Big)\,dx\Big]^\gamma,
\end{aligned}
$$

where

$$
\gamma = \begin{cases} \frac{1}{q^+} & \text{if } \|\varphi\| < 1, \\ \frac{1}{q^-} & \text{if } \|\varphi\| \geq 1. \end{cases}
$$

Using the fact that $PAA_0(\mathbb{X})$ is translation invariant and the (usual) Dominated Convergence Theorem, it follows that

$$
\begin{aligned}
&\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \inf\Big\{\lambda > 0 : \int_0^1 \|\frac{\varphi(x+t)}{\lambda}\|^{q(x+t)} dx \leq 1\Big\} dt \\
&\leq \Big(\|\varphi\|_\infty + 1\Big)^{\frac{q^+-1}{\gamma}} \Big[\int_0^1 \Big(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \|\varphi(t+x)\|\,dt\Big)\,dx\Big]^\gamma = 0.
\end{aligned}
$$

□

Using similar argument as in [22], the following Lemma can be established.

**Lemma 4.14.** *Let $p, q \geq 1$ be a constants. If $f = h + \varphi \in S^{p,q}_{paa}(\mathbb{X})$ such that $h^b \in AA\big(L^p((0,1),\mathbb{X})\big)$ and $\varphi^b \in PAA_0\big(L^q((0,1),\mathbb{X})\big)$, then*

$$
\big\{h(t+.) : t \in \mathbb{R}\big\} \subset \overline{\big\{f(t+.) : t \in \mathbb{R}\big\}}, \quad \text{in } S^{p,q}(\mathbb{X}).
$$

*Proof.* We prove it by contradiction. Indeed, if this is not true, then there exist a $t_0 \in \mathbb{R}$ and an $\varepsilon > 0$ such that

$$
\|h(t_0 + \cdot) - f(t + \cdot)\|_{S^{p,q}} \geq 2\varepsilon, \quad t \in \mathbb{R}.
$$

Since $h^b \in AA\big(L^p((0,1),\mathbb{X})\big)$ and $\big(BS^p(\mathbb{X}), \|\cdot\|_{S^p}\big) \hookrightarrow \big(BS^{p,q}(\mathbb{X}), \|\cdot\|_{S^{p,q}}\big)$, fix $t_0 \in \mathbb{R}, \varepsilon > 0$ and write, $B_\varepsilon := \{\tau \in \mathbb{R}; \|h(t_0 + \tau + \cdot) - g(t_0 + \cdot)\|_{S^{p,q}} < \varepsilon\}$. By [22, Lemma 2.1], there exist $s_1, \dots, s_m \in \mathbb{R}$ such that

$$
\cup_{i=1}^m (s_i + B_\varepsilon) = \mathbb{R}.
$$

Write

$$
\hat{s}_i = s_i - t_0 \quad (1 \leq i \leq m), \quad \eta = \max_{1\leq i\leq m} |\hat{s}_i|.
$$

For $T \in \mathbb{R}$ with $|T| > \eta$; we put

$$
B^{(i)}_{\varepsilon,T} = [-T + \eta - \hat{s}_i, T - \eta - \hat{s}_i] \cap (t_0 + B_\varepsilon), \quad 1 \leq i \leq m,
$$

one has $\cup_{i=1}^m (\hat{s}_i + B^{(i)}_{\varepsilon,T}) = [-T + \eta, T - \eta]$.

Using the fact that $B^{(i)}_{\varepsilon,T} \subset [-T, T] \cap (t_0 + B_\varepsilon)$, $i = 1, \dots, m$, we obtain

$$
\begin{aligned}
2(T - \eta) &= \operatorname{meas}([-T + \eta, T - \eta]) \\
&\leq \sum_{i=1}^m \operatorname{meas}(\hat{s}_i + B^{(i)}_{\varepsilon,T}) \\
&= \sum_{i=1}^m \operatorname{meas}(B^{(i)}_{\varepsilon,T}) \\
&\leq m \max_{1\leq i\leq m} \big\{\operatorname{meas}(B^{(i)}_{\varepsilon,T})\big\}
\end{aligned}
$$

$$\le m \ \text{meas}([-T,T] \cap (t_0 + B_\varepsilon)),$$

On the other hand, by using the Minkowski inequality, for any $t \in t_0 + B_\varepsilon$, one has

$$\begin{aligned}
\|\varphi(t+\cdot)\|_{S^q} &= \|\varphi(t+\cdot)\|_{S^{p,q}} \\
&= \|f(t+\cdot) - h(t+\cdot)\|_{S^{p,q}} \\
&\ge \|h(t_0+\cdot) - f(t+\cdot)\|_{S^{p,q}} - \|h(t+\cdot) - h(t_0+\cdot)\|_{S^{p,q}} > \varepsilon.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T} \|\varphi(t+\cdot)\|_{S^q}\,dt &\ge \frac{1}{2T}\int_{[-T,T]\cap(t_0+B_\varepsilon)} \|\varphi(t+\cdot)\|_{S^q}\,dt \\
&\ge \varepsilon(T-\eta)(mT)^{-1} \to \varepsilon m^{-1}, \quad \text{as } T \to \infty.
\end{aligned}$$

This is a contradiction, since $\varphi^b \in PAA_0\big(L^q((0,1),\mathbb{X})\big)$. □

**Theorem 4.15.** *Let $p, q \ge 1$ be constants. The space $S^{p,q}_{paa}(\mathbb{X})$ equipped with the norm $\|\cdot\|_{S^{p,q}}$ is a Banach space.*

*Proof.* It is sufficient to prove that $S^{p,q}_{paa}(\mathbb{X})$ is a closed subspace of $BS^{p,q}(\mathbb{X})$. Let $f_n = h_n + \varphi_n$ be a Cauchy sequence in $S^{p,q}_{paa}(\mathbb{X})$ with $(h^b_n)_{n\in\mathbb{N}} \subset AA\big(L^p((0,1),\mathbb{X})\big)$ and $(\varphi^b_n)_{n\in\mathbb{N}} \subset PAA_0\big(L^q((0,1),\mathbb{X})\big)$ such that $\|f_n - f\|_{S^{p,q}} \to 0$ as $n \to \infty$. By Lemma 4.14, one has

$$\{h_n(t+.) : t \in \mathbb{R}\} \subset \overline{\{f_n(t+.) : t \in \mathbb{R}\}},$$

and hence

$$\|h_n\|_{S^p} = \|h_n\|_{S^{p,q}} \le \|f_n\|_{S^{p,q}} \quad \text{for all } n \in \mathbb{N}.$$

Consequently, there exists a function $h \in S^p_{aa}(\mathbb{X})$ such that $\|h_n - h\|_{S^p} \to 0$ as $n \to \infty$. Using the previous fact, it easily follows that the function $\varphi := f - h \in BS^q(\mathbb{X})$ and that $\|\varphi_n - \varphi\|_{S^q} = \|(f_n - h_n) - (f-h)\|_{S^q} \to 0$ as $n \to \infty$. Using the fact that $\varphi = (\varphi - \varphi_n) + \varphi_n$ it follows that

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}\Big(\int_0^1 \|\varphi(\tau+t)\|^q d\tau\Big)^{1/q} dt \\
&\le \frac{1}{2T}\int_{-T}^{T}\Big(\int_0^1 \|\varphi(\tau+t) - \varphi_n(\tau+t)\|^q d\tau\Big)^{1/q} dt \\
&\quad + \frac{1}{2T}\int_{-T}^{T}\Big(\int_0^1 \|\varphi_n(\tau+t)\|^q d\tau\Big)^{1/q} dt \\
&\le \|\varphi_n - \varphi\|_{S^q} + \frac{1}{2T}\int_{-T}^{T}\Big(\int_0^1 \|\varphi_n(\tau+t)\|^q d\tau\Big)^{1/q} dt.
\end{aligned}$$

Letting $T \to \infty$ and then $n \to \infty$ in the previous inequality, we obtain that $\varphi^b \in PAA_0\big(L^q((0,1),\mathbb{X})\big)$; that is, $f = h + \varphi \in S^{p,q}_{paa}(\mathbb{X})$. □

Using similar arguments as in the proof of [15, Theorem 3.4], we obtain the next theorem.

**Theorem 4.16.** *If $u \in S^{p,q}_{paa}(\mathbb{Y})$ and if $C \in B(\mathbb{Y},\mathbb{X})$, then the function $t \mapsto Cu(t)$ belongs to $S^{p,q}_{paa}(\mathbb{X})$.*

**Definition 4.17.** *Let $p \geq 1$ and $q \in C_+(\mathbb{R})$. A function $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ with $F(.,u) \in BS^{p,q(x)}(\mathbb{X})$ for each $u \in \mathbb{Y}$, is said to be $S^{p,q(x)}$-pseudo-almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{Y}$ if $t \mapsto F(t,u)$ is $S^{p,q(x)}$-pseudo-almost automorphic for each $u \in B$ where $B \subset \mathbb{Y}$ is an arbitrary bounded set. This means, there exist two functions $G, H : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ such that $F = G + H$, where $G^b \in AA(\mathbb{Y}, L^p((0,1),\mathbb{X}))$ and $H^b \in PAA_0(\mathbb{Y}, L^{q^b(x)}((0,1),\mathbb{X}))$; that is,*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \inf\Big\{\lambda > 0 : \int_0^1 \big\|\frac{H(x+t,u)}{\lambda}\big\|^{q(x+t)} dx \leq 1\Big\} dt = 0,$$

*uniformly in $u \in B$ where $B \subset \mathbb{Y}$ is an arbitrary bounded set. The collection of such functions will be denoted by $S^{p,q(x)}_{paa}(\mathbb{Y},\mathbb{X})$.*

Let $Lip^r(\mathbb{Y},\mathbb{X})$ denote the collection of functions $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ satisfying: there exists a nonnegative function $L_f \in L^r(\mathbb{R})$ such that

$$\|f(t,u) - f(t,v)\| \leq L_f(t)\|u - v\|_{\mathbb{Y}} \quad \text{for all } u, v \in \mathbb{Y},\ t \in \mathbb{R}.$$

Now, we recall the following composition theorem for $S^p_{aa}$ functions.

**Theorem 4.18** ([17]). *Let $p > 1$ be a constant. We suppose that the following conditions hold:*

(a) *$f \in S^p_{aa}(\mathbb{Y},\mathbb{X}) \cap Lip^r(\mathbb{Y},\mathbb{X})$ with $r \geq \max\{p, \frac{p}{p-1}\}$.*
(b) *$\phi \in S^p_{aa}(\mathbb{X})$ and there exists a set $E \subset \mathbb{R}$ such that $K := \overline{\{\phi(t) : t \in \mathbb{R} \setminus E\}}$ is compact in $\mathbb{X}$.*

*Then there exists $m \in [1,p)$ such that $f(\cdot,\phi(\cdot)) \in S^m_{aa}(\mathbb{X})$.*

To obtain a composition theorem for $S^{p,q}_{paa}$ functions, we need the following lemma.

**Lemma 4.19.** *Let $p, q > 1$ be a constants. Assume that $f = g + h \in S^{p,q}_{paa}(\mathbb{R} \times \mathbb{X})$ with $g^b \in AA\big(\mathbb{R} \times L^p((0,1),\mathbb{X})\big)$ and $h^b \in PAA_0\big(\mathbb{R} \times L^q((0,1),\mathbb{X})\big)$. If $f \in Lip^p(\mathbb{R},\mathbb{X})$, then $g$ satisfies*

$$\Big(\int_0^1 \|g(t+s,u(s)) - g(t+s,v(s))\|^p\, ds\Big)^{1/p} \leq c\|L_f\|_{S^p}\|u - v\|_{\mathbb{Y}}.$$

*for all $u, v \in \mathbb{Y}$ and $t \in \mathbb{R}$, where $c$ is a nonnegative constant.*

*Proof.* Let $f = g+h \in S^{p,q(x)}_{paa}(\mathbb{R}\times\mathbb{X})$ with $g^b(\cdot,u) \in AA\big(L^p((0,1),\mathbb{X})\big)$ and $h^b(\cdot,u) \in PAA_0\big(L^q((0,1),\mathbb{X})\big)$ for each $u \in \mathbb{Y}$. Using Lemma 4.14 it follows that

$$\{g(t+\cdot,u) : t \in \mathbb{R}\} \subset \overline{\{f(t+\cdot,u) : t \in \mathbb{R}\}} \quad \text{in} \quad S^{p,q}(\mathbb{X})$$

for each $u \in \mathbb{Y}$.

Since $f \in Lip^p(\mathbb{R},\mathbb{X})$ and $\big(BS^p(\mathbb{X}), \|\cdot\|_{S^p}\big) \hookrightarrow \big(BS^{p,q}(\mathbb{X}), \|\cdot\|_{S^{p,q}}\big)$, it follows that

$$\begin{aligned}\Big(\int_0^1 \|g(t+s,u(s)) - g(t+s,v(s))\|^p\, ds\Big)^{1/p} &\leq \|g(\cdot,u) - g(\cdot,v)\|_{S^p}\\ &= \|g(\cdot,u) - g(\cdot,v)\|_{S^{p,q}}\\ &\leq \|f(\cdot,u) - f(\cdot,v)\|_{S^{p,q}}\\ &\leq c\|f(\cdot,u) - f(\cdot,v)\|_{S^p}\\ &\leq c\|L_f\|_{S^p}\|u - v\|_{\mathbb{Y}}.\end{aligned}$$

for all $u, v \in \mathbb{Y}$ and $t \in \mathbb{R}$. ∎

**Theorem 4.20.** *Let $p, q > 1$ be a constants such that $p \le q$. Assume that the following conditions hold:*

(a) $f = g + h \in S^{p,q}_{paa}(\mathbb{R} \times \mathbb{X})$ *with* $g \in S^p_{aa}(\mathbb{R} \times \mathbb{X})\big)$ *and* $h \in S^q_{paa_0}(\mathbb{R} \times \mathbb{X})$. *Moreover,* $f, g \in Lip^r(\mathbb{R}, \mathbb{X})$ *with* $r \ge \max\{p, \frac{p}{p-1}\}$*;*

(b) $\phi = \alpha + \beta \in S^{p,q}_{paa}(\mathbb{Y})$ *with* $\alpha \in S^p_{aa}(\mathbb{Y})$ *and* $\beta \in S^q_{paa_0}(\mathbb{Y})$*, and* $K := \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ *is compact in* $\mathbb{Y}$.

*Then there exists* $m \in [1, p)$ *such that* $f(\cdot, \phi(\cdot)) \in S^{m,m}_{paa}(\mathbb{R} \times \mathbb{X})$.

*Proof.* First of all, write

$$f^b(\cdot, \phi^b(\cdot)) = g^b(\cdot, \alpha^b(\cdot)) + f^b(\cdot, \phi^b(\cdot)) - f^b(\cdot, \alpha^b(\cdot)) + h^b(\cdot, \alpha^b(\cdot)).$$

From Lemma 4.19, one has $g \in S^p_{aa}(\mathbb{R}\times\mathbb{X})$. Now using the theorem of composition of $S^p$-almost automorphic functions (Theorem 4.18), it is easy to see that there exists $m \in [1, p)$ with $\frac{1}{m} = \frac{1}{p} + \frac{1}{r}$ such that $g^b(\cdot, \alpha^b(\cdot)) \in AA(\mathbb{R} \times L^m((0, 1), \mathbb{X}))$.

Set $\Phi^b(\cdot) = f^b(\cdot, \phi^b(\cdot)) - f^b(\cdot, \alpha^b(\cdot))$. Clearly, $\Phi^b \in PAA_0(\mathbb{R} \times L^m((0, 1), \mathbb{X}))$. Now, for $T > 0$,

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}\Big(\int_t^{t+1}\|\Phi^b(s)\|^m ds\Big)^{1/m} dt\\
&= \frac{1}{2T}\int_{-T}^{T}\Big(\int_t^{t+1}\|f^b(s, \phi^b(s)) - f^b(s, \alpha^b(s))\|^m ds\Big)^{1/m} dt\\
&\le \frac{1}{2T}\int_{-T}^{T}\Big(\int_t^{t+1}\Big(L^b_f(s)\|\beta^b(s)\|_{\mathbb{Y}}\Big)^m ds\Big)^{1/m} dt\\
&\le \|L^b_f\|_{S^r}\Big[\frac{1}{2T}\int_{-T}^{T}\Big(\int_t^{t+1}\|\beta^b(s)\|^p_{\mathbb{Y}} ds\Big)^{1/p} dt\Big]\\
&\le \|L^b_f\|_{S^r}\Big[\frac{1}{2T}\int_{-T}^{T}\Big(\int_t^{t+1}\|\beta^b(s)\|^q_{\mathbb{Y}} ds\Big)^{1/q} dt\Big].
\end{aligned}$$

Using the fact that $\beta^b \in PAA_0(L^q((0, 1), \mathbb{Y})\big)$, it follows that $\Phi^b \in PAA_0(\mathbb{R} \times L^m((0, 1), \mathbb{X}))$.

On the other hand, since $f, g \in Lip^r(\mathbb{R}, \mathbb{X}) \subset Lip^p(\mathbb{R}, \mathbb{X})$, one has

$$\begin{aligned}
&\Big(\int_0^1 \|h(t+s, u(s)) - h(t+s, v(s))\|^m ds\Big)^{1/m}\\
&\le \Big(\int_0^1 \|f(t+s, u(s)) - f(t+\cdot, v(s))\|^m ds\Big)^{1/m}\\
&\quad + \Big(\int_0^1 \|g(t+s, u(s)) - g(t+s, v(s))\|^m ds\Big)^{1/m}\\
&\le \Big(\int_0^1 \Big(L_f(t+s)\|u(s) - v(s)\|_{\mathbb{Y}}\Big)^m ds\Big)^{1/m}\\
&\quad + \Big(\int_0^1 \Big(L_g(t+s)\|u(s) - v(s)\|_{\mathbb{Y}}\Big)^m ds\Big)^{1/m}\\
&\le \Big(\|L_f\|_{S^r} + \|L_g\|_{S^r}\Big)\|u(s) - v(s)\|_p.
\end{aligned}$$

Since $K := \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact in $\mathbb{Y}$, then for each $\varepsilon > 0$, there exists a finite number of open balls $B_k = B(x_k, \varepsilon)$, centered at $x_k \in K$ with radius $\varepsilon$ such that

$$\{\alpha(t) : t \in \mathbb{R}\} \subset \cup_{k=1}^{m} B_k.$$

Therefore, for $1 \le k \le m$, the set $U_k = \{t \in \mathbb{R} : \alpha \in B_k\}$ is open and $\mathbb{R} = \cup_{k=1}^{m} U_k$. Now, for $2 \le k \le m$, set $V_k = U_k - \cup_{i=1}^{k-1} U_i$ and $V_1 = U_1$. Clearly, $V_i \cap V_j = \emptyset$ for all $i \neq j$. Define the step function $\overline{x} : \mathbb{R} \to \mathbb{Y}$ by $\overline{x}(t) = x_k, t \in V_k, k = 1, 2, \dots, m$. It easy to see that

$$\|\alpha(s) - \overline{x}(s)\|_{\mathbb{Y}} \le \varepsilon, \quad \text{for all } s \in \mathbb{R}.$$

which yields

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}\Big(\int_t^{t+1} \|h(s,\alpha(s))\|^m ds\Big)^{1/m} dt \\
&\le \frac{1}{2T}\int_{-T}^{T}\Big(\int_t^{t+1} \|h(s,\alpha(s)) - h(s,\overline{x}(s))\|^m ds\Big)^{1/m} dt \\
&\quad + \frac{1}{2T}\int_{-T}^{T}\Big(\int_t^{t+1} \|h(s,\overline{x}(s))\|^m ds\Big)^{1/m} dt \\
&\le \Big(\|L_f\|_{S^r} + \|L_g\|_{S^r}\Big)\varepsilon + \frac{1}{2T}\int_{-T}^{T}\Big(\sum_{k=1}^{m}\int_{V_k\cap[t,t+1]} \|h(s,\overline{x}(s))\|^m ds\Big)^{1/m} dt \\
&\le \Big(\|L_f\|_{S^r} + \|L_g\|_{S^r}\Big)\varepsilon + \frac{1}{2T}\int_{-T}^{T}\Big(\sum_{k=1}^{m}\int_{V_k\cap[t,t+1]} \|h(s,\overline{x}(s))\|^q ds\Big)^{1/q} dt.
\end{aligned}$$

Since $\varepsilon$ is arbitrary and $h^b \in PAA_0(\mathbb{R} \times L^q((0,1),\mathbb{X}))$, it follows that the function $h^b(\cdot, \alpha^b(\cdot))$ belongs to $PAA_0(\mathbb{R} \times L^m((0,1),\mathbb{X}))$. □

**Remark 4.21.** A general composition theorem in $S_{paa}^{p,q(x)}(\mathbb{R} \times \mathbb{X})$ is unlikely as compositions of elements of $S_{paa}^{p,q(x)}(\mathbb{R} \times \mathbb{X})$ may not be well-defined unless $q(\cdot)$ is the constant function.

## 5. Existence of pseudo-almost automorphic solutions

Let $p, q > 1$ be constants such that $p \le q$. In this section, we discuss the existence and uniqueness of pseudo-almost automorphic solutions to the first-order linear differential equation (1.1) and to the semilinear equation (1.2). For that, we make the following assumptions:

- (H1) The family of closed linear operators $A(t)$ satisfy Acquistapace–Terreni conditions.
- (H2) The evolution family $(U(t,s))_{t\ge s}$ generated by $A(t)$ has an exponential dichotomy with constants $M > 0, \delta > 0$, dichotomy projections $P(t), t \in \mathbb{R}$, and Green's function $\Gamma(t,s)$.
- (H3) $\Gamma(t,s) \in bAA(\mathbb{R} \times \mathbb{R}, B(\mathbb{X}))$.
- (H4) $B : \mathbb{X} \mapsto \mathbb{X}$ is a bounded linear operator and let $\|B\|_{B(\mathbb{X})} = c$.
- (H5) $F = G + H \in S_{paa}^{p,q}(\mathbb{R} \times \mathbb{X}) \cap C(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ with $G^b \in AA(\mathbb{R} \times L^p((0,1),\mathbb{X}))$ and $H^b \in PAA_0(\mathbb{R} \times L^q((0,1),\mathbb{X}))$. Moreover, $F, G \in Lip^r(\mathbb{R}, \mathbb{X})$ with

$$r \ge \max\Big\{p, \frac{p}{p-1}\Big\}.$$

Let us also mention that (H1) was introduced in the literature by Acquistapace and Terreni in [2, 3]. Among other things, from [1, Theorem 2.3] (see also [3, 24, 25]), assumption (H1) does ensure that the family of operators $A(t)$ generates a unique strongly continuous evolution family on $\mathbb{X}$, which we will denote by $(U(t,s))_{t\geq s}$.

**Definition 5.1.** Under (H1), if $f:\mathbb{R}\to\mathbb{X}$ is a bounded continuous function, then a mild solution to (1.1) is a continuous function $u:\mathbb{R}\to\mathbb{X}$ satisfying

$$u(t)=U(t,s)u(s)+\int_s^t U(t,\sigma)f(\sigma)d\sigma \tag{5.1}$$

for all $(t,s)\in\mathbb{T}:=\big\{(t,s)\in\mathbb{R}\times\mathbb{R}:\quad t\geq s\big\}$.

**Definition 5.2.** Suppose (H1) and (H4) hold. If $F:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ is a bounded continuous function, then a mild solution to (1.2) is a continuous function $u:\mathbb{R}\to\mathbb{X}$ satisfying

$$u(t)=U(t,s)u(s)+\int_s^t U(t,\sigma)F(\sigma,Bu(\sigma))d\sigma \tag{5.2}$$

for all $(t,s)\in\mathbb{T}$.

**Theorem 5.3.** *Let $p>1$ be a constant and let $q\in C_+(\mathbb{R})$. Suppose that* (H1)–(H3) *hold. If $f\in S^{p,q(x)}_{paa}(\mathbb{X})\cap C(\mathbb{R},\mathbb{X})$, then the* (1.1) *has a unique pseudo-almost automorphic solution given by*

$$u(t)=\int_{-\infty}^{+\infty}\Gamma(t,\sigma)f(\sigma)d\sigma,\quad t\in\mathbb{R}. \tag{5.3}$$

*Proof.* Define the function $u:\mathbb{R}\mapsto\mathbb{X}$ by

$$u(t):=\int_{-\infty}^t U(t,\sigma)P(\sigma)f(\sigma)d\sigma-\int_t^{+\infty}U_Q(t,\sigma)Q(\sigma)f(\sigma)d\sigma,\quad t\in\mathbb{R}.$$

Let us show that $u$ satisfies (5.1) for all $t\geq s$, all $t,s\in\mathbb{R}$. Indeed, applying $U(t,s)$ for all $t\geq s$, to both sides of the expression of $u$, we obtain,

$$\begin{aligned}
U(t,s)u(s)&=\int_{-\infty}^s U(t,\sigma)P(\sigma)f(\sigma)d\sigma-\int_s^{+\infty}U_Q(t,\sigma)Q(\sigma)f(\sigma)d\sigma\\
&=\int_{-\infty}^t U(t,\sigma)P(\sigma)f(\sigma)d\sigma-\int_s^t U(t,\sigma)P(\sigma)f(\sigma)d\sigma\\
&\quad-\int_t^{+\infty}U_Q(t,\sigma)Q(\sigma)f(\sigma)d\sigma-\int_s^t U_Q(t,\sigma)Q(\sigma)f(\sigma)d\sigma\\
&=u(t)-\int_s^t U(t,\sigma)f(\sigma)d\sigma
\end{aligned}$$

and hence $u$ is a mild solution to (1.1).

Let us show that $u\in PAA(\mathbb{X})$. Indeed, since $f\in S^{p,q(x)}_{paa}(\mathbb{X})\cap C(\mathbb{R},\mathbb{X})$, then $f=g+\varphi$, where $g^b\in AA(L^p((0,1),\mathbb{X}))$ and $\varphi^b\in PAA_0(L^{q^b(x)}((0,1),\mathbb{X}))$. Then $u$ can be decomposed as $u(t)=X(t)+Y(t)$, where

$$X(t)=\int_{-\infty}^t U(t,s)P(s)g(s)ds+\int_{+\infty}^t U_Q(t,s)Q(s)g(s)ds,$$

$$Y(t) = \int_{-\infty}^{t} U(t,s)P(s)\varphi(s)ds + \int_{+\infty}^{t} U_Q(t,s)Q(s)\varphi(s)ds.$$

The proof that $X \in AA(\mathbb{X})$ is obvious and hence is omitted. To prove that $Y \in PAA_0(\mathbb{X})$, we define for all $n = 1, 2, \dots$, the sequence of integral operators

$$\begin{aligned} Y_n(t) :&= \int_{t-n}^{t-n+1} U(t,s)P(s)\varphi(s)ds + \int_{t+n-1}^{t+n} U_Q(t,s)Q(s)\varphi(s)ds \\ &= \int_{n-1}^{n} U(t,t-s)P(t-s)\varphi(t-s)ds + \int_{n-1}^{n} U_Q(t,t+s)Q(t+s)\varphi(t+s)ds \end{aligned}$$

for each $t \in \mathbb{R}$.

Let $d \in m(\mathbb{R})$ such that $q^{-1}(x) + d^{-1}(x) = 1$. From exponential dichotomy of $(U(t,s))_{t\geq s}$ and Hölder's inequality (Theorem 3.8), it follows that

$$\begin{aligned} \|Y_n(t)\| &\leq M \int_{t-n}^{t-n+1} e^{-\delta(t-s)}\|\varphi(s)\|ds + M\int_{t+n-1}^{t+n} e^{\delta(t-s)}\|\varphi(s)\|ds \\ &\leq M\big(\frac{1}{d^-} + \frac{1}{q^-}\big)\Big[\inf\Big\{\lambda > 0 : \int_{t-n}^{t-n+1} \Big(\frac{e^{-\delta(t-s)}}{\lambda}\Big)^{d(s)} ds \leq 1\Big\}\Big] \\ &\quad \times \Big[\inf\Big\{\lambda > 0 : \int_{t-n}^{t-n+1} \|\frac{\varphi(s)}{\lambda}\|^{q(s)} ds \leq 1\Big\}\Big] \\ &\quad + M\big(\frac{1}{d^-} + \frac{1}{q^-}\big)\Big[\inf\Big\{\lambda > 0 : \int_{t+n-1}^{t+n} \Big(\frac{e^{\delta(t-s)}}{\lambda}\Big)^{d(s)} ds \leq 1\Big\}\Big] \\ &\quad \times \Big[\inf\Big\{\lambda > 0 : \int_{t+n-1}^{t+n} \Big\|\frac{\varphi(s)}{\lambda}\Big\|^{q(s)} ds \leq 1\Big\}\Big]. \end{aligned}$$

Now since

$$\begin{aligned} \int_{t-n}^{t-n+1} \Big[\frac{e^{-\delta(t-s)}}{e^{-\delta(n-1)}}\Big]^{d(s)} ds &= \int_{t-n}^{t-n+1} \Big[e^{\delta(s-t+n-1)}\Big]^{d(s)} ds \\ &\leq \int_{t-n}^{t-n+1} \Big[1\Big]^{d(s)} ds \leq 1 \end{aligned}$$

it follows that

$$e^{-\delta(n-1)} \in \Big\{\lambda > 0 : \int_{t-n}^{t-n+1} \Big(\frac{e^{-\delta(t-s)}}{\lambda}\Big)^{d(s)} ds \leq 1\Big\},$$

which shows that

$$\Big[\inf\Big\{\lambda > 0 : \int_{t-n}^{t-n+1} \Big(\frac{e^{-\delta(t-s)}}{\lambda}\Big)^{d(s)} ds \leq 1\Big\}\Big] \leq e^{-\delta(n-1)}.$$

Consequently,

$$\begin{aligned} \|Y_n(t)\| &\leq M\big(\frac{1}{d^-} + \frac{1}{q^-}\big)e^{-\delta(n-1)}\|\varphi\|_{S^{q(x)}} + M\big(\frac{1}{d^-} + \frac{1}{q^-}\big)e^{\delta(1-n)}\|\varphi\|_{S^{q(x)}} \\ &\leq 2M\big(\frac{1}{d^-} + \frac{1}{q^-}\big)e^{-\delta(n-1)}\|\varphi\|_{S^{q(x)}}. \end{aligned}$$

Since the series $\sum_{n=1}^{\infty} e^{-\delta(n-1)}$ converges, we deduce from the well-known Weierstrass test that the series $\sum_{n=1}^{\infty} Y_n(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$Y(t) = \int_{-\infty}^{t} U(t,s)P(s)\varphi(s)ds + \int_{+\infty}^{t} U_Q(t,s)Q(s)\varphi(s)ds = \sum_{n=1}^{\infty} Y_n(t),$$

$Y \in C(\mathbb{R}, \mathbb{X})$, and

$$\|Y(t)\| \le \sum_{n=1}^{\infty} \|Y_n(t)\| \le 2M\big(\frac{1}{d^-} + \frac{1}{q^-}\big) \sum_{n=1}^{\infty} e^{-\delta(n-1)} \|\varphi\|_{S^{q(x)}}.$$

Next, we will show that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|Y(s)\|\, ds = 0.$$

Indeed,

$$\begin{aligned} &\frac{1}{2T} \int_{-T}^{T} \|Y_n(t)\|\, dt \\ &\le 2M\big(\frac{1}{d^-} + \frac{1}{q^-}\big) e^{-\delta(n-1)} \Big[\frac{1}{2T} \int_{-T}^{T} \inf\Big\{\lambda > 0 : \int_{t+n-1}^{t+n} \Big\|\frac{\varphi(s)}{\lambda}\Big\|^{q(s)} ds \le 1\Big\}\Big]. \end{aligned}$$

Since $\varphi^b \in PAA_0(L^{q^b(x)}((0,1), \mathbb{X}))$, the above inequality leads to $Y_n \in PAA_0(\mathbb{X})$. Using the following inequality

$$\frac{1}{2T} \int_{-T}^{T} \|Y(s)\|\, ds \le \frac{1}{2T} \int_{-T}^{T} \Big\|Y(s) - \sum_{n=1}^{\infty} Y_n(s)\Big\|\, dt + \sum_{n=1}^{\infty} \frac{1}{2T} \int_{-T}^{T} \|Y_n(s)\|\, ds,$$

we deduce that the uniform limit $Y(\cdot) = \sum_{n=1}^{\infty} Y_n(\cdot) \in PAA_0(\mathbb{X})$. Therefore $u \in PAA(\mathbb{X})$.

It remains to prove the uniqueness of $u$ as a mild solution. This has already been done by Diagana [6, 10]. However, for the sake of clarity let us reproduce it here. Let $u, v$ be two bounded mild solutions to (1.1). Setting $w = u - v$, one can easily see that $w$ is bounded and that $w(t) = U(t,s)w(s)$ for all $(t,s) \in \mathbb{T}$. Now using property (i) from exponential dichotomy (Definition 2.11) it follows that $P(t)w(t) = P(t)U(t,s)w(s) = U(t,s)P(s)w(s)$, and hence

$$\|P(t)w(t)\| = \|U(t,s)P(s)w(s)\| \le Me^{-\delta(t-s)}\|w(s)\| \le Me^{-\delta(t-s)}\|w\|_\infty.$$

for all $(t,s) \in \mathbb{T}$.

Now, given $t \in \mathbb{R}$ with $t \ge s$, if we let $s \to -\infty$, we then obtain that $P(t)w(t) = 0$, that is, $P(t)u(t) = P(t)v(t)$. Since $t$ is arbitrary it follows that $P(t)w(t) = 0$ for all $t \ge s$. Similarly, from $w(t) = U(t,s)w(s)$ for all $t \ge s$ and property (i) from exponential dichotomy (Definition 2.11) it follows that $Q(t)w(t) = Q(t)U(t,s)w(s) = U(t,s)Q(s)w(s)$, and hence $U_Q(s,t)Q(t)w(t) = Q(s)w(s)$ for all $t \ge s$. Moreover,

$$\|Q(s)w(s)\| = \|U_Q(s,t)Q(t)w(t)\| \le Me^{-\delta(t-s)}\|w\|_\infty.$$

for all $t \ge s$.

Now, given $s \in \mathbb{R}$ with $t \ge s$, if we let $t \to +\infty$, we then obtain that $Q(t)w(t) = 0$, that is, $Q(s)u(s) = Q(s)v(s)$. Since $s$ is arbitrary it follows that $Q(s)w(s) = 0$ for all $t \ge s$. $\square$

Using Theorem 5.3 one easily proves the following theorem.

**Theorem 5.4.** *Let $p, q > 1$ be constants such that $p \leq q$. Under assumptions* (H1)–(H5)*, then* (1.2) *has a unique solution whenever $\|L_F\|_{S^r}$ is small enough. And the solution satisfies the integral equation*

$$u(t) = \int_{-\infty}^{t} U(t,\sigma)P(\sigma)F(\sigma, Bu(\sigma))d\sigma - \int_{t}^{+\infty} U_Q(t,\sigma)Q(\sigma)F(\sigma, Bu(\sigma))d\sigma, \ t \in \mathbb{R}.$$

*Proof.* Define $\Xi : PAA(\mathbb{X}) \to PAA(\mathbb{X})$ as

$$(\Xi u)(t) = \int_{-\infty}^{t} U(t,\sigma)P(\sigma)F(\sigma, Bu(\sigma))d\sigma - \int_{t}^{+\infty} U_Q(t,\sigma)Q(\sigma)F(\sigma, Bu(\sigma))d\sigma$$

Let $u \in PAA(\mathbb{X}) \subset S^{p,q}_{paa}(\mathbb{X})$. From (H4) and Theorem 4.16 it is clear that $Bu(.) \in S^{p,q}_{paa}(\mathbb{X})$. Using the composition theorem for $S^{p,q}_{paa}$ functions, we deduce that there exists $m \in [1,p)$ such that $F(., Bu(.)) \in S^{m,m}_{paa}(\mathbb{X})$. applying the proof of Theorem 5.3, to $f(.) = F(., Bu(.))$, one can easily see that the operator $\Xi$ maps $PAA(\mathbb{X})$ into its self. Moreover, for all $u, v \in PAA(\mathbb{X})$, it is easy to see that

$$\begin{aligned}
&\|(\Xi u)(t) - (\Xi v)(t)\| \\
&\leq \int_{\mathbb{R}} \|\Gamma(t-s)\| \|F(s, Bu(s)) - F(s, Bv(s))\| \, ds \\
&\leq \int_{-\infty}^{t} cMe^{-\delta(t-s)} L_F(s) \, ds \|u-v\|_\infty + \int_{t}^{+\infty} cMe^{\delta(t-s)} L_F(s) \, ds \|u-v\|_\infty \\
&\leq \sum_{n=1}^{\infty} \int_{t-n}^{t-n+1} cMe^{-\delta(t-s)} L_F(s) \, ds \|u-v\|_\infty \\
&\quad + \sum_{n=1}^{\infty} \int_{t+n-1}^{t+n} cMe^{\delta(t-s)} L_F(s) \, ds \|u-v\|_\infty \\
&\leq cM \sum_{n=1}^{\infty} \Big( \int_{t-n}^{t-n+1} e^{-r_0\delta(t-s)} \, ds \Big)^{\frac{1}{r_0}} \|L_F\|_{S^r} \|u-v\|_\infty \\
&\quad + cM \sum_{n=1}^{\infty} \Big( \int_{t+n-1}^{t+n} e^{r_0\delta(t-s)} \, ds \Big)^{\frac{1}{r_0}} \|L_F\|_{S^r} \|u-v\|_\infty \\
&\leq 2cM \sum_{n=1}^{\infty} \Big( \frac{e^{-r_0(n-1)\delta} - e^{-r_0 n\delta}}{r_0\delta} \Big)^{\frac{1}{r_0}} \|L_F\|_{S^r} \|u-v\|_\infty \\
&\leq 2cM \sqrt[r_0]{\frac{1+e^{r_0\delta}}{r_0\delta}} \sum_{n=1}^{\infty} e^{-n\delta} \|L_F\|_{S^r} \|u-v\|_\infty,
\end{aligned}$$

for each $t \in \mathbb{R}$, where $\frac{1}{r} + \frac{1}{r_0} = 1$. Hence whenever $\|L_F\|_{S^r}$ is small enough, that is,

$$2cM \sqrt[r_0]{\frac{1+e^{r_0\delta}}{r_0\delta}} \sum_{n=1}^{\infty} e^{-n\delta} \|L_F\|_{S^r} < 1,$$

then $\Xi$ has a unique fixed point, which obviously is the unique pseudo-almost automorphic solution to (1.2). □

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