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EXISTENCE OF SOLUTIONS FOR EIGENVALUE PROBLEMS WITH NONSTANDARD GROWTH CONDITIONS

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Abstract. We prove the existence of weak solutions for some eigenvalue problems involving variable exponents. Our main tool is critical point theory.

1. Introduction and statement of main results

In this article, we are concerned with the quasilinear problem

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u=\lambda\varphi(x)|u|^{\alpha(x)-2}u+h,\quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, p and $\alpha \in \{v \in C(\mathbb{R}^N,\mathbb{R}) \cap L^\infty(\mathbb{R}^N), \inf_{x\in\mathbb{R}^N} v(x) > 1\}$, $\varphi \in C(\mathbb{R}^N,\mathbb{R})$, $\varphi(x) > 0$ for all $x \in \mathbb{R}^N$, λ is a positive parameter and h is a function which belongs to the dual of the Sobolev space with variable exponent $W^{1,p(\cdot)}(\mathbb{R}^N)$.

The study of eigenvalue problems involving variable exponents growth conditions has been an interesting topic of research in last years. We can for example refer to [6, 9, 12, 13, 14, 15, 16]. A first contribution in this sense is due to Fan, Zhand and Zhao [9] who studied the problem

$$\begin{gathered}-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)=\lambda|u|^{p(x)-2}u \quad \text{in } \Omega\\ u=0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $p : \overline{\Omega} \to (1,\infty)$ is a continuous function and λ is a real number. In [9], the authors established the existence of infinitely many eigenvalues for problem (1.2). Denoting Λ the set of all nonnegative eigenvalues, it was proved in [9] that $\sup(\Lambda) = +\infty$. It was also proved that only under special conditions concerning the monotony of the variable exponent $p(\cdot)$, we have $\inf(\Lambda) > 0$ which is in contrast with the case when p is a constant. Mihăilescu and Rădulescu [13] studied the problem

$$\begin{gathered}-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)=\lambda|u|^{q(x)-2}u \quad \text{in } \Omega\\ u=0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $p, q : \overline{\Omega} \to (1,+\infty)$ are two continuous functions and λ is a real number. Using Ekeland's variational

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principle, they proved that under the assumption

$$\min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} q(x) < \max_{x\in\overline{\Omega}} q(x), \quad \max_{x\in\overline{\Omega}} q(x) < N, \quad q(x) < \frac{Np(x)}{N-p(x)} \quad \forall x \in \overline{\Omega},$$

there exists a continuous family of eigenvalues which lies in a neighborhood of the origin. The case when $\max_{x\in\overline{\Omega}} p(x) < \min_{x\in\overline{\Omega}} q(x)$ was treated by Fan and Zhang [8] using the Mountain-Pass Theorem. Finally, in the case when $\max_{x\in\overline{\Omega}} p(x) < \min_{x\in\overline{\Omega}} q(x)$ and by combining results of [8] and [14], it is easy to see that there exists two positive constants λ^* and λ^{**} such that any $\lambda \in (0,\lambda^*) \cup (\lambda^{**},+\infty)$ is an eigenvalue of the problem. Another important eigenvalue problem is the following

$$\begin{gathered} -\operatorname{div}((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u) = \lambda |u|^{q(x)-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. Provided that $p_1, p_2, q : \overline{\Omega} \to (1,+\infty)$ are continuous functions such that q has a sub-critical growth with respect to p_2 and the following condition is verified

$$1 < p_2(x) < \min_{\overline{\Omega}} q \le \max_{\overline{\Omega}} q < p_1(x) \quad \forall x \in \overline{\Omega},$$

problem (1.4) was discussed in [15] and it was shown that there exist two positive constants λ_0 and λ_1 with $\lambda_0 \le \lambda_1$ such that any $\lambda \in [\lambda_1,+\infty)$ is an eigenvalue of the problem (1.4) while for any $\lambda \in (0,\lambda_0)$, problem (1.4) does not admit any nontrivial solution. The novelty in this article lies in the fact that we divide $\mathbb{R}^N$ into three parts

$$\begin{gathered} \Omega_1 = \{x \in \mathbb{R}^N : \alpha(x) < p(x)\}, \quad \Omega_2 = \{x \in \mathbb{R}^N : \alpha(x) > p(x)\}, \\ \Omega_3 = \{x \in \mathbb{R}^N : \alpha(x) = p(x)\}. \end{gathered}$$

We assume that $\operatorname{meas}(\Omega_3) = 0$ where "meas" denotes the Lebesgue measure in $\mathbb{R}^N$. In this work, we are interested in the case when $\operatorname{meas}(\Omega_1) > 0$ and $\operatorname{meas}(\Omega_2) > 0$. Thus, possibly we could have $\operatorname{meas}(\Omega_1) = +\infty$ and $\operatorname{meas}(\Omega_2) = +\infty$. We have to notice that this possibility to divide $\mathbb{R}^N$ into Ω_1, Ω_2 and Ω_3 is so related to quasilinear equations involving variable exponents because we cannot find such a phenomenon when treating quasilinear equations with constant exponents. On the other hand, in the majority of works dealing with nonlinear equations involving variable exponents, a precise comparison between the extrema of involved variable exponents is provided. So, the situation that we are treating is rather new.

Throughout this paper, we denote

$$\begin{gathered} \alpha^-_{\Omega_1} = \inf_{x\in\Omega_1} \alpha(x), \quad \alpha^-_{\Omega_2} = \inf_{x\in\Omega_2} \alpha(x), \\ p^-_{\Omega_1} = \inf_{x\in\Omega_1} p(x), \quad p^+_{\Omega_1} = \sup_{x\in\Omega_1} p(x), \\ p^-_{\Omega_2} = \inf_{x\in\Omega_2} p(x), \quad p^+_{\Omega_2} = \sup_{x\in\Omega_2} p(x), \end{gathered}$$

$p^+ = \sup_{x\in\mathbb{R}^N} p(x)$, $\|h\|_{-1}$ is the norm of h in the dual of $W^{1,p(\cdot)}(\mathbb{R}^N)$. Set

$$E = \big\{u \in W^{1,p(\cdot)}(\mathbb{R}^N), \int_{\mathbb{R}^N} \varphi(x)|u|^{\alpha(x)}dx < +\infty\big\}.$$

We equip the functional space E with the norm

$$\|u\|_E = \|u\|_{W^{1,p(\cdot)}(\mathbb{R}^N)} + |(\varphi(\cdot))^{\frac{1}{\alpha(\cdot)}}u|_{L^{\alpha(\cdot)}(\mathbb{R}^N)}.$$

Definition A function $u \in E$ is said to be a weak solution of the problem (1.1) if it satisfies

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_{\mathbb{R}^N} |u|^{p(x)-2} uv dx\\ &= \lambda \int_{\mathbb{R}^N} \varphi(x)|u|^{\alpha(x)-2} uv dx + \int_{\mathbb{R}^N} hv dx, \quad \forall v \in E.\end{aligned}$$

This article is divided into two parts. In the first part, we will study problem (1.1) under the following hypotheses:

- (H1) $\int_{\Omega_1} (\varphi(x))^{\frac{p(x)}{p(x)-\alpha(x)}} dx < +\infty$;
- (H2) $p(x) < N$ for all $x \in \Omega_2$, and there exists $r \in C_+(\overline{\Omega_2})$ such that $\varphi \in L^{r(\cdot)}(\Omega_2)$ and

$$p(x) \leq \frac{\alpha(x) r(x)}{r(x)-1} \leq p^*(x) \quad \forall x \in \Omega_2, \quad \text{where } p^*(x) = \frac{Np(x)}{N-p(x)};$$

- (H3) There exists $\psi \in W^{1,p(\cdot)}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} h(x)\psi(x) > 0$.

The main result of this first part is given by the following theorem.

Theorem 1.1. *Assume that (H1), (H2) hold. Assume also that $\alpha^-_{\Omega_2} \geq p^+_{\Omega_2}$. Then, we have: if* (H3) *holds, or $h = 0$, then there exists $\lambda_* > 0$ such that for all $0 < \lambda < \lambda_*$, there exists $\eta_\lambda >$ verifying that: if $\|h\|_{-1} < \eta_\lambda$, then problem* (1.1) *admits at least one nontrivial weak solution $u_{0,\lambda}$. Moreover, if $h = 0$, then $u_{0,\lambda} \to 0$ strongly in $W^{1,p(\cdot)}(\mathbb{R}^N)$ when $\lambda \to 0$.*

In the second part of this article, we will remove the assumptions (H1) and (H2) and we will study (1.1) under the following hypotheses:

- (H4) The exponent $p(\cdot)$ is log-Hölder continuous; i.e., there exists a positive constant $D > 0$ such that

$$|p(x) - p(y)| \leq \frac{D}{-\log(|x-y|)}, \quad \text{for every } x, y \in \mathbb{R}^N \text{ with } |x-y| \leq 1/2;$$

- (H5) $\inf_{x\in\mathbb{R}^N} \alpha(x) = \alpha^- > 2$.

Theorem 1.2. *Assume that* (H4), (H5) *hold. If $h = 0$, then there exists $0 < \lambda_{**}$ such that for every $0 < \lambda < \lambda_{**}$, then problem* (1.1) *admits at least one nontrivial weak solution.*

Remark 1.3. The importance of the hypothesis (H4) lies in the fact that if p verifies the logarithmic Hölder continuity condition (also called the Dini-Lipschitz condition), the space $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^N)$ (see [4, 19]).

2. Preliminaries

First, we give some background facts from the variable exponent Lebesgue and Sobolev spaces. For details, we refer to the books [2, 17] and the papers [3, 7, 11, 20]. Assume $\Omega \subset \mathbb{R}^N$ is a (bounded or unbounded) open domain. Set $C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) \cap L^\infty(\Omega),\, h(x) > 1,\, \forall x \in \overline{\Omega}\}$. For any $p \in C_+(\overline{\Omega})$, we define

$$p^+ = \sup_{x\in\Omega} p(x) \quad \text{and } p^- = \inf_{x\in\Omega} p(x).$$

For each $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) = \{u;\ u : \Omega \to \mathbb{R} \text{ measurable such that } \int_\Omega |u(x)|^{p(x)} dx < +\infty\}.$$

This space becomes a Banach space with respect to the Luxemburg norm,

$$|u|_{L^{p(\cdot)}(\Omega)} = \inf\{\mu > 0 : \int_\Omega |\frac{u(x)}{\mu}|^{p(x)} dx \le 1\}.$$

Moreover, $L^{p(\cdot)}(\Omega)$ is a reflexive space provided that $1 < p^- \le p^+ < +\infty$. Denoting by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$; for any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ we have the following Hölder type inequality

$$|\int_\Omega uv dx| \le 2|u|_{L^{p(\cdot)}(\Omega)} |v|_{L^{p'(\cdot)}(\Omega)}. \tag{2.1}$$

Now, we introduce the modular of the Lebesgue-Sobolev space $L^{p(\cdot)}(\Omega)$ as the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)} dx, \quad u \in L^{p(\cdot)}(\Omega).$$

Here, we give some relations which could be established between the Luxemburg norm and the modular. If $(u_n)_n, u \in L^{p(\cdot)}(\Omega)$ and $1 < p^- \le p^+ < +\infty$, then the following relations hold:

$$|u|_{L^{p(\cdot)}(\Omega)} > 1 \Rightarrow\ |u|^{p^-}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le |u|^{p^+}_{L^{p(\cdot)}(\Omega)}, \tag{2.2}$$

$$|u|_{L^{p(\cdot)}(\Omega)} < 1 \Rightarrow\ |u|^{p^+}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le |u|^{p^-}_{L^{p(\cdot)}(\Omega)}, \tag{2.3}$$

$$|u_n - u|_{L^{p(\cdot)}(\Omega)} \to 0 \Leftrightarrow \rho_{p(\cdot)}(u_n - u) \to 0. \tag{2.4}$$

Next, we define $W^{1,p(\cdot)}(\Omega)$ as the space

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\}$$

and it can be equipped with the norm $\|u\|_{1,p(\cdot)} = |u|_{L^{p(\cdot)}(\Omega)} + |\nabla u|_{L^{p(\cdot)}(\Omega)}$. The space $W^{1,p(\cdot)}(\Omega)$ is a Banach space which is reflexive under condition $1 < p^- \le p^+ < +\infty$.

Let $p, q \in C_+(\overline{\Omega})$. If we have $p(x) \le q(x) \le p^*(x)$ for all $x \in \overline{\Omega}$, where

$$(p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \ge N; \end{cases}$$

then there is a continuous embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$. This last embedding is compact provided that Ω is bounded in $\mathbb{R}^N$ and that $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

3. Proof of Theorem 1.1

Here, we notice that since $\alpha(\cdot)$ satisfies the conditions (H1) and (H2), it is easy to see that $E = W^{1,p(\cdot)}(\mathbb{R}^N)$. In this first part, we will equip E with the norm

$$\|u\| = \|u\|_{W^{1,p(\cdot)}(\Omega_1)} + \|u\|_{W^{1,p(\cdot)}(\Omega_2)}$$

which is clearly equivalent to the norm $\|\cdot\|_E$ or $\|\cdot\|_{W^{1,p(\cdot)}(\mathbb{R}^N)}$.

Let $J_\lambda : W^{1,p(\cdot)}(\mathbb{R}^N) \to \mathbb{R}$ be the energy functional given by

$$J_\lambda(u) = \int_{\mathbb{R}^N} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx - \lambda \int_{\mathbb{R}^N} \frac{\varphi(x)}{\alpha(x)} |u|^{\alpha(x)} dx - \int_{\mathbb{R}^N} hu dx.$$

Using inequality (2.1) and hypotheses (H1) and (H2), it is easy to see that the functional J_λ is well defined on $W^{1,p(\cdot)}(\mathbb{R}^N)$. Moreover, by classical arguments we have that $J_\lambda \in C^1(W^{1,p(\cdot)}(\mathbb{R}^N), \mathbb{R})$ and

$$\begin{aligned}\langle J'_\lambda(u), v\rangle &= \int_{\mathbb{R}^N} |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_{\mathbb{R}^N} |u|^{p(x)-2} uv dx \\ &\quad - \lambda \int_{\mathbb{R}^N} \varphi(x)|u|^{\alpha(x)-2} uv dx - \int_{\mathbb{R}^N} hv dx, \quad \forall u, v \in E.\end{aligned}$$

Hence, in order to obtain weak solutions of the problem (1.1) we will look for critical points of the functional J_λ. Now, we have to note that since $\operatorname{meas}(\Omega_2) \neq 0$, then one cannot show that the functional J_λ is coercive and consequently we cannot find a global minimum of the functional J_λ. The existence of a first critical point should be established using the Ekeland's variational principle.

Lemma 3.1. *Under the assumptions of Theorem 1.1, there exists $\lambda_* > 0$ such that for any $0 < \lambda < \lambda_*$, there exists $\gamma_\lambda > 0$ and $\eta_\lambda > 0$ such that*

$$J_\lambda(u) \geq \gamma_\lambda \text{ for } \|u\| = \frac{1}{2} \quad \text{provided that } \|h\|_{-1} < \eta_\lambda.$$

Proof. Let $u \in W^{1,p(\cdot)}(\mathbb{R}^N)$ be such that $\|u\| < 1$. By (2.1), (2.2) and (2.3) we have

$$\begin{aligned}\int_{\Omega_1} \frac{\varphi(x)}{\alpha(x)}|u|^{\alpha(x)} dx &\leq 2|\varphi(\cdot)|_{L^{\frac{p(\cdot)}{p(\cdot)-\alpha(\cdot)}}(\Omega_1)} ||u|^{\alpha(\cdot)}|_{L^{\frac{p(\cdot)}{\alpha(\cdot)}}(\Omega_1)} \\ &\leq c_1(|u|^{\alpha^+_{\Omega_1}}_{L^{p(\cdot)}(\Omega_1)} + |u|^{\alpha^-_{\Omega_1}}_{L^{p(\cdot)}(\Omega_1)}) \\ &\leq c_2\|u\|^{\alpha^-_{\Omega_1}}_{W^{1,p(\cdot)}(\Omega_1)},\end{aligned} \tag{3.1}$$

and

$$\int_{\Omega_2} \frac{\varphi(x)}{\alpha(x)}|u|^{\alpha(x)} dx \leq 2|\varphi(\cdot)|_{L^{r(\cdot)}(\Omega_2)} ||u|^{\alpha(\cdot)}|_{L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega_2)} \leq c_3\|u\|^{\alpha^-_{\Omega_2}}_{W^{1,p(\cdot)}(\Omega_2)}. \tag{3.2}$$

Using again (2.2) and (2.3), and taking (3.1) and (3.2) into account, we obtain

$$\begin{aligned}J_\lambda(u) &\geq \frac{1}{p^+}(\|u\|^{p^+_{\Omega_1}}_{W^{1,p(\cdot)}(\Omega_1)} + \|u\|^{p^+_{\Omega_2}}_{W^{1,p(\cdot)}(\Omega_2)}) \\ &\quad - \lambda c_2\|u\|^{\alpha^-_{\Omega_1}}_{W^{1,p(\cdot)}(\Omega_1)} - \lambda c_3\|u\|^{\alpha^-_{\Omega_2}}_{W^{1,p(\cdot)}(\Omega_2)} - \|h\|_{-1}\|u\| \\ &\geq \|u\|^{p^+_{\Omega_2}}_{W^{1,p(\cdot)}(\Omega_2)}(\frac{1}{p^+} - \lambda c_3\|u\|^{\alpha^-_{\Omega_2}-p^+_{\Omega_2}}_{W^{1,p(\cdot)}(\Omega_2)}) \\ &\quad + \frac{1}{p^+}\|u\|^{p^+_{\Omega_1}}_{W^{1,p(\cdot)}(\Omega_1)} - \lambda c_2\|u\|^{\alpha^-_{\Omega_1}}_{W^{1,p(\cdot)}(\Omega_1)} - \|h\|_{-1}\|u\|.\end{aligned} \tag{3.3}$$

For $\lambda \leq \frac{1}{2p^+c_3}$, we have

$$\frac{1}{p^+} - \lambda c_3\|u\|^{\alpha^-_{\Omega_2}-p^+_{\Omega_2}}_{W^{1,p(\cdot)}(\Omega_2)} \geq \frac{1}{p^+} - \lambda c_3 \geq \frac{1}{2p^+}.$$

Putting that inequality in (3.3), it yields

$$J_\lambda(u) \geq c_4\|u\|^{\sup(p^+_{\Omega_1},\, p^+_{\Omega_2})} - c_2\lambda\|u\|^{\alpha^-_{\Omega_1}} - \|h\|_{-1}\|u\|. \tag{3.4}$$

Set

$$\lambda_* = \inf(\frac{1}{2p^+c_3}, \frac{c_4}{c_2}(\frac{1}{2})^{\sup(p^+_{\Omega_1},\, p^+_{\Omega_2})-\alpha^-_{\Omega_1}}).$$

For $0 < \lambda < \lambda_*$, set

$$\gamma_\lambda = c_4(\frac{1}{2})^{\sup(p^+_{\Omega_1},\ p^+_{\Omega_2})} - c_2\lambda(\frac{1}{2})^{\alpha^-_{\Omega_1}} - \frac{\|h\|_{-1}}{2},$$
$$\eta_\lambda = 2(c_4(\frac{1}{2})^{\sup(p^+_{\Omega_1}, p^+_{\Omega_2})} - c_2\lambda(\frac{1}{2})^{\alpha^-_{\Omega_1}}).$$

The claimed result can be deduced from (3.4). □

Lemma 3.2. *Let $(u_n)_n \subset W^{1,p(\cdot)}(\mathbb{R}^N)$ be a bounded sequence such that $J'_\lambda(u_n) \to 0$. Then, $(u_n)_n$ is relatively compact.*

Proof. Let u be the weak limit of $(u_n)_n$ in $W^{1,p(\cdot)}(\mathbb{R}^N)$. We claim that, up to a subsequence, $(u_n)_n$ is strongly convergent to u in $W^{1,p(\cdot)}(\mathbb{R}^N)$. For $t > 0$, denote $B_t = \{x \in \mathbb{R}^N : |x| < t\}$. We have

$$\int_{\Omega_2 \setminus B_t} \varphi(x)|u_n - u|^{\alpha(x)} dx \le 2||u_n - u|^{\alpha(\cdot)}|_{L^{\frac{r(\cdot)}{r(\cdot)-1}}(\mathbb{R}^N)} |\varphi(\cdot)|_{L^{r(\cdot)}(\Omega_2 \setminus B_t)}. \tag{3.5}$$

Now, since $\varphi \in L^{r(\cdot)}(\Omega_2)$, it follows that $|\varphi(\cdot)|_{L^{r(\cdot)}(\Omega_2\setminus B_t)} \to 0$ as $t \to +\infty$. Taking into account that $(u_n)_n$ is bounded in $W^{1,p(\cdot)}(\mathbb{R}^N)$, it follows from (3.5) that for all $\epsilon > 0$ there exists $t_\epsilon > 0$ large enough such that

$$\int_{\Omega_2 \setminus B_{t_\epsilon}} \varphi(x)|u_n - u|^{\alpha(x)} dx < \frac{\epsilon}{2}. \tag{3.6}$$

On the other hand, we have

$$\int_{\Omega_2 \cap B_{t_\epsilon}} \varphi(x)|u_n - u|^{\alpha(x)} dx \le \sup_{x \in B_{t_\epsilon}} |\varphi(x)| \int_{\Omega_2 \cap B_{t_\epsilon}} |u_n - u|^{\alpha(x)} dx. \tag{3.7}$$

Since $\alpha(x) < \frac{\alpha(x)r(x)}{r(x)-1} \le p^*(x)$ for all $x \in \Omega_2$ and $(\Omega_2 \cap B_{t_\epsilon})$ is a bounded open set of Ω_2, we obtain

$$\lim_{n\to+\infty} \int_{\Omega_2 \cap B_{t_\epsilon}} |u_n - u|^{\alpha(x)} dx = 0.$$

Having in mind that φ is continuous, then $\sup_{x\in B_{t_\epsilon}} |\varphi(x)| < +\infty$ and consequently we deduce from (3.7) that

$$\lim_{n\to+\infty} \int_{\Omega_2 \cap B_{t_\epsilon}} \varphi(x)|u_n - u|^{\alpha(x)} dx = 0.$$

This implies that there exists $n_0(\epsilon) \ge 1$ such that for all $n \ge n_0(\epsilon)$, we have

$$\int_{\Omega_2 \cap B_{t_\epsilon}} \varphi(x)|u_n - u|^{\alpha(x)} dx < \frac{\epsilon}{2}. \tag{3.8}$$

Combining (3.6) and (3.8), it yields

$$\int_{\Omega_2} \varphi(x)|u_n - u|^{\alpha(x)} dx < \epsilon \quad \forall n \ge n_0(\epsilon).$$

Hence,

$$\lim_{n\to+\infty} \int_{\Omega_2} \varphi(x)|u_n - u|^{\alpha(x)} dx = 0. \tag{3.9}$$

Next, if we replace $r(\cdot)$ by $\frac{p(\cdot)}{p(\cdot)-\alpha(\cdot)}$ and $\frac{r(\cdot)}{r(\cdot)-1}$ by $p(\cdot)$, proceeding as previously (i.e. for the open set Ω_2), we can so easily infer

$$\lim_{n\to+\infty}\int_{\Omega_1}\varphi(x)|u_n-u|^{\alpha(x)}dx=0. \tag{3.10}$$

On the other hand, since $J'_\lambda(u_n)\to 0$, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)dx+\int_{\mathbb{R}^N}|u_n|^{p(x)-2}u_n(u_n-u)dx\\&-\int_{\mathbb{R}^N}\varphi(x)|u_n|^{\alpha(x)-2}u_n(u_n-u)dx-\int_{\mathbb{R}^N}h(u_n-u)dx\to 0,\end{aligned} \tag{3.11}$$

as $n\to+\infty$. Having in mind that $u_n\rightharpoonup u$ weakly in $W^{1,p(\cdot)}(\mathbb{R}^N)$, we deduce from (3.11), (3.10) and (3.9) that

$$\begin{aligned}0\le&\int_{\mathbb{R}^N}(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)\nabla(u_n-u)dx\\&+\int_{\mathbb{R}^N}(|u_n|^{p(x)-2}u_n-|u|^{p(x)-2}u)(u_n-u)dx\to 0,\quad\text{as } n\to+\infty.\end{aligned} \tag{3.12}$$

Observe now that (see [1, 8, 10]), we have the following relations satisfied for ξ and η in $\mathbb{R}^N$,

$$[(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)(\xi-\eta)]^{\frac{p}{2}}(|\xi|^p+|\eta|^p)^{\frac{2-p}{2}}\ge(p-1)|\xi-\eta|^p \tag{3.13}$$

for $1<p<2$ and

$$(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)(\xi-\eta)\ge 2^{-p}|\xi-\eta|^p,\quad p\ge 2. \tag{3.14}$$

Divide $\mathbb{R}^N$ into two parts:

$$D_1=\{x\in\mathbb{R}^N,\ p(x)<2\},\quad D_2=\{x\in\mathbb{R}^N,\ p(x)\ge 2\}.$$

By (3.12), (3.14) and (2.4), it yields

$$\lim_{n\to+\infty}\int_{D_2}(|\nabla u_n-\nabla u|^{p(x)}+|u_n-u|^{p(x)})dx=0. \tag{3.15}$$

On the other hand, by (3.13) we have

$$\begin{aligned}&\int_{D_1}|\nabla u_n-\nabla u|^{p(x)}dx\\&\le(\frac{1}{p^--1})\int_{D_1}(p(x)-1)|\nabla u_n-\nabla u|^{p(x)}dx\\&\le(\frac{1}{p^--1})\int_{D_1}((|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u))^{\frac{p(x)}{2}}\\&\quad\times(|\nabla u_n|^{p(x)}+|\nabla u|^{p(x)})^{\frac{2-p(x)}{2}}dx.\end{aligned}$$

Using (3.12) and (2.4) and having in mind that $(u_n)_n$ is bounded in E, we deduce

$$\int_{D_1}|\nabla u_n-\nabla u|^{p(x)}dx\to 0,\quad\text{as } n\to+\infty.$$

Similarly, we obtain

$$\int_{D_1}|u_n-u|^{p(x)}dx\to 0,\quad\text{as } n\to+\infty.$$

Thus,

$$\int_{D_1} (|\nabla u_n - \nabla u|^{p(x)} + |u_n - u|^{p(x)})dx \to 0, \quad \text{as } n \to +\infty. \tag{3.16}$$

From (3.15), (3.16) and (2.4), we conclude that $u_n \to u$ strongly in $W^{1,p(\cdot)}(\mathbb{R}^N)$. $\square$

Completion of the proof of Theorem 1.1. Let

$$m_\lambda = \inf\{J_\lambda(u),\ u \in W^{1,p(\cdot)}(\mathbb{R}^N) \text{ and } \|u\| \le \frac{1}{2}\}.$$

The set

$$\overline{B_{1/2}^{W^{1,p(\cdot)}(\mathbb{R}^N)}}(0) = \{u \in W^{1,p(\cdot)}(\mathbb{R}^N),\ \|u\| \le \frac{1}{2}\}$$

is a complete metric space with respect to the distance

$$\operatorname{dist}(u,v) = \|u - v\|,\ u,\ v \in W^{1,p(\cdot)}(\mathbb{R}^N).$$

The functional J_λ is lower semi-continuous and bounded from below in the set $\overline{B_{1/2}^{W^{1,p(\cdot)}(\mathbb{R}^N)}}(0)$. Note, that $\inf_{\|v\|<1/2} J_\lambda(v) \le J_\lambda(0) = 0$ and $\inf_{\|v\|=1/2} J_\lambda(v) \ge \gamma_\lambda > 0$ (provided that $\|h\|_{-1} < \eta_\lambda$). Let

$$0 < \epsilon < \inf_{\|v\|=1/2} J_\lambda(v) - \inf_{\|v\|<1/2} J_\lambda(v).$$

Applying Ekeland's variational principle (see [5]), we can find $u_\epsilon \in \overline{B_{1/2}^{W^{1,p(\cdot)}(\mathbb{R}^N)}}(0)$ such that

$$J_\lambda(u_\epsilon) < m_\lambda + \epsilon, \quad J_\lambda(u_\epsilon) < J_\lambda(w) + \epsilon\|w - u_\epsilon\|, \quad \forall w \ne u_\epsilon.$$

Since, $J_\lambda(u_\epsilon) \le m_\lambda + \epsilon \le \inf_{\|v\|<1/2} J_\lambda(v) + \epsilon < \inf_{\|v\|=1/2} J_\lambda(v)$, it follows that

$$u_\epsilon \in B_{1/2}^{W^{1,p(\cdot)}(\mathbb{R}^N)}(0) = \{u \in W^{1,p(\cdot)}(\mathbb{R}^N),\ \|u\| < \frac{1}{2}\}.$$

Define $I_\lambda^\epsilon : \overline{B_{1/2}^{W^{1,p(\cdot)}(\mathbb{R}^N)}}(0) \to \mathbb{R}$ by $I_\lambda^\epsilon(u) = J_\lambda(u) + \epsilon\|u - u_\epsilon\|$. Obviously, u_ϵ is a minimum of I_λ^ϵ. Then

$$\frac{I_\lambda^\epsilon(u_\epsilon + tv) - I_\lambda^\epsilon(u_\epsilon)}{|t|} \ge 0, \quad \forall 0 < |t| < 1 \text{ and } v \in B_{1/2}^{W^{1,p(\cdot)}(\mathbb{R}^N)}(0),$$

which implies

$$\frac{J_\lambda(u_\epsilon + tv) - J_\lambda(u_\epsilon)}{|t|} + \epsilon\|v\| \ge 0.$$

Let $t \to 0^+$, it follows that $\langle J'_\lambda(u_\epsilon), v\rangle + \epsilon\|v\| \ge 0$. Next, let $t \to 0^-$; it follows that $-\langle J'_\lambda(u_\epsilon), v\rangle + \epsilon\|v\| \ge 0$. Consequently, we obtain that $\|J'_\lambda(u_\epsilon)\| \le \epsilon$. Hence, there exists a sequence $(u_n)_n \subset B_{1/2}^{W^{1,p(\cdot)}(\mathbb{R}^N)}(0)$ such that

$$J_\lambda(u_n) \to m_\lambda, \quad J'_\lambda(u_n) \to 0.$$

Observing that $(u_n)_n$ is bounded in $W^{1,p(\cdot)}(\mathbb{R}^N)$ and using Lemma 3.2, we have that $(u_n)_n$ is strongly convergent to its weak limit denoted, for example, by $u_{0,\lambda} \in W^{1,p(\cdot)}(\mathbb{R}^N)$. Moreover, since $J_\lambda \in C^1(W^{1,p(\cdot)}(\mathbb{R}^N), \mathbb{R})$, it yields $J_\lambda(u_{0,\lambda}) = m_\lambda$ and $J'_\lambda(u_{0,\lambda}) = 0$. Hence, $u_{0,\lambda}$ is a weak solution of the problem (1.1). Now, we claim that $m_\lambda < 0$. We distinguish two cases.

* If (H3) holds. Let ψ be as in (H3). For $0 < t < 1$, we have

$$J_\lambda(t\psi) \le t^{\inf(p^-_{\Omega_1}, p^-_{\Omega_2})} \int_{\mathbb{R}^N} (|\nabla\psi|^{p(x)} + |\psi|^{p(x)})dx - t\int_{\mathbb{R}^N} h(x)\psi(x)dx.$$

Since $\inf(p^-_{\Omega_1}, p^-_{\Omega_2}) > 1$, we deduce that there exists $0 < t_0 < \inf(1, \frac{1}{2\|\psi\|})$ such that $J_\lambda(t_0\psi) < 0$. Taking into account that $t_0\psi \in \overline{B^{W^{1,p(\cdot)}(\mathbb{R}^N)}_{1/2}}(0)$, it follows that $m_\lambda < 0$.

* Assume that $h = 0$. Let $a_0 \in \Omega_1$ and $r_0 > 0$ small enough be such that $\overline{B_{r_0}(a_0)} \subset \Omega_1$ and $p_0 = \inf_{x\in\overline{B_{r_0}(a_0)}} p(x) > \alpha_0 = \sup_{x\in\overline{B_{r_0}(a_0)}} \alpha(x)$. Consider $\xi \in C_0^\infty(B_{r_0}(a_0)), \xi \neq 0$. For $0 < t < 1$, we have

$$
\begin{aligned}
J_\lambda(t\xi) &\leq t^{p_0}\int_{\Omega_1}\Big(|\nabla\xi|^{p(x)} + |\xi|^{p(x)}\Big)dx - \lambda t^{\alpha_0}\int_{\Omega_1}\frac{\varphi(x)}{\alpha(x)}|\xi|^{\alpha(x)}dx \\
&\leq c_8 t^{p_0} - c_9\lambda t^{\alpha_0} \\
&\leq t^{\alpha_0}(c_8 t^{p_0-\alpha_0} - c_9\lambda).
\end{aligned}
$$

Since, $p_0 - \alpha_0 > 0$, there exists $0 < t_1(\lambda) < \inf(1, \frac{1}{2\|\xi\|})$ such that $J_\lambda(t_1(\lambda)\xi) < 0$. Hence, $m_\lambda \leq J_\lambda(t_1(\lambda)\xi) < 0$. In this last case, by (3.1) and (3.2), we have

$$
\begin{aligned}
\int_{\mathbb{R}^N}(|\nabla u_{0,\lambda}|^{p(x)} + |u_{0,\lambda}|^{p(x)})dx &= \lambda\Big(\int_{\Omega_1}\varphi(x)|u_{0,\lambda}|^{\alpha(x)}dx + \int_{\Omega_2}\varphi(x)|u_{0,\lambda}|^{\alpha(x)}dx\Big) \\
&\leq \lambda\Big(c_{10}\|u_{0,\lambda}\|^{\alpha^-_{\Omega_1}}_{W^{1,p(\cdot)}(\Omega_1)} + c_{11}\|u_{0,\lambda}\|^{\alpha^-_{\Omega_2}}_{W^{1,p(\cdot)}(\Omega_2)}\Big) \\
&\leq \lambda\Big(c_{10}(\frac{1}{2})^{\alpha^-_{\Omega_1}} + c_{11}(\frac{1}{2})^{\alpha^-_{\Omega_2}}\Big).
\end{aligned}
$$

Using this inequality, it follows that $\lim_{\lambda\to 0}\|u_{0,\lambda}\| = 0$. This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Here, clearly $E \neq W^{1,p(\cdot)}(\mathbb{R}^N)$. Moreover, the arguments used in the proof of Theorem 1.1 are no longer valid. In fact, we cannot establish the existence of weak solution as a global neither a local minimum for the energy functional corresponding to the problem (1.1) and the Mountain-Pass is not useful as well. Hence, some new ideas have to be introduced and some new tools have to be employed. We shall adapt arguments used in [21].

Lemma 4.1. *There is $\lambda_{**} > 0$ such that if $0 < \lambda < \lambda_{**}$, then there exists a nonnegative and nontrivial function $\overline{U_\lambda} \in E \cap L^\infty(\mathbb{R}^N)$ satisfying*

$$
\int_{\mathbb{R}^N}|\nabla\overline{U_\lambda}|^{p(x)-2}\nabla\overline{U_\lambda}\nabla w\,dx + \int_{\mathbb{R}^N}(\overline{U_\lambda})^{p(x)-1}w\,dx \geq \lambda\int_{\mathbb{R}^N}\varphi(x)(\overline{U_\lambda})^{\alpha(x)-1}w\,dx,
$$

for every $w \in E$ with $w \geq 0$. ($\overline{U_\lambda}$ is called a weak super-solution of (1.1)).

Proof. For $\lambda > 0$, define $\overline{U_\lambda} : \mathbb{R}^N \to \mathbb{R}$ by

$$
\overline{U_\lambda}(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 2 - |x| & \text{if } 1 \leq |x| \leq 2 \\ 0 & \text{if } |x| > 2. \end{cases}
$$

For $1 \leq j \leq N$, we have

$$
\frac{\partial\overline{U_\lambda}}{\partial x_j}(x) = \begin{cases} 0 & \text{if } |x| < 1 \text{ or } |x| > 2 \\ -x_j/|x| & \text{if } 1 \leq |x| \leq 2, \end{cases}
$$

where $x=(x_1,\cdots,x_N)$. Thus,

$$|\nabla\overline{U_\lambda}(x)|=\begin{cases}0 & \text{if } |x|<1 \text{ or } |x|>2\\ 1 & \text{if } 2\le|x|\le 2.\end{cases}$$

Hence,

$$\begin{aligned}-\operatorname{div}(|\nabla\overline{U_\lambda}|^{p(x)-2}\nabla\overline{U_\lambda}) &= -\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\Big(|\nabla\overline{U_\lambda}|^{p(x)-2}\frac{\partial\overline{U_\lambda}}{\partial x_j}\Big)\\ &=\begin{cases}0 & \text{if } |x|<1 \text{ or } |x|>2\\ \frac{N-1}{|x|} & \text{if } 1\le|x|\le 2.\end{cases}\end{aligned}$$

Set

$$\lambda_{**}=\min\Big(\frac{1}{\max_{|x|<1}\varphi(x)},\frac{N-1}{\max_{1\le|x|\le2}(2^{\alpha(x)}\varphi(x))}\Big).$$

Then, for every $0<\lambda<\lambda_{**}$, we have

$$\begin{gathered}1\ge\lambda\varphi(x)\quad\text{if } |x|<1\\ \frac{N-1}{|x|}\ge\lambda\varphi(x)(2-|x|)^{\alpha(x)-1}\quad\text{if } 1\le|x|\le 2.\end{gathered}$$

Therefore,

$$-\operatorname{div}(|\nabla\overline{U_\lambda}|^{p(x)-2}\nabla\overline{U_\lambda})+(\overline{U_\lambda})^{p(x)-1}\ge\lambda\varphi(x)(\overline{U_\lambda})^{\alpha(x)-1}.$$

This completes the proof. □

Completion of the proof of Theorem 1.2. For $0<\lambda<\lambda_{**}$, set

$$f_\lambda(x,s)=\lambda\ \varphi(x)|s|^{\alpha(x)-2}s,\quad x\in\mathbb{R}^N,\ s\in\mathbb{R}.$$

Note that there exists $L_\lambda>0$ such that, for every $s\in[-1,1]$ and $x\in\overline{B(0,2)}=\{x\in\mathbb{R}^N,\ |x|\le 2\}$, we have

$$|\frac{\partial f_\lambda}{\partial s}(x,s)|\le L_\lambda.$$

Thus, $(x,s)\longmapsto f_\lambda(x,s)$ is $L_\lambda-$Lipschitz continuous with respect to $s\in[-1,1]$ uniformly for $x\in\overline{B(0,2)}$; i.e., we have

$$f_\lambda(x,s_1)-f_\lambda(x,s_2)\le L_\lambda(s_2-s_1),\tag{4.1}$$

for any $s_1,s_2\in[-1,1]$ with $s_1\le s_2$ and $x\in\overline{B(0,2)}$. Now, define

$$\tilde{f_\lambda}(x,s)=\begin{cases}-f(x,\overline{U_\lambda}(x))-L_\lambda\overline{U_\lambda}(x) & \text{if } s\le-\overline{U_\lambda}(x)\\ f_\lambda(x,s)+L_\lambda s & \text{if } -\overline{U_\lambda}(x)<s\le\overline{U_\lambda}(x)\\ f_\lambda(x,\overline{U_\lambda}(x))+L_\lambda\overline{U_\lambda}(x) & \text{if } s>\overline{U_\lambda}(x),\end{cases}$$

and $\tilde{F_\lambda}(x,s)=\int_0^s\tilde{f_\lambda}(x,t)dt$. If $s\le-\overline{U_\lambda}(x)$, we have

$$\tilde{F_\lambda}(x,s)\le(-s)(f_\lambda(x,\overline{U_\lambda}(x))+L_\lambda\overline{U_\lambda}(x)).$$

If $0\le s\le\overline{U_\lambda}(x)$, using (4.1) and the fact that $\|\overline{U_\lambda}\|_\infty=\sup_{x\in\mathbb{R}^N}|\overline{U_\lambda}(x)|=1$, we have

$$\tilde{F_\lambda}(x,s)\le(f_\lambda(x,s)+L_\lambda s)s\le(f_\lambda(x,\overline{U_\lambda}(x))+L_\lambda\overline{U_\lambda}(x))s.$$

If $-\overline{U_\lambda}(x)<s<0$, we have

$$\tilde{F_\lambda}(x,s)\le(f_\lambda(x,s)+L_\lambda s)s\le(f_\lambda(x,-\overline{U_\lambda}(x))-L_\lambda\overline{U_\lambda}(x))s$$

$$\le (f_\lambda(x,\overline{U_\lambda}(x))+L_\lambda\overline{U_\lambda}(x))(-s).$$

If $s>\overline{U_\lambda}(x)$, we have

$$\begin{aligned}\tilde{F_\lambda}(x,s)&=\int_0^{\overline{U_\lambda}(x)}(f_\lambda(x,t)+L_\lambda t)dt+\int_{\overline{U_\lambda}(x)}^s(f_\lambda(x,\overline{U_\lambda}(x))+L_\lambda\overline{U_\lambda}(x))dt\\&\le(f_\lambda(x,\overline{U_\lambda}(x))+L_\lambda\overline{U_\lambda}(x))\overline{U_\lambda}(x)+(f_\lambda(x,\overline{U_\lambda}(x))+L_\lambda\overline{U_\lambda}(x))(s-\overline{U_\lambda}(x))\\&\le(f_\lambda(x,\overline{U_\lambda}(x))+L_\lambda\overline{U_\lambda}(x))s.\end{aligned}$$

Therefore, for all $(x,s)\in\mathbb{R}^N\times\mathbb{R}$,

$$\tilde{F_\lambda}(x,s)\le(f_\lambda(x,\overline{U_\lambda}(x))+L_\lambda\overline{U_\lambda}(x))|s|. \tag{4.2}$$

Next, we introduce the functional space $X=W^{1,p(\cdot)}(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$ equipped with the norm

$$\|u\|_X=\|u\|_{W^{1,p(\cdot)}(\mathbb{R}^N)}+|u|_{L^2(\mathbb{R}^N)}.$$

For any $u\in X$, we define

$$\tilde{J_\lambda}(u)=\int_{\mathbb{R}^N}\frac{|\nabla u|^{p(x)}+|u|^{p(x)}}{p(x)}dx+\frac{L_\lambda}{2}\int_{\mathbb{R}^N}u^2dx-\int_{\mathbb{R}^N}\tilde{F_\lambda}(x,u)dx.$$

Set $\psi_\lambda(x)=(f_\lambda(x,\overline{U_\lambda}(x))+L_\lambda\overline{U_\lambda}(x))$. Clearly, $\psi_\lambda\in L^2(\mathbb{R}^N)$ and it becomes easy to verify that $\tilde{J_\lambda}\in C^1(X,\mathbb{R})$. By (4.2), for $\epsilon>0$, there exists $c_\epsilon>0$ such that

$$\tilde{J_\lambda}(u)\ge\int_{\mathbb{R}^N}\frac{|\nabla u|^{p(x)}+|u|^{p(x)}}{p(x)}dx+\frac{L_\lambda}{2}\int_{\mathbb{R}^N}u^2dx-\epsilon\int_{\mathbb{R}^N}u^2dx-c_\epsilon\int_{\mathbb{R}^N}(\psi_\lambda(x))^2dx.$$

Choosing $\epsilon>0$ such that $\frac{L_\lambda}{2}-\epsilon>0$, we infer that $\tilde{J_\lambda}$ is coercive. Let $(u_n)_n$ be a minimizing sequence of $\tilde{J_\lambda}$, i.e. $(u_n)_n\subset X$ and $\tilde{J_\lambda}(u_n)\to\inf_{v\in X}\tilde{J_\lambda}(v)>-\infty$. Since $\tilde{J_\lambda}$ is coercive, then $(u_n)_n$ is bounded and there exists $u\in E$ such that $u_n\rightharpoonup u$ weakly in X. By the mean value theorem, there exists some θ_n between 0 and 1 such that

$$\begin{aligned}|\int_{\mathbb{R}^N}(\tilde{F_\lambda}(x,u_n)-\tilde{F_\lambda}(x,u))dx|&=|\int_{\mathbb{R}^N}\tilde{f_\lambda}(x,\theta_n(u_n-u))(u_n-u)dx|\\&\le\int_{\mathbb{R}^N}\psi_\lambda(x)|u_n-u|dx.\end{aligned}\tag{4.3}$$

Let A be a measurable subset of $\mathbb{R}^N$. Using Hölder's inequality we have

$$\int_A\psi_\lambda(x)|u_n-u|dx\le2|\psi_\lambda(\cdot)|_{L^2(A)}|u_n-u|_{L^2(\mathbb{R}^N)}.$$

Since $(u_n-u)_n$ is bounded in $L^2(\mathbb{R}^N)$ and $\psi_\lambda\in L^2(\mathbb{R}^N)$, it follows that the integral $\int_A\psi_\lambda(x)|u_n-u|dx$ is small uniformly in n when the measure of A is small.
On the other hand, for $R>0$, we have

$$\int_{\mathbb{R}^N\setminus B_R}\psi_\lambda(x)|u_n-u|dx\le2|u_n-u|_{L^2(\mathbb{R}^N)}|\psi_\lambda(\cdot)|_{L^2(\mathbb{R}^N\setminus B_R)}.$$

Since $\psi_\lambda(\cdot)\in L^2(\mathbb{R}^N)$,

$$\lim_{R\to+\infty}|\psi_\lambda(\cdot)|_{L^2(\mathbb{R}^N\setminus B_R)}=0.$$

This fact together with the boundedness of the sequence $(|u_n - u|_{L^2(\mathbb{R}^N)})_n$ implies that for every $\epsilon > 0$, there exists $R_\epsilon > 0$ large enough such that

$$\int_{\mathbb{R}^N \setminus B_{R_\epsilon}} \psi_\lambda(x)|u_n - u|dx < \epsilon.$$

Therefore, we get the equi-integrability of the sequence $(\psi_\lambda(\cdot)|u_n - u|)_n$. By the virtue of Vitali's Theorem, we obtain

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \psi_\lambda(x)|u_n - u|dx = 0.$$

By (4.3), we deduce that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \tilde{F}_\lambda(u_n)dx = \int_{\mathbb{R}^N} \tilde{F}_\lambda(u)dx.$$

This implies

$$\inf_{v\in X} \tilde{J}_\lambda(v) \le \tilde{J}_\lambda(u) \le \liminf_{n\to+\infty} \tilde{J}_\lambda(u_n).$$

Consequently, $\tilde{J}_\lambda(u) = \inf_{v\in X} \tilde{J}_\lambda(v)$ and we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u|^{p(x)-2}\nabla u \nabla w\, dx + \int_{\mathbb{R}^N} |u|^{p(x)-2} uw\, dx + L_\lambda \int_{\mathbb{R}^N} uw\, dx \\ = \int_{\mathbb{R}^N} \tilde{f}_\lambda(x,u) w\, dx, \ \forall w \in X. \end{aligned} \tag{4.4}$$

Now take $w = (u - \overline{U_\lambda})^+ = \max(u - \overline{U_\lambda}, 0)$ in (4.4), and having in mind the definition of $\overline{U_\lambda}$, we get

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla \overline{U_\lambda}|^{p(x)-2}\nabla \overline{U_\lambda}\nabla(u - \overline{U_\lambda})^+ dx + \int_{\mathbb{R}^N} (\overline{U_\lambda})^{p(x)-1}(u - \overline{U_\lambda})^+ dx \\ &+ L_\lambda \int_{\mathbb{R}^N} \overline{U_\lambda}(u - \overline{U_\lambda})^+ dx \\ &\ge \int_{\mathbb{R}^N} (f_\lambda(x, \overline{U_\lambda}) + L_\lambda \overline{U_\lambda})(u - \overline{U_\lambda})^+ dx \\ &\ge \int_{\mathbb{R}^N} \tilde{f}_\lambda(x,u)(u - \overline{U_\lambda})^+ dx \\ &\ge \int_{\mathbb{R}^N} |\nabla u|^{p(x)-2}\nabla u \nabla(u - \overline{U_\lambda})^+ dx + \int_{\mathbb{R}^N} |u|^{p(x)-2}u(u - \overline{U_\lambda})^+ dx \\ &+ L_\lambda \int_{\mathbb{R}^N} u(u - \overline{U_\lambda})^+ dx. \end{aligned}$$

Thus,

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla u|^{p(x)-2}\nabla u - |\nabla \overline{U_\lambda}|^{p(x)-2}\nabla \overline{U_\lambda})\nabla(u - \overline{U_\lambda})^+ dx \\ &+ \int_{\mathbb{R}^N} (|u|^{p(x)-2}u - |\overline{U_\lambda}|^{p(x)-2}\overline{U_\lambda})(u - \overline{U_\lambda})^+ dx \\ &+ L_\lambda \int_{\mathbb{R}^N} ((u - \overline{U_\lambda})^+)^2 dx \le 0. \end{aligned}$$

Taking into account that the terms

$$\int_{\mathbb{R}^N} (|\nabla u|^{p(x)-2}\nabla u - |\nabla \overline{U_\lambda}|^{p(x)-2}\nabla \overline{U_\lambda})\nabla(u - \overline{U_\lambda})^+ dx$$

and

$$\int_{\mathbb{R}^N}(|u|^{p(x)-2}u-|\overline{U_\lambda}|^{p(x)-2}\overline{U_\lambda})(u-\overline{U_\lambda})^+dx$$

are nonnegative, then $u\le \overline{U_\lambda}$ a.e. in $\mathbb{R}^N$. On the other hand, define $-\overline{U_\lambda}=\overline{V_\lambda}$, and take $w=(\overline{V_\lambda}-u)^+=\max(\overline{V_\lambda}-u,0)$ in (4.4), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}|\nabla\overline{V_\lambda}|^{p(x)-2}\nabla\overline{V_\lambda}\nabla(\overline{V_\lambda}-u)^+dx+\int_{\mathbb{R}^N}|\overline{V_\lambda}|^{p(x)-2}\overline{V_\lambda}(\overline{V_\lambda}-u)^+dx\\
&+L_\lambda\int_{\mathbb{R}^N}\overline{V_\lambda}(\overline{V_\lambda}-u)^+dx\\
&\le\int_{\mathbb{R}^N}(f_\lambda(x,\overline{V_\lambda})+L_\lambda\overline{V_\lambda})(\overline{V_\lambda}-u)^+dx\\
&\le\int_{\mathbb{R}^N}\tilde f_\lambda(x,u)(\overline{V_\lambda}-u)^+dx\\
&\le\int_{\mathbb{R}^N}|\nabla u|^{p(x)-2}\nabla u\nabla(\overline{V_\lambda}-u)^+dx+\int_{\mathbb{R}^N}|u|^{p(x)-2}u(\overline{V_\lambda}-u)^+dx\\
&\quad+L_\lambda\int_{\mathbb{R}^N}u(\overline{V_\lambda}-u)^+dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^N}(|\nabla\overline{V_\lambda}|^{p(x)-2}\nabla\overline{V_\lambda}-|\nabla u|^{p(x)-2}\nabla u)\nabla(\overline{V_\lambda}-u)^+dx\\
&+\int_{\mathbb{R}^N}(|\overline{V_\lambda}|^{p(x)-2}\overline{V_\lambda}-|u|^{p(x)-2}u)(\overline{V_\lambda}-u)^+dx\\
&+L_\lambda\int_{\mathbb{R}^N}((\overline{V_\lambda}-u)^+)^2dx\le 0.
\end{aligned}$$

Hence, $(\overline{V_\lambda}-u)^+=0$, which implies $-\overline{U_\lambda}\le u$ a.e. in $\mathbb{R}^N$. Therefore, $\tilde f_\lambda(x,u)=f_\lambda(x,u)+L_\lambda u$ and by (4.4), for all $w\in X$ we have

$$\int_{\mathbb{R}^N}|\nabla u|^{p(x)-2}\nabla u\nabla w\,dx+\int_{\mathbb{R}^N}|u|^{p(x)-2}uw\,dx=\int_{\mathbb{R}^N}f_\lambda(x,u)w\,dx.$$

Now, without loss of generality, we could assume that $0\in\Omega_1$. Taking into account that Ω_1 is an open set, one can find $0<r<1$ small enough such that $\overline{B_r(0)}\subset\Omega_1$ and $p_1=\inf_{x\in\overline{B_r(0)}}p(x)>\alpha_1=\sup_{x\in\overline{B_r(0)}}\alpha(x)$. Let $\vartheta\in C_0^\infty(B_r(0))$ be such that $\vartheta\ne0$ and $\vartheta\ge0$. Take $0<t<1$ such that $t\vartheta(x)\le1$, for all $x\in B_r(0)$. We have $\tilde F_\lambda(x,t\vartheta(x))=\int_0^{t\vartheta(x)}\tilde f_\lambda(x,s)ds$. For $x\notin B_r(0)$, $\tilde F_\lambda(x,t\vartheta(x))=0$. For $x\in B_r(0)$, $0\le t\vartheta(x)\le\overline{U_\lambda}(x)$ and $\tilde F_\lambda(x,t\vartheta(x))=\lambda\frac{\varphi(x)}{\alpha(x)}t^{\alpha(x)}|\vartheta(x)|^{\alpha(x)}+\frac{L_\lambda}{2}t^2(\vartheta(x))^2$. Thus, we have

$$\begin{aligned}
\tilde J_\lambda(t\vartheta)&\le t^{p_1}\int_{B_r(0)}(|\nabla\vartheta|^{p(x)}+|\vartheta|^{p(x)})dx-\lambda t^{\alpha_1}\int_{B_r(0)}\frac{\varphi(x)}{\alpha(x)}|\vartheta|^{\alpha(x)}dx\\
&\le t^{\alpha_1}(c_{12}t^{p_1-\alpha_1}-\lambda c_{13}).
\end{aligned}$$

Since $p_1-\alpha_1>0$, then there exists $0<t(\lambda)<1$ small enough such that $\tilde J_\lambda(t(\lambda)\vartheta)<0$. Therefore, $\tilde J_\lambda(u)=\inf_{v\in X}\tilde J_\lambda(v)<0$ and $u\ne0$. Now, note that u satisfies

$$\int_{\mathbb{R}^N}|\nabla u|^{p(x)-2}\nabla u\nabla w\,dx+\int_{\mathbb{R}^N}|u|^{p(x)-2}uw\,dx=\int_{\mathbb{R}^N}f_\lambda(x,u)w\,dx,$$

for all $w \in C_0^\infty(\mathbb{R}^N)$. On the other hand, since $|u| \le \overline{U_\lambda}$, then $u \in E$. Having in mind that $p(\cdot)$ satisfies the logarithmic Hölder inequality, we could immediately deduce that $C_0^\infty(\mathbb{R}^N)$ is dense in E and we infer

$$\int_{\mathbb{R}^N} |\nabla u|^{p(x)-2}\nabla u \nabla w\,dx + \int_{\mathbb{R}^N} |u|^{p(x)-2}uw\,dx = \lambda \int_{\mathbb{R}^N} \varphi(x)|u|^{p(x)-2}uw\,dx,$$

for all $w \in E$. This competes the proof of Theorem 1.2.

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Corrigendum posted on September 12, 2013

The author would like to make the following corrections to the proof of Theorem 1.2. The choice of the function

$$\overline{U_\lambda}(x)=\begin{cases}1 & \text{if } |x|<1\\ 2-|x| & \text{if } 1\le |x|\le 2\\ 0 & \text{if } |x|>2\end{cases}$$

as a super-solution of the problem (1.1) is not appropriate since the identity

$$-\operatorname{div}\left(|\nabla\overline{U_\lambda}|^{p(x)-2}\nabla\overline{U_\lambda}\right)=\begin{cases}0 & \text{if } |x|<1 \text{ or } |x|>2\\ \frac{N-1}{|x|} & \text{if } 1\le |x|\le 2\end{cases}$$

is wrong. Some Dirac measures appear when computing $-\operatorname{div}\left(|\nabla\overline{U_\lambda}|^{p(x)-2}\nabla\overline{U_\lambda}\right)$, in the sense of distributions. Thus, we have to change the choice of this function. For this purpose, we add the following assumption to Theorem 1.2,

(H6) There exists a nonnegative and nontrivial function e in the space $L^\infty(\mathbb{R}^N)\cap W^{-1,p'(\cdot)}(\mathbb{R}^N)$ (where $W^{-1,p'(\cdot)}(\mathbb{R}^N)$ is the dual space of $W^{1,p(\cdot)}(\mathbb{R}^N)$) such that
$$e(x)\ge\varphi(x),\quad \forall x\in\mathbb{R}^N.$$

Concerning the construction of a super-solution of problem (1.1), we note that the problem

$$-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)+|u|^{p(x)-2}u=e$$

has a nontrivial and nonnegative weak solution $U_e\in W^{1,p(\cdot)}(\mathbb{R}^N)$; i.e., U_e satisfies

$$\int_{\mathbb{R}^N}|\nabla U_e|^{p(x)-2}\nabla U_e\nabla w dx+\int_{\mathbb{R}^N}(U_e)^{p(x)-1}\,w dx=\int_{\mathbb{R}^N}e(x)w(x)dx,$$

for all $w\in W^{1,p(\cdot)}(\mathbb{R}^N)$. Moreover, it is easy to see that $U_e\in L^\infty(\mathbb{R}^N)$ and that $U_e\in E$. Let

$$\lambda_{**}=\frac{1}{\|U_e\|_\infty^{\alpha^+-1}+\|U_e\|_\infty^{\alpha^--1}}.$$

If $0<\lambda<\lambda_{**}$, we have $e(x)\ge\varphi(x)\ge\lambda\varphi(x)\,(U_e)^{\alpha(x)-1}$. By the definition of U_e, it follows immediately that U_e is a super-solution of the problem (1.1) provided that $h=0$ and $0<\lambda<\lambda_{**}$. Therefore, in the proof of Theorem 1.2 we can take $\overline{U_\lambda}=U_e$, for all $0<\lambda<\lambda_{**}$. Consequently, we can easily find a constant L_λ such that $f_\lambda(x,s)$ is L_λ-Lipschitz continuous with respect to $s\in[-\|U_e\|_\infty,\|U_e\|_\infty]$ uniformly for $x\in\mathbb{R}^N$.

End of corrigendum.

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