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EXISTENCE OF SOLUTIONS FOR QUASILINEAR ELLIPTIC DEGENERATE SYSTEMS WITH L^1 DATA AND NONLINEARITY IN THE GRADIENT

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Abstract. In this article we show the existence of weak solutions for some quasilinear degenerate elliptic systems arising in modeling chemotaxis and angiogenesis. The nonlinearity we consider has critical growth with respect to the gradient and the data are in L^1.

1. Introduction

Reaction-diffusion systems are important for a wide range of applied areas such as cell processes, drug release, ecology, spread of diseases, industrial catalytic processes, transport of contaminants in the environment, chemistry in interstellar media, to mention a few. Some of these applications, especially in chemistry and biology, are explained in books by Murray [26, 27] and Baker [10]. While a general theory of reaction-diffusion systems is detailed in the books of Rothe [34] and Grzybowski [21]. Various forms of this problems have been proposed in the literature. Most discussions in the current literature are for linear or nonlinear systems and different methods for the existence problem have been used, see Alaa et al [1]–[9], Baras [11, 12], Boccardo et al [15], Boudiba [16] and Pierre et al [29]-[32]. This is a relatively recent subject of mathematical and applied research. Most of the work that has been done so far is concerned with the exploration of particular aspects of very specific systems and equations. This is because these systems are usually very complex and display a wide range of phenomena remain poorly understood. Consequently, there is no established program for solving a large class of systems. For example a system of Chemotaxis, which is a biological phenomenon describing the change of motion of a population densities or of single particles (such as amoebae, bacteria, endothelial cells, any cell, animals, etc.) in response (taxis) to an external chemical stimulus spread in the environment where they reside see for example [28]. The simple mathematical model which describes such a phenomenon

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reads as follows

$$\begin{gathered}\frac{\partial u}{\partial t}-D_u\Delta u+\nabla(\kappa(u)\nabla u+\chi(v)\nabla v)=0 \quad \text{in } \Omega\times(0,T)\\ \frac{\partial v}{\partial t}-D_v\Delta v+\nabla(\zeta(u)\nabla u+\eta(v)\nabla v)=0 \quad \text{in } \Omega\times(0,T)\\ u(0)=u_0,\quad v_0(0)=v_0\end{gathered}\tag{1.1}$$

here u and v are the population densities. For a simple expansion, we include

$$\begin{gathered}a(x)=\frac{\partial\kappa(u)}{\partial u},\quad b(x)=\frac{\partial\chi(v)}{\partial v},\quad c(x)=\frac{\partial\zeta(u)}{\partial u},\quad d(x)=\frac{\partial\eta(v)}{\partial v}\\ f=-(\kappa(u)\Delta u+\chi(v)\Delta)v,\quad g=-(\zeta(u)\Delta u+\eta(v)\Delta v)\,.\end{gathered}$$

Then the system can be written as

$$\begin{gathered}\frac{\partial u}{\partial t}-D_u\Delta u+a(x)|\nabla u|^2+b(x)|\nabla v|^2=f \quad \text{in } \Omega\times(0,T)\\ \frac{\partial v}{\partial t}-D_v\Delta v+c(x)|\nabla u|^2+d(x)|\nabla v|^2=g \quad \text{in } \Omega\times(0,T)\\ u(0)=u_0,\quad v_0(0)=v_0\,.\end{gathered}\tag{1.2}$$

In this work we are interested in the quasilinear elliplic degenerate problem

$$\begin{gathered}u-D_1\Delta u+a(x)|\nabla u|^2+b(x)|\nabla v|^\alpha=f(x) \quad \text{in } \Omega\\ v-D_2\Delta v+c(x)|\nabla u|^\beta+d(x)|\nabla v|^2=g(x) \quad \text{in } \Omega\\ u=v=0 \quad \text{on } \partial\Omega\end{gathered}\tag{1.3}$$

where Ω is an open bounded set of $\mathbb{R}^N$, $N\geq 1$, with smooth boundary $\partial\Omega$, the diffusion coefficients D_1 and D_2 are positive constants, $a,b,c,d,f,g:\Omega\to[0,+\infty)$ are a non-negative integrable functions and $1\leq\alpha,\beta\leq 2$.

We are interested in the case where the data are non-regular and where the growth of the nonlinear terms is arbitrary with respect to the gradient. To help understanding the situation, let us mention some previous works concerning the problem when $a,b,c,d\in L^\infty(\Omega)$.

• if f,g are regular enough ($f,g\in W^{1,\infty}(\Omega)$) and for all $\alpha,\beta\geq 1$, the method of sub- and super-solution can be used to prove the existence of solutions to (1.3). For instance $(0,0)$ is a subsolution and a solution, $w=(w_1,w_2)$, of the linear problem

$$\begin{gathered}w_1-D_1\Delta w_2=f(x) \quad \text{in } \Omega\\ w_1-D_1\Delta w_2=g(x) \quad \text{in } \Omega\\ w_1=w_2=0 \quad \text{on } \partial\Omega,\end{gathered}\tag{1.4}$$

is a supersolution. Then (1.3) has a solution $(u,v)\in W_0^{1,\infty}(\Omega)\cap W^{2,p}(\Omega)$; see Lions [23].

• If $f,g\in L^2(\Omega)$ and $1\leq\alpha,\beta\leq 2$, then $|\nabla u|^\alpha,|\nabla v|^\beta\in L^1(\Omega)$. Many authors have studied this problem and showed that (1.3) has a solution $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)$, see Bensoussan et al [14], Boccardo et al [15] and the references there in.

• If $f,g\in L^1(\Omega)$ and $1\leq\alpha,\beta<2$, Alaa and Mesbahi [1] proved that (1.3) has a non negative solution $(u,v)\in W_0^{1,1}(\Omega)\times W_0^{1,1}(\Omega)$.

• The case where $f,g\in M_B^+(\Omega)$ (f,g are a finite non negative measures on Ω) has treated by Alaa and Pierre [9]. They proved that if $1\leq\alpha,\beta\leq 2$ and the supersolution $w=(w_1,w_2)\in H_0^1(\Omega)\times H_0^1(\Omega)$, then the problem (1.3) has a non negative solution $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)$.

We are particularly interested in the case of a system (1.3) when a, b, c, d, f, g are not regular, more precisely, a, b, c, d, f, g are in $L^1(\Omega)$.

Let us make some specifications on the model problem

$$\begin{gathered} u - D_1\Delta u + b(r)|\nabla v|^\alpha = f \quad \text{in } B \\ v - D_2\Delta v + c(r)|\nabla u|^\beta = g \quad \text{in } B \\ u = v = 0 \quad \text{on } \partial B \end{gathered} \tag{1.5}$$

where B is the unit ball in $\mathbb{R}^N$, $r = \|x\|$ and

$$b(r) = c(r) = \begin{cases} -\ln r & \text{if } N = 2 \\ r^{2-N} & \text{if } N \geq 3\,. \end{cases} \tag{1.6}$$

In this case, $b(r), c(r)$ are in $L^1_{\rm loc}(B)$ but not in $L^\infty(B)$. As a consequence the techniques usually used to prove existence and based on a priori L^∞-estimates on u and ∇u fail. To overcome this difficulty, we will develop a new method which differ completely of the previous approach.

We have organized this article as follows. In section 2 we give the precise setting of the problem and state the main result. In section 3 we present an approximate problem and we give suitable estimates to prove that (1.3) has a solution in the case where the growth of the nonlinearity with respect to the gradient is arbitrary.

2. Assumptions and statement of main results

Let f, g, a, b, c, d are functions that satisfies the following assumptions

$$f, g \in L^1(\Omega), \quad f, g \geq 0 \tag{2.1}$$

$$a, b, c, d \in L^1_{\rm loc}(\Omega), \quad a, b, c, d \geq 0 \tag{2.2}$$

First, we have to clarify in which sense we want to solved problem (1.3).

Definition 2.1. We say that (u, v) is a weak solution of (1.3) if

$$\begin{gathered} u, v \in W^{1,1}_0(\Omega) \\ a(x)|\nabla u|^2, \quad b(x)|\nabla v|^\alpha, \quad c(x)|\nabla u|^\beta, \quad d(x)|\nabla v|^2 \in L^1_{\rm loc}(\Omega) \\ u - D_1\Delta u + a(x)|\nabla u|^2 + b(x)|\nabla v|^\alpha = f(x) \quad \text{in } D'(\Omega) \\ v - D_2\Delta v + c(x)|\nabla u|^\beta + d(x)|\nabla v|^2 = g(x) \quad \text{in } D'(\Omega) \end{gathered} \tag{2.3}$$

We are interested to proving the existence of weak positive solutions of the problem (1.3). For this, we define the truncation function $T_k \in C^2$, such that

$$\begin{gathered} T_k(r) = r \quad \text{if } 0 \leq r \leq k \\ T_k(r) \leq k + 1 \quad \text{if } r \geq k \\ 0 \leq T'_k(r) \leq 1 \quad \text{if } r \geq 0 \\ T'_k(r) = 0 \quad \text{if } r \geq k + 1 \\ 0 \leq -T''_k(r) \leq C(k)\,. \end{gathered} \tag{2.4}$$

For example, the function T_k can be defined as

$$
\begin{gathered}
T_k(r) = r \quad \text{in } [0,k] \\
T_k(r) = \frac{1}{2}(r-k)^4 - (r-k)^3 + r \quad \text{in } [k,k+1] \\
T_k(r) = \frac{1}{2}(k+1) \quad \text{for } r > k+1\,.
\end{gathered} \tag{2.5}
$$

Then we define the the space

$$
\tau^{1,2}(\Omega) = \big\{ w : \Omega \to \mathbb{R} \text{ measurable, such that } T_k(w) \in H^1(\Omega) \text{ for all } k > 0 \big\}
$$

This enables us to state the main result of this paper.

Theorem 2.2. *Assume that* (2.1) *and* (2.2) *hold, and* $1 \le \alpha, \beta < 2$. *If there exists a function* $\theta \in \tau^{1,2}(\Omega)$ *and a sequence* $\theta_n \in L^\infty(\Omega)$ *such that*

$$
\begin{gathered}
0 \le a,b,c,d \le \theta \quad \text{in } \Omega \\
\theta_n \to \theta \quad \text{a.e. } \Omega \\
\nabla T_k(\theta_n) \to \nabla T_k(\theta) \quad \text{strongly in } L^2(\Omega) \\
\lim_{k\to\infty} \sup_n \Big(\frac{1}{k} \int_\Omega |\nabla T_k(\theta_n)|^2 \Big) = 0
\end{gathered} \tag{2.6}
$$

Then the problem (1.3) *has a non negative weak solution.*

Remark 2.3. (i) If $a,b,c,d \in L^\infty(\Omega)$, then (2.6) is satisfied. Indeed, θ can take the value of any non negative constant C, such that

$$
C \ge \max\big\{ \|a\|_{L^\infty}, \|b\|_{L^\infty}, \|c\|_{L^\infty}, \|d\|_{L^\infty} \big\} \tag{2.7}
$$

(ii) Hypothesis (2.6) holds for the functions $\xi = b$ or c given in (1.6). Indeed $-\Delta\xi = \lambda$ is in this case the measure of Dirac which is a finite non negative measure on Ω. By consequent, we take $\theta = \xi$ and θ_n solution of

$$
\begin{gathered}
-\Delta\theta_n = \lambda_n \quad \text{in } \Omega \\
\theta_n = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{2.8}
$$

where $\lambda_n \in C_0^\infty(\Omega)$, $\lambda_n \to \lambda$ in $L^1(\Omega)$ and $\lambda_n \le \lambda$. Then, we can applied Theorem 2.2 and conclude the existence of the non negative weak solution for our model problem (2.1).

3. Proof of theorem 2.2

3.1. **An approximation scheme.** In this paragraph, we define an approximated system of (1.3). For this, we truncate the functions a,b,c,d,f,g by introducing the sequence $a_n, b_n, c_n, d_n, f_n, g_n$ defined as follows

$$
a_n = \min\{a, \theta_n\}, \quad b_n = \min\{b, \theta_n\}, \quad c_n = \min\{c, \theta_n\}, \quad d_n = \min\{d, \theta_n\}
$$

and

$$
\begin{gathered}
f_n \in C_0^\infty(\Omega),\ f_n \to f \text{ in } L^1(\Omega),\ f_n \le f \\
g_n \in C_0^\infty(\Omega),\ g_n \to g \text{ in } L^1(\Omega),\ g_n \le g
\end{gathered} \tag{3.1}
$$

Then the approximate problem is

$$
\begin{gathered}
u_n, v_n \in W_0^{1,\infty}(\Omega) \\
u_n - D_1\Delta u_n + a_n(x)|\nabla u_n|^2 + b_n(x)|\nabla v_n|^\alpha = f_n(x) \quad \text{in } D'(\Omega) \\
v_n - D_2\Delta v_n + c_n(x)|\nabla u_n|^\beta + d_n(x)|\nabla v_n|^2 = g_n(x) \quad \text{in } D'(\Omega)\,.
\end{gathered} \tag{3.2}
$$

One can see that a_n, b_n, c_n, d_n are in $L^\infty(\Omega)$. On the other hand, $(0,0)$ is a subsolution of (3.2) and (U_n, V_n) a solution of the linear problem

$$\begin{gathered} U_n - D_1\Delta U_n = f_n \quad \text{in } \Omega \\ V_n - D_2\Delta V_n = g_n \quad \text{in } \Omega \\ U_n, V_n \in W_0^{1,\infty}(\Omega) \end{gathered} \tag{3.3}$$

is a supersolution, then by the classical results in Amann and Grandall [13] and Lions [23, 24], there exists (u_n, v_n) solution of (3.2) such that

$$\begin{gathered} 0 \le u_n \le U_n \quad \text{for all } n \\ 0 \le v_n \le V_n \quad \text{for all } n \end{gathered}$$

3.2. **A priori estimates.** To prove theorem 2.2, we propose to send n to infinity in (3.2). For this we will need some estimates passing to the limit.

Lemma 3.1. *Let $u_n, v_n, a_n, b_n, c_n, d_n$ be sequences defined as above. Then (i)*

$$\begin{gathered} \int_\Omega |\nabla T_k(u_n)|^2 \le k\|f\|_{L^1(\Omega)} \\ \int_\Omega |\nabla T_k(v_n)|^2 \le k\|g\|_{L^1(\Omega)} \end{gathered}$$

and (ii)

$$\begin{gathered} \int_\Omega b_n.|\nabla T_k(v_n)|^\alpha \le k\|f\|_{L^1(\Omega)} \\ \int_\Omega c_n.|\nabla T_k(u_n)|^\beta \le k\|g\|_{L^1(\Omega)} \end{gathered}$$

Proof. (i) By multiplying the first equation of (3.2) by $T_k(u_n)$ and the second equation by $T_k(v_n)$ and integrating over Ω, we have

$$\begin{aligned} &\int_\Omega |T_k(u_n)|^2 + D_1 \int_\Omega |\nabla T_k(u_n)|^2 \\ &+ \int_\Omega a_n T_k(u_n)|\nabla T_k(u_n)|^2 + \int_\Omega b_n T_k(u_n)|\nabla T_k(v_n)|^\alpha \le \int_\Omega f_n T_k(u_n) \end{aligned}$$

and

$$\begin{aligned} &\int_\Omega |T_k(v_n)|^2 + D_2 \int_\Omega |\nabla T_k(v_n)|^2 \\ &+ \int_\Omega c_n T_k(v_n)|\nabla T_k(u_n)|^\beta + \int_\Omega d_n T_k(v_n)|\nabla T_k(v_n)|^2 \le \int_\Omega g_n T_k(v_n) \end{aligned}$$

Thanks to the positivity of a_n, b_n, c_n, d_n, the assumptions on f_n and g_n, the definition of the function T_k, we deduce the result.

(ii) Integrating the first equation of (3.2) over Ω, we obtain

$$\int_\Omega u_n - D_1 \int_\Omega \Delta u_n + \int_\Omega a_n(x)|\nabla u_n|^2 + \int_\Omega b_n(x)|\nabla v_n|^\alpha = \int_\Omega f_n(x) \tag{3.4}$$

On the other hand, it is well know that for every function y in $W_0^{1,1}(\Omega)$ such that

$$\begin{gathered} -\Delta y = H,\ H \in L^1(\Omega) \\ y \ge 0 \end{gathered}$$

there exists a sequence y_n in $C^2(\Omega) \cap C_0(\overline{\Omega})$ which satisfies

$$y_n \to y \quad \text{strongly in } W_0^{1,1}(\Omega)$$
$$\Delta y_n \to \Delta y \quad \text{strongly in } L^1(\Omega)$$

The regularity of y_n allows us to write

$$\int_\Omega \Delta y_n = \int_{\partial\Omega} \frac{\partial y_n}{\partial \upsilon} d\sigma,$$

but $y_n \geq 0$ on Ω and $y_n = 0$ in $\partial\Omega$. Then $\frac{\partial y_n}{\partial \upsilon} \leq 0$. We deduce by passing to the limit that $\int_\Omega \Delta y \leq 0$. Therefore

$$\int_\Omega \Delta u_n \leq 0$$

The relation (3.4) yields

$$\int_\Omega u_n + \int_\Omega a_n(x)|\nabla u_n|^2 + \int_\Omega b_n(x)|\nabla v_n|^\alpha \leq \int_\Omega f_n(x)\,.$$

By (3.1); we conclude that

$$\int_\Omega u_n + \int_\Omega a_n(x)|\nabla u_n|^2 + \int_\Omega b_n(x)|\nabla v_n|^\alpha \leq \|f\|_{L^1(\Omega)}\,.$$

In the same way, if we integrate the second equation of (3.2) over Ω, we obtain

$$\int_\Omega v_n + \int_\Omega c_n(x)|\nabla u_n|^\beta + \int_\Omega d_n(x)|\nabla v_n|^2 \leq \|g\|_{L^1(\Omega)},$$

hence the result follows. □

Remark 3.2. (1) Using the assertion (ii) of lemma 3.1, and the compactness of the operator

$$L^1(\Omega) \to W_0^{1,q}(\Omega)$$
$$G \mapsto \vartheta$$

where $1 \leq q < \frac{N}{N-1}$, and ϑ is the solution of the problem

$$\vartheta \in W_0^{1,q}(\Omega)$$
$$\alpha\vartheta - \Delta\vartheta = G \quad \text{in } D'(\Omega)$$

we conclude the existence of u, up to a subsequence, still denoted by u_n for simplicity, such that

$$u_n \to u \quad \text{strongly in } W_0^{1,q}(\Omega), \quad 1 \leq q < \frac{N}{N-1},$$
$$(u_n, \nabla u_n) \to (u, \nabla u) \quad \text{a.e. in } \Omega$$

see Brezis [17]

(2) Assertion (i) implies that

$$(T_k(u_n), T_k(v_n)) \to (T_k(u), T_k(v)) \quad \text{weakly in } H_0^1(\Omega) \times H_0^1(\Omega)$$

Lemma 3.3. *Let (u_n, v_n) be a solution of* (3.2)*, then*

$$\lim_{h\to+\infty} \sup_n \Big(\frac{1}{h}\int_\Omega |\nabla T_h(u_n)|^2 dx\Big) = \lim_{h\to+\infty} \sup_n \Big(\frac{1}{h}\int_\Omega |\nabla T_h(v_n)|^2 dx\Big) = 0$$

Proof. We first remark that u_n satisfies

$$-\Delta u_n \leq f_n \quad \text{in } D'(\Omega)$$

If we multiply this inequality by $T_h(u_n)$ and integrate on Ω, we obtain for every $0 < M < h$,

$$\begin{aligned}\int_\Omega |\nabla T_h(u_n)|^2 &\leq \int_{\Omega\cap\{u_n\leq M\}} fT_h(u_n) + \int_{\Omega\cap\{u_n>M\}} fT_h(u_n)\\ &\leq M\int_\Omega f + h\int_\Omega f\chi_{\{u_n>M\}}\end{aligned}$$

hence

$$\begin{gathered}\frac{1}{h}\int_\Omega |\nabla T_h(u_n)|^2 \leq \frac{M}{h}\int_\Omega f + \int_\Omega f\chi_{\{u_n>M\}}\\ |\{u_n > M\}| = \int_{\{u_n>M\}} dx \leq \frac{1}{M}\|u_n\|_{L^1} \leq \frac{C}{M}\end{gathered}$$

Then $\lim_{M\to+\infty}\big(\sup_n |\{u_n > M\}|\big) = 0$

On other hand, since $f \in L^1(\Omega)$, we have for each $\varepsilon > 0$ there exists δ such that for for all $E \subset \Omega$,

$$|E| < \delta \int_E |f| \leq \frac{\varepsilon}{2}.$$

Taking into account the above limit, we obtain that for each $\varepsilon > 0$, there exists M_ε such that for all $M \geq M_\varepsilon$,

$$\sup_n \Big(\int_\Omega f\chi_{[u_n>M]}\Big) \leq \frac{\varepsilon}{2}$$

Taking $M = M_\varepsilon$ and letting h tend to infinity, we obtain

$$\lim_{h\to\infty}\sup_n \Big(\frac{1}{h}\int_\Omega |\nabla T_h(u_n)|^2\Big) = 0\,.$$

$\square$

Lemma 3.4. *Let η_n be sequence such that $\eta_n \to \eta$, a.e. in Ω and $\int_\Omega |\eta_n|^2 \leq C$ then $\eta_n \to \eta$ in $L^\alpha(\Omega)$ for all $1 \leq \alpha < 2$.*

Proof. We show that η_n is equi-integrable in $L^\alpha(\Omega)$. Let E be a measurable subset of Ω; we have

$$\int_E |\eta_n|^\alpha \leq |E|^{(2-\alpha)/2}\Big(\int_E |\eta_n|^2\Big)^{\alpha/2} \leq C|E|^{(2-\alpha)/2}$$

Since $1 \leq \alpha < 2$ then $0 < 2-\alpha \leq 1$. We choose $|E| = (\frac{\varepsilon}{C})^{2/(2-\alpha)}$, we obtain $\int_E |\eta_n|^\alpha \leq \varepsilon$. $\square$

3.3. **Convergence.** The aim of this paragraph is to prove that (u, v) (obtained in the previous section) is in fact a solution of problem (1.3). According to definition 2.1, we have to show only that

$$\begin{gathered}u - D_1\Delta u + a(x)|\nabla u|^2 + b(x)|\nabla v|^\alpha = f(x) \quad \text{in } D'(\Omega)\\ v - D_2\Delta v + c(x)|\nabla u|^\beta + d(x)|\nabla v|^2 = g(x) \quad \text{in } D'(\Omega)\end{gathered}$$

By lemma 3.1, we know that $a_n(x)|\nabla u_n|^2$, $d_n(x)|\nabla v_n|^2$, $b_n(x)|\nabla v_n|^\alpha$, $c_n(x)|\nabla u_n|^\beta$ are uniformly bounded in $L^1(\Omega)$. Moreover

$$a_n(x)|\nabla u_n|^2 \geq 0, \quad d_n(x)|\nabla u_n|^2 \geq 0, \quad b_n(x)|\nabla u_n|^\alpha \geq 0, \quad c_n(x)|\nabla u_n|^\beta \geq 0$$

and for almost every x in Ω, we have

$$\begin{gathered}
a_n(x)|\nabla u_n(x)|^2 \to a(x)|\nabla u(x)|^2 \\
d_n(x)|\nabla v_n(x)|^2 \to d(x)|\nabla v(x)|^2 \\
b_n(x)|\nabla v_n(x)|^\alpha \to b(x)|\nabla v(x)|^\alpha \\
c_n(x)|\nabla u_n(x)|^\beta \to c(x)|\nabla u(x)|^\beta
\end{gathered}$$

Then there exists μ_1, μ_2 non negative measures, see Schwartz [35], such that

$$\begin{aligned}
&\lim_{n\to+\infty}(u_n - D_1\Delta u_n + a_n(x)|\nabla u_n|^2 + b_n(x)|\nabla v_n|^\alpha) \\
&= u - D_1\Delta u + a(x)|\nabla u|^2 + b(x)|\nabla v|^\alpha + \mu_1 \quad \text{in } D'(\Omega) \\
&\lim_{n\to+\infty}(v_n - D_2\Delta v_n + c_n(x)|\nabla u_n|^\beta + d_n(x)|\nabla v_n|^2) \\
&= v - D_2\Delta v + c(x)|\nabla u|^\beta + d(x)|\nabla v|^2 + \mu_2 \quad \text{in } D'(\Omega)
\end{aligned}$$

Consequently,

$$\begin{gathered}
u - D_1\Delta u + a(x)|\nabla u|^2 + b(x)|\nabla v|^\alpha \leq f \quad \text{in } D'(\Omega) \\
v - D_2\Delta v + c(x)|\nabla u|^\beta + d(x)|\nabla v|^2 \leq g \quad \text{in } D'(\Omega)
\end{gathered}$$

Therefore, to conclude the proof of Theorem 2.2, we must establish the opposite inequality. For this, Let H be a function in $C^1(\mathbb{R})$, such that

$$\begin{gathered}
0 \leq H(s) \leq 1 \\
H(s) = \begin{cases} 0 & \text{if } |s| \geq 1 \\ 1 & \text{if } |s| \leq \frac{1}{2} \end{cases}
\end{gathered}$$

To this end, we introduce the test functions

$$\begin{gathered}
\Phi_1 = \psi_1 \exp[-\frac{\theta_n}{D_1}u_n]H(\frac{\theta_n}{k})H(\frac{u_n}{k}), \\
\Phi_2 = \psi_2 \exp[-\frac{\theta_n}{D_2}v_n]H(\frac{\theta_n}{k})H(\frac{v_n}{k})
\end{gathered}$$

where H denotes the function defined above and $\psi_1,\ \psi_2 \leq 0$, $\psi_1,\ \psi_2 \in H_0^1(\Omega) \cap L^\infty(\Omega)$. We multiply the first equation in (3.2) by Φ_1 and we integrate on Ω, we obtain

$$\int_\Omega f_n\Phi_1 = \sum_{1\leq j\leq 7} I_j,$$

where

$$\begin{gathered}
I_1 = \int_\Omega u_n\Phi_1, \\
I_2 = D_1\int_\Omega \nabla u_n\nabla\psi_1 \exp[-\frac{\theta_n}{D_1}u_n]H(\frac{\theta_n}{k})H(\frac{u_n}{k}) \\
I_3 = -\int_\Omega u_n\nabla u_n\nabla\theta_n\Phi_1
\end{gathered}$$

$$I_4 = \frac{D_1}{k}\int_\Omega \nabla u_n \nabla\theta_n \psi_1 \exp[-\frac{\theta_n}{D_1}u_n]H'(\frac{\theta_n}{k})H(\frac{u_n}{k})$$
$$I_5 = \int_\Omega (a_n-\theta_n)|\nabla u_n|^2\Phi_1$$
$$I_6 = \frac{D_1}{k}\int_\Omega |\nabla u_n|^2 \psi_1 \exp[-\frac{\theta_n}{D_1}u_n]H(\frac{\theta_n}{k})H'(\frac{u_n}{k})$$
$$I_7 = \int_\Omega b_n.|\nabla v_n|^\alpha \Phi_1$$

Next we study each term. For the first term, we have

$$\begin{aligned}\lim_{n\to+\infty} I_1 &= \lim_{n\to+\infty}\int_\Omega T_k(u_n)\psi_1 \exp[-\frac{\theta_n}{D_1}u_n]H(\frac{\theta_n}{k})H(\frac{u_n}{k})\\ &= \int_\Omega u\psi_1\exp[-\frac{\theta}{D_1}u]H(\frac{\theta}{k})H(\frac{u}{k})\end{aligned}$$

since

$$\psi_1 \exp[-\frac{\theta_n}{D_1}u_n]H(\frac{\theta_n}{k})H(\frac{u_n}{k})$$

converges strongly in $L^2(\Omega)$ to

$$\psi_1 \exp[-\frac{\theta}{D_1}u]H(\frac{\theta}{k})H(\frac{u}{k}) \quad \text{in } L^2(\Omega)$$

and $\nabla T_k(u_n)$ converges weakly to $\nabla T_k(u)$ in $L^2(\Omega)$, (see [24, lemma 1.3, p 12]).

Concerning the second term, we get

$$\begin{aligned}\lim_{n\to+\infty} I_2 &= \lim_{n\to+\infty} D_1\int_\Omega \nabla T_k(u_n)\nabla\psi_1 \exp[-\frac{\theta_n}{D_1}u_n]H(\frac{\theta_n}{k})H(\frac{u_n}{k})\\ &= D_1\int_\Omega \nabla u\nabla\psi_1\exp[-\frac{\theta}{D_1}u]H(\frac{\theta}{k})H(\frac{u}{k})\end{aligned}$$

since

$$\nabla\psi_1 \exp[-\frac{\theta_n}{D_1}u_n]H(\frac{\theta_n}{k})H(\frac{u_n}{k})$$

converges strongly in $L^2(\Omega)$ to

$$\nabla\psi_1 \exp[-\frac{\theta}{D_1}u]H(\frac{\theta}{k})H(\frac{u}{k})\,.$$

For I_3, we first remark that

$$\begin{aligned}\lim_{n\to+\infty} I_3 &= -\lim_{n\to+\infty}\int_\Omega T_k(u_n)\nabla T_k(u_n)\nabla T_k(\theta_n)\psi_1 \exp[-\frac{\theta_n}{D_1}u_n]H(\frac{\theta_n}{k})H(\frac{u_n}{k})\\ &= -\int_\Omega T_k(u)\nabla T_k(u)\nabla T_k(\theta)\psi_1\exp[-\frac{\theta}{D_1}u]H(\frac{\theta}{k})H(\frac{u}{k})\end{aligned}$$

since

$$\begin{gathered} T_k(u_n)\to T_k(u) \quad \text{weakly in } H_0^1(\Omega)\\ T_k(\theta_n)\to T_k(\theta) \quad \text{strongly in } H_0^1(\Omega)\end{gathered}$$

To study I_4 and I_6 we use Lemma 3.3. For I_4, we have

$$I_4 \le D_1\Big[\frac{1}{k}\int_\Omega |\nabla T_k(u_n)|^2\psi_1 \exp[-\frac{\theta_n}{D_1}u_n]H'(\frac{\theta_n}{k})H(\frac{u_n}{k})\Big]^{1/2}$$

$$\times [\frac{1}{k}\int_\Omega |\nabla T_k(\theta_n)|^2 \psi_1 \exp[-\frac{\theta_n}{D_1}u_n]H'(\frac{\theta_n}{k})H(\frac{u_n}{k})]^{1/2}$$
$$\leq D_1[\|\psi_1\|_{L^\infty(\Omega)}\frac{1}{k}\int_\Omega |\nabla T_k(u_n)|^2]^{1/2}[\|\psi_1\|_{L^\infty(\Omega)}\frac{1}{k}\int_\Omega |\nabla T_k(\theta_n)|^2]^{12}$$

since $\exp[-\frac{\theta_n}{D_1}u_n] \leq 1$, thus

$$I_4 \leq D_1[\|\psi_1\|_{L^\infty}\delta_k]^{1/2}[\|\psi_1\|_{L^\infty}\rho_k]^{1/2}$$

Where

$$\delta_k = \sup_n(\frac{1}{k}\int_\Omega |\nabla T_k(u_n)|^2) \quad \text{and} \quad \rho_k = \sup_n(\frac{1}{k}\int_\Omega |\nabla T_k(\theta_n)|^2)$$

By Lemma 3.3, we have

$$\lim_{k\to\infty}\delta_k = 0, \quad \lim_{k\to\infty}\rho_k = 0$$

Then

$$\lim_{k\to\infty}\sup_n(I_4) = 0$$

Similarly, for I_6, we have

$$I_6 \leq D_1\|\psi_1\|_{L^\infty}\delta_k$$

Then

$$\lim_{k\to\infty}\sup_n(I_6) = 0$$

Now we investigate the remaining term I_5. Since $a_n \leq \theta_n$ and $\psi_1 \leq 0$, we have

$$(a_n - \theta_n)|\nabla u_n|^2 \exp[-\frac{\theta_n}{D_1}u]H(\frac{\theta_n}{k})H(\frac{u_n}{k}) \geq 0 \quad \text{in } \Omega$$

Therefore, by Fatou's lemma, we obtain

$$\lim_{n\to+\infty} I_5 \geq \int_\Omega (a-\theta)|\nabla u|^2 \exp[-\frac{\theta}{D_1}u]H(\frac{\theta}{k})H(\frac{u}{k})$$

For I_7, we obtain

$$\lim_{n\to+\infty} I_7 = \lim_{n\to+\infty}\int_\Omega T_k(b_n)|\nabla T_k(v_n)|^\alpha \psi_1 \exp[-\frac{\theta_n}{D_1}u_n]H(\frac{\theta_n}{k})H(\frac{u_n}{k})$$

By a direct application of Lemma 3.3, we have $|\nabla T_k(v_n)|^\alpha \to |\nabla T_k(v)|^\alpha$strongly in $L^1(\Omega)$, then

$$\lim_{n\to+\infty} I_7 = \int_\Omega b|\nabla v|^\alpha \psi_1 \exp[-\frac{\theta}{D_1}u]H(\frac{\theta}{k})H(\frac{u}{k})$$

We have shown that

$$\begin{aligned}
&\omega(\frac{1}{k}) + \int_\Omega u\psi_1 \exp[-\frac{\theta}{D_1}u]H(\frac{\theta}{k})H(\frac{u}{k}) + D_1\int_\Omega \nabla u\nabla\psi_1 \exp[-\frac{\theta}{D_1}u]H(\frac{\theta}{k})H(\frac{u}{k})\\
&- \int_\Omega u\nabla u\nabla\theta\psi_1 \exp[-\frac{\theta}{D_1}u]H(\frac{\theta}{k})H(\frac{u}{k}) + \int_\Omega (a-\theta)|\nabla u|^2 \exp[-\frac{\theta}{D_1}u]H(\frac{\theta}{k})H(\frac{u}{k})\\
&+ \int_\Omega b|\nabla v|^\alpha \psi_1 \exp[-\frac{\theta}{D_1}u]H(\frac{\theta}{k})H(\frac{u}{k})\\
&\leq \int_\Omega f\psi_1 \exp[-\frac{\theta}{D_1}u]H(\frac{\theta}{k})H(\frac{u}{k})
\end{aligned}$$

where $\omega(\varepsilon)$ denotes a quantity that tends to 0 when ε tends to 0. Now we choose

$$\psi_1 = -\varphi_1 \exp[\frac{\theta}{D_1}u]H(\frac{\theta}{k})H(\frac{u}{k})$$

where $\varphi_1 \geq 0$, $\varphi_1 \in D(\Omega)$ and we replace ψ_1 by this value in the previous inequality to obtain

$$\begin{aligned}
&w(\frac{1}{k}) - \int_\Omega u\varphi_1 H^2(\frac{\theta}{k})H^2(\frac{u}{k}) - D_1\int_\Omega \nabla u\nabla\varphi_1 H^2(\frac{\theta}{k})H^2(\frac{u}{k})\\
&- \int_\Omega \varphi_1 u\nabla u\nabla\theta H^2(\frac{\theta}{k})H^2(\frac{u}{k}) - \int_\Omega \varphi_1|\nabla u|^2\theta H^2(\frac{\theta}{k})H^2(\frac{u}{k})\\
&- \frac{D_1}{k}\int_\Omega \varphi_1\nabla u\nabla\theta H'(\frac{\theta}{k})H(\frac{\theta}{k})H^2(\frac{u}{k}) - \frac{D_1}{k}\int_\Omega \varphi_1|\nabla u|^2 H^2(\frac{\theta}{k})H(\frac{u}{k})H'(\frac{u}{k})\\
&+ \int_\Omega u\nabla u\nabla\theta\varphi_1 H^2(\frac{\theta}{k})H^2(\frac{u}{k}) - \int_\Omega b|\nabla v|^\alpha\varphi_1 H^2(\frac{\theta}{k})H^2(\frac{u}{k})\\
&- \int_\Omega a|\nabla u|^2\varphi_1 H^2(\frac{\theta}{k})H^2(\frac{u}{k}) + \int_\Omega \theta\varphi_1|\nabla u|^2 H^2(\frac{\theta}{k})H^2(\frac{u}{k})\\
&\leq -\int_\Omega f\varphi_1 H^2(\frac{\theta}{k})H^2(\frac{u}{k})
\end{aligned}$$

By developing calculations and remarking that the sixth and seventh terms are equivalent to $\omega(\frac{1}{k})$, we can write

$$\begin{aligned}
&-\int_\Omega u\varphi_1 H^2(\frac{\theta}{k})H^2(\frac{u}{k}) - D_1\int_\Omega \nabla u\nabla\varphi_1 H^2(\frac{\theta}{k})H^2(\frac{u}{k})\\
&-\int_\Omega [a|\nabla u|^2 + b|\nabla v|^\alpha]\varphi_1 H^2(\frac{\theta}{k})H^2(\frac{u}{k}) + \omega(\frac{1}{k})\\
&\leq -\int_\Omega f\varphi_1 H^2(\frac{\theta}{k})H^2(\frac{u}{k})
\end{aligned}$$

Finally passing to the limit as k tends to infinity, we use the fact that

$$\lim_{k\to\infty} H(\frac{\theta}{k}) = 1, \quad \lim_{k\to\infty} H(\frac{u}{k}) = 1$$

to conclude that for every $\varphi_1 \geq 0,\ \varphi_1 \in D(\Omega)$,

$$\int_\Omega [u - D_1\Delta u + a(x)|\nabla u|^2 + b(x)|\nabla v|^\alpha]\varphi_1 \geq \int_\Omega f\varphi_1$$

In the same way, we multiply the second equation in (3.2) by Φ_2 and we integrate on Ω. By studying separately each term as in the previous case still using Lemmas 3.1, 3.3 and 3.4, we choose

$$\psi_2 = -\varphi_2\exp[\frac{\theta}{D_2}u]H(\frac{\theta}{k})H(\frac{v}{k})$$

where $\varphi_2 \geq 0$, $\varphi_2 \in D(\Omega)$ and we replace ψ_2 by this value in the inequality obtained to conclude that for every $\varphi_2 \geq 0$, $\varphi_2 \in D(\Omega)$ that

$$\int_\Omega [v - D_2\Delta v + c(x)|\nabla u|^\beta + d(x)|\nabla v|^2]\varphi_2 \geq \int_\Omega g\varphi_2$$

This completes the proof of theorem 2.2.

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