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PICONE'S IDENTITY FOR A SYSTEM OF FIRST-ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We established a Picone identity for systems of nonlinear partial differential equations of first-order. With the help of this formula, we obtain qualitative results such as an integral inequality of Wirtinger type and the existence of zeros for the first components of solutions in a given bounded domain.

1. INTRODUCTION

The purpose of this article is to establish a Picone-type identity for the nonlinear differential system

$$\begin{aligned} \nabla u &= uA(x) + B(x)\|v\|^{q-2}v, \\ \operatorname{div} v &= -C(x)|u|^{p-2}u - D(x)\cdot v, \end{aligned} \tag{1.1}$$

where $p > 1$ is a constant, $q = p/(p-1)$ is its conjugate, $A(x), D(x) \in C(\overline{\Omega};\mathbb{R}^n)$, $C(x) \in C(\overline{\Omega},\mathbb{R})$, $B(x) = \operatorname{diag}\{B_1(x),\dots,B_n(x)\}$ is a diagonal matrix with the positive entries defined and continuous in a bounded domain $\Omega \subset \mathbb{R}^n$ with a piecewise smooth boundary $\partial\Omega$ and u and v denote real- and vector-valued functions of $x = (x_1,\dots,x_n)$, respectively, which are continuously differentiable in their domains of definition. Here div and ∇ are the usual divergence and nabla operators, $\|\cdot\|$ is the Euclidean length of a vector in $\mathbb{R}^n$ and the dot is used to denote the scalar product of two vectors in $\mathbb{R}^n$.

If the special case $A(x) \equiv 0$ in $\overline{\Omega}$, the system (1.1) is equivalent with the second-order half-linear partial differential equation

$$\operatorname{div}\big(P(x)\|\nabla u\|^{p-2}\nabla u\big) + R(x)\cdot\|\nabla u\|^{p-2}\nabla u + Q(x)|u|^{p-2}u = 0, \tag{1.2}$$

where

$$P(x) = B(x)^{1-p}, \quad R(x) = B(x)^{1-p}D(x), \quad Q(x) = C(x).$$

If the coefficient $P(x)$ is a scalar function, then (1.2) reduces to the equation studied in [9] where the following theorem was proved.

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Theorem 1.1. *Suppose that there exists a nontrivial function $y \in C^1(\overline{\Omega};\mathbb{R})$ such that $y=0$ on $\partial\Omega$ and*

$$M_\Omega[y] \equiv \int_\Omega \big[P(x)\big\|\nabla y - \frac{1}{p}\frac{R(x)}{P(x)}y\big\|^p - Q(x)|y|^p\big]dx \le 0. \tag{1.3}$$

Then every solution u of (1.2) must have a zero in $\overline{\Omega}$.

The proof of the above theorem was based on an identity which says that if u is a solution of (1.2) satisfying $u(x) \neq 0$ in $\overline{\Omega}$ and $y \in C^1(\overline{\Omega};\mathbb{R})$ is not identically zero in Ω, then

$$\begin{aligned} &\operatorname{div}\Big[|y|^p P(x)\frac{\|\nabla u\|^{p-2}}{|u|^{p-2}u}\nabla u\Big] \\ &= P(x)\big\|\nabla y - \frac{y}{pP(x)}R(x)\big\|^p - Q(x)|y|^p - P(x)\Big\{\big\|\nabla y - \frac{y}{pP(x)}R(x)\big\|^p \\ &\quad - p\big(\nabla y - \frac{y}{pP(x)}R(x)\big)\cdot\big\|\frac{y}{u}\nabla u\big\|^{p-2}\frac{y}{u}\nabla u + (p-1)\big\|\frac{y}{u}\nabla u\big\|^p\Big\}. \end{aligned} \tag{1.4}$$

Moreover, if $D(x) \equiv 0$ in Ω, then (1.2) reduces to

$$\operatorname{div}\big[P(x)\|\nabla u\|^{p-2}\nabla u\big] + Q(x)|u|^{p-2}u = 0. \tag{1.5}$$

Identities of Picone type for (1.5) (or its special case where $P(x) \equiv 1$ in $\overline{\Omega}$) were established by several authors including Allegretto [1], Dunninger [3], Kusano et al [6] and Yoshida [10] who obtained a variety of qualitative results based on these formulas. For an extension of Picone's identity to the case of pseudo-p-Laplacian and anisotropic p-Laplacian see Došlý [2] and Fišnarová et al [4], respectively. As was demonstrated in Mařík [7], an alternative approach to (1.2) and (1.5) can be based upon Riccati-type equations and inequalities.

While comparison and oscillation theory for equations of the type (1.2) and (1.5) is well-developed, there appears to be little known for general systems such as (1.1), particularly in the case where $A(x) \neq 0$ or $A(x) \neq D(x)$ in Ω (for some results concerning the case $p=2$ see Wong [11]).

The purpose of this article is to generalize Picone's identity for nonlinear partial differential systems of the form (1.1) and illustrate its applications by deriving Wirtinger-type inequalities formulated in terms of solutions of the system (1.1) and obtaining results about the existence and distribution of zeros of the first component of the solution of (1.1). Our results involve an arbitrary continuous vector-valued function $G(x)$ and particular choices of this function lead to different integral inequalities or criteria for the existence of zeros of first components of solutions of (1.1). They are new even when they are specialized to the case of the damped equation (1.2).

This article is organized as follows. In Section 2, the desired generalization of Picone's formula to nonlinear system (1.1) is derived and some particular cases of this new identity are discussed. Section 3 contains some applications of the basic formula which include the integral inequalities of the Wirtinger type and theorems about the existence of zeros for components of solutions of system (1.1).

2. Picone's identity

Define $\varphi_p(s) := |s|^{p-2}s,\ s \in \mathbb{R}$, and $\Phi_p(\xi) := \|\xi\|^{p-2}\xi,\ \xi \in \mathbb{R}^n$. Let $\xi,\eta \in \mathbb{R}^n$ and B be a diagonal matrix with positive entries $B_i,\ i=1,\dots,n$. Define the form

F_B by

$$F_B[\xi,\eta] = \xi \cdot B^{1-p}\Phi_p(\xi) - p\xi \cdot B^{1-p}\Phi_p(\eta) + (p-1)\eta \cdot B^{1-p}\Phi_p(\eta). \tag{2.1}$$

where $B^{1-p} = \mathrm{diag}\{B_1^{1-p}, \dots, B_n^{1-p}\}$. The next lemma establishes the generalization of Picone's identity for the nonlinear system (1.1).

Lemma 2.1. *Let (u,v) be a solution of (1.1) with $u(x) \neq 0$ in $\overline{\Omega}$. Then, for any $y \in C^1(\overline{\Omega};\mathbb{R})$ and $G \in C(\overline{\Omega},\mathbb{R}^n)$,*

$$\begin{aligned} \operatorname{div}\Big[|y|^p \frac{v}{\varphi_p(u)}\Big] &= \big[\nabla y - yG(x)\big] \cdot B(x)^{1-p}\Phi_p(\nabla y - yG(x)) - C(x)|y|^p \\ &\quad - \big[p\big(A(x) - G(x)\big) + D(x) - A(x)\big] \cdot \frac{|y|^p}{\varphi_p(u)} v \\ &\quad - F_B[\nabla y - yG(x), B(x)y\Phi_q(v)/u]. \end{aligned} \tag{2.2}$$

Proof. If (u,v) is a solution of (1.1) with $u(x) \neq 0$ and $y \in C^1(\overline{\Omega},\mathbb{R})$, then a direct computation yields

$$\operatorname{div}\big[|y|^p \frac{v}{\varphi_p(u)}\big] = p\frac{\varphi_p(y)}{\varphi_p(u)}\nabla y \cdot v - (p-1)\frac{|y|^p}{|u|^p}\nabla u.v + \frac{|y|^p}{\varphi_p(u)} \operatorname{div} v\,. \tag{2.3}$$

Using (1.1), adding and subtracting the terms $[\nabla y - yG(x)] \cdot B(x)^{1-p}\Phi_p(\nabla y - yG(x))$ and $pyG(x).B(x)^{1-p}\Phi_p(B(x)\frac{y}{u}\Phi_q(v))$ $\big(=pyG(x) \cdot \frac{\varphi_p(y)}{\varphi_p(u)} v\big)$ on the right-hand side of (2.3), we obtain

$$\begin{aligned} \operatorname{div}\Big[|y|^p \frac{v}{\varphi_p(u)}\Big] &= \big[\nabla y - yG(x)\big] \cdot B(x)^{1-p}\Phi_p(\nabla y - yG(x)) \\ &\quad - C(x)|y|^p - \big[p\big(A(x) - G(x)\big) + D(x) - A(x)\big] \cdot \frac{|y|^p}{\varphi_p(u)} v \\ &\quad - \big\{\big[\nabla y - yG(x)\big] \cdot B(x)^{1-p}\Phi_p\big(\nabla y - yG(x)\big) \\ &\quad - p\big[\nabla y - yG(x)\big] \cdot B(x)^{1-p}\Phi_p\big(B(x)\frac{y}{u}\Phi_q(v)\big) \\ &\quad + (p-1)B(x)\frac{y}{u}\Phi_q(v) \cdot B(x)^{1-p}\Phi_p\big(B(x)\frac{y}{u}\Phi_q(v)\big)\big\}, \end{aligned}$$

which is the desired identity (2.2). ∎

Remark 2.2. If we put $y(x) \equiv 1$ in (2.2) and denote $w = v/\varphi_p(u)$, then (2.2) becomes the generalized Riccati equation

$$\begin{aligned} &\operatorname{div} w + \big[pG(x) + (p-1)B(x)\Phi_q(w)\big] \cdot B(x)^{1-p}\Phi_p\big(B(x)\Phi_q(w)\big) \\ &+ \big[p\big(A(x) - G(x)\big) + D(x) - A(x)\big] \cdot w + C(x) = 0\,. \end{aligned} \tag{2.4}$$

Moreover, if $G(x) \equiv 0$ and $B(x)$ is a scalar function, then the Riccati-type equation (2.4) reduces to

$$\operatorname{div} w + (p-1)B(x)\|w\|^q + \big[(p-1)A(x) + D(x)\big] \cdot w + C(x) = 0. \tag{2.5}$$

In the particular case where $A(x) \equiv 0$ and $B(x) \equiv 1$ in $\overline{\Omega}$, Equation (2.5) has been employed by Mařík [8] as a tool for studying oscillatory properties of damped half-linear PDEs of the form (1.2).

Remark 2.3. If $G(x) \equiv 0$ in $\overline{\Omega}$, then (2.2) simplifies to

$$\begin{aligned} \operatorname{div}\Big[|y|^p \frac{v}{\varphi_p(u)}\Big] &= \nabla y \cdot B(x)^{1-p}\Phi_p(\nabla y) - C(x)|y|^p \\ &\quad - \big[(p-1)A(x) + D(x)\big] \cdot \frac{|y|^p}{\varphi_p(u)} v - F_B\big[\nabla y, B(x)\frac{y}{u}\Phi_q(v)\big]. \end{aligned} \tag{2.6}$$

In the particular case $p = 2$, the identity (2.6) reduces to the formula used (implicitly) by Wong [11] in establishing an integral inequality of the Wirtinger type and comparison theorems based on this inequality for the linear system

$$\nabla u = uA(x) + B(x)v, \quad \operatorname{div} v = -C(x)u - D(x) \cdot v, \tag{2.7}$$

and its Sturmian minorant

$$\nabla y = ya(x) + b(x)z, \quad \operatorname{div} z = -c(x)y - d(x) \cdot z, \tag{2.8}$$

where the coefficient functions satisfy the same assumptions as above with the only difference that because of the linearity of the problem the matrices $b(x)$ and $B(x)$ are not necessarily diagonal, but are allowed to be any continuous symmetric and positive definite matrices.

The choice $G(x) = (1/q)A(x) + (1/p)D(x)$ in (2.2) yields

$$\begin{aligned} &\operatorname{div}\big[|y|^p \frac{v}{\varphi_p(u)}\big] \\ &= \Big[\nabla y - y\Big(\frac{A(x)}{q} + \frac{D(x)}{p}\Big)\Big] \cdot B(x)^{1-p}\Phi_p\Big(\nabla y - y\Big(\frac{A(x)}{q} + \frac{D(x)}{p}\Big)\Big) \\ &- C(x)|y|^p - F_B\Big[\nabla y - y\Big(\frac{A(x)}{q} + \frac{D(x)}{p}\Big), B(x)\frac{y}{u}\Phi_q(v)\Big]. \end{aligned} \tag{2.9}$$

Under the further restriction $A(x) \equiv 0$ and $B_1(x) = \dots = B_n(x) =: B(x)$ in $\overline{\Omega}$, the identity (2.9) reduces to the following Yoshida's formula for partial differential equations with p-gradient terms (see[9, Theorem 8.3.1]):

$$\begin{aligned} &\operatorname{div}\big[|y|^p \frac{v}{\varphi_p(u)}\big] \\ &= B(x)^{1-p}\big\|\nabla y - \frac{y}{p}D(x)\big\|^p - C(x)|y|^p - F_B\big[\nabla y - \frac{y}{p}D(x), B(x)\frac{y}{u}\Phi_q(v)\big] \end{aligned} \tag{2.10}$$

which was used in proving Theorem 1.1.

3. Applications

In what follows, for simplicity we restrict our considerations to the "isotropic" case where $B_1(x) = \dots = B_n(x) =: B(x)$. In this special case it follows from [6, Lemma 2.1] that the form $F_B[\xi, \eta]$ defined by (2.1) is positive semi-definite and the equality in $F_B[\xi, \eta] \ge 0$ occurs if and only if $\xi = \eta$.

As the first application of the identity (2.2) we establish an inequality of the Wirtinger type.

Theorem 3.1. *If there exists a solution (u, v) of (1.1) such that $u(x) \ne 0$ in $\overline{\Omega}$ and*

$$\Big[p\big(A(x) - G(x)\big) + D(x) - A(x)\Big] \cdot \frac{v}{\varphi_p(u)} \ge 0 \tag{3.1}$$

in $\overline{\Omega}$, then the inequality

$$J_\Omega[y] := \int_\Omega \big[B(x)^{1-p}\big\|\nabla y - yG(x)\big\|^p - C(x)|y|^p\big]\,dx \geq 0 \tag{3.2}$$

holds for any nontrivial function $y \in C^1(\overline{\Omega};\mathbb{R})$ such that $y = 0$ on $\partial\Omega$. Moreover, if $\big[p(A-G)+D-A\big]\cdot v/\varphi_p(u) \equiv 0$ in $\overline{\Omega}$, then equality in (3.2) occurs if and only if $y(x)$ is a solution of

$$\nabla y = \Big[G(x) + B(x)\frac{\Phi_q(v)}{u}\Big]y. \tag{3.3}$$

Proof. Assume that (1.1) has a solution (u,v) with $u(x) \neq 0$ in $\overline{\Omega}$ which satisfies (3.1). Let $y(x)$ be a nontrivial continuously differentiable real-valued function such that $y = 0$ on $\partial\Omega$. Integrating (2.2) on Ω and using the divergence theorem we get

$$\begin{aligned} 0 = J_\Omega[y] &- \int_\Omega \big[p\big(A(x)-G(x)\big) + D(x) - A(x)\big]\cdot\frac{|y|^p}{\varphi_p(u)}v\,dx \\ &- \int_\Omega F_B[\nabla y - yG(x), B(x)y\Phi_q(v)/u]dx. \end{aligned}$$

Since the form F_B is positive semi-definite and the condition (3.1) holds, we conclude that

$$0 \leq J_\Omega[y]$$

as claimed. Clearly, if $\big[p(A-G)+D-A\big]v/\varphi_p(u) \equiv 0$ in $\overline{\Omega}$, then the equality holds in (3.2) if and only if $F_B[\nabla y - yG(x), B(x)y\Phi_q(v)/u] \equiv 0$ in $\overline{\Omega}$ which is equivalent with the condition (3.3). □

As an immediate consequence of the above theorem we have the following result.

Corollary 3.2. *Let (u,v) be a solution of (1.1) such that $u(x) \neq 0$ in $\overline{\Omega}$ and*

$$\Big[p\big(A(x)-G(x)\big) + D(x) - A(x)\Big]\cdot\frac{v}{\varphi_p(u)} \equiv 0 \tag{3.4}$$

in $\overline{\Omega}$. Then, for every nontrivial $y \in C^1(\overline{\Omega};\mathbb{R})$ such that $y = 0$ on $\partial\Omega$, the inequality (3.2) is valid. Moreover, the equality holds in (3.2) if and only if

$$\nabla\Big(\frac{y}{u}\Big) = \frac{y}{u}\big(G(x) - A(x)\big) \tag{3.5}$$

in Ω.

Proof. We need to show only that (3.5) is equivalent to (3.3). Using the first equation in (1.1), it is easily seen that

$$\begin{aligned} \nabla y - \big[G(x) + B(x)\frac{y}{u}\Phi_q(v)\big]y &= \nabla y - \frac{y}{u}\nabla u + y\big[A(x) - G(x)\big] \\ &= u\nabla\Big(\frac{y}{u}\Big) + y\big[A(x) - G(x)\big] \\ &= u\Big[\nabla\Big(\frac{y}{u}\Big) - \frac{y}{u}\big(G(x) - A(x)\big)\Big], \end{aligned}$$

from which the assertion follows. □

In the case where $G(x) \equiv A(x) \equiv D(x)$ in $\overline{\Omega}$, condition (3.4) is trivially satisfied and inequality (3.2) reduces to

$$\int_\Omega \big[B(x)^{1-p}\big\|\nabla y - yA(x)\big\|^p - C(x)|y|^p\big]dx \geq 0.$$

Clearly, in this special case the equality in (3.2) occurs if and only if $y(x)$ is a constant multiple of $u(x)$.

Another choice of $G(x)$ which guarantees the satisfaction of (3.4) is

$$G(x) = \frac{(p-1)A(x) + D(x)}{p}.$$

The last result specializes as follows.

Corollary 3.3. *If (u, v) is a solution of (1.1) with $u(x) \neq 0$ in $\overline{\Omega}$ and a nontrivial $y \in C^1(\overline{\Omega}; \mathbb{R})$ is such that $y = 0$ on $\partial\Omega$, then*

$$J_\Omega[y] = \int_\Omega \Big[B(x)^{1-p} \big\|\nabla y - y\frac{(p-1)A(x) + D(x)}{p}\big\|^p - C(x)|y|^p\Big] dx \geq 0. \tag{3.6}$$

Furthermore, equality in (3.6) occurs if and only if

$$y(x) = Ku(x)\exp\{f(x)\} \quad \text{on } \overline{\Omega} \tag{3.7}$$

for some constant $K \neq 0$ and some continuous function $f(x)$.

Proof. It suffices to prove (3.7). If (3.5) holds, then from [5, Lemma 2.3] if follows that there exists a continuous function $f(x)$ such that $y(x)$ is proportional to $u(x)\exp\{f(x)\}$. The proof is complete. □

The above result can be reformulated as the following theorem which generalizes [9, Theorem 8.3.2].

Corollary 3.4. *If for some nontrivial C^1-function $y(x)$ defined on $\overline{\Omega}$ and satisfying $y = 0$ on $\partial\Omega$, the condition*

$$J_\Omega[y] = \int_\Omega \Big[B(x)^{1-p} \big\|\nabla y - y\frac{(p-1)A(x) + D(x)}{p}\big\|^p - C(x)|y|^p\Big] dx \leq 0 \tag{3.8}$$

holds, then for any solution (u, v) of (1.1) the first component $u(x)$ must have a zero in $\overline{\Omega}$.

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