Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 132, pp. 1–23.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

CONSTRUCTION OF WAVE OPERATOR FOR TWO-DIMENSIONAL KLEIN-GORDON-SCHRÖDINGER SYSTEMS WITH YUKAWA COUPLING

KAI TSURUTA

ABSTRACT. We prove the existence of the wave operator for the Klein-Gordon-Schrödinger system with Yukawa coupling. This non-linearity type is below Strichartz scaling, and therefore classic perturbation methods will fail in any Strichartz space. Instead, we follow the "first iteration method" to handle these critical non-linearities.

1. Introduction and Overview

We study the Klein-Gordon-Schrödinger system

$$\begin{aligned} i\partial_t u + \frac{1}{2}\Delta u &= \pm uv \\ (\Box + 1)v &= \pm|\partial_x u|^2 \end{aligned} \tag{1.1}$$

on $\mathbb{R}^2$ and show that wave operators exist under smallness conditions and a control assumption on a single frequency band of the "final data".

A common choice of coupling, known as the Yukawa interaction, is to replace the non-linearity $\pm|\partial_x u|^2$ with $\pm|u|^2$. With the Yukawa coupling, the system describes the interaction of a complex scalar nucleon field u with a real scalar meson field v (see [17]). Our method only relies on the precise form of phase functions in frequency space and not on any conservation laws particular to our choice of non-linearity; therefore, the same method can be used to construct the wave operator in the case of Yukawa interaction.

Our purpose in this paper is to construct the wave operators for (1.1). Roughly, the positive-time wave operator of a non-linear dispersive equation is defined as follows: Suppose that for a solution ψ_{lin} to the dispersive equation with no non-linearity and initial data ψ_+, there exists a unique solution ψ to the non-linear equation with initial data ψ_0 such that ψ behaves as ψ_{lin} as $t \to \infty$ (this is known as scattering), then the positive-time wave operator is the map W_+ that takes ψ_+ to ψ_0. We similarly define the negative-time wave operator W_- to consider behavior as $t \to -\infty$. In this paper, we will only construct W_+; W_- can be constructed analogously.

2000 *Mathematics Subject Classification.* 35Q40, 35Q55.

Key words and phrases. Klein-Gordon-Schrödinger; Yukawa coupling; wave operator.

Submitted November 2, 2012. Published May 29, 2013.

As an explicit example, we discuss the positive-time wave operator for the Schrödinger equation with non-linearity $F(u)$. Solutions to the linear equation have the form $e^{\frac{1}{2}it\Delta}u_+$, where u_+ is the initial data at $t = 0$. Suppose there is a space X such that if for any $u_+ \in X$, there is a global strong X solution u to the non-linear Schrödinger equation with initial data u_0, such that

$$\|u - e^{\frac{1}{2}it\Delta}u_+\|_X \to 0 \quad \text{as } t \to \infty.$$

Then $W_+ : X \to X$ is defined by $W_+(u_+) = u_0$.

In a space where the linear operator is an isometry, one can view the wave operator problem as an initial value problem in the limiting case $t_0 = \infty$. For instance, in the case of the Schrödinger equation with non-linearity $F(u)$, the linear operator is an isometry in any H^s space. In these spaces, we formally expect solutions to have the form

$$u(t) = e^{\frac{1}{2}it\Delta}u_+ - i\int_t^\infty e^{\frac{1}{2}i(t-s)\Delta}F(u)(s)ds, \tag{1.2}$$

where u_+ is the scattered state of u.

The reason one expects a non-linear solution to have this linear behavior as $t \to \infty$ is that if the solutions tend to zero over time, then the non-linearity should tend to zero even faster as t increases. Intuitively, this means that it should be easier to establish scattering for a higher degree non-linearity. However, high degree non-linearities augment the large values of the solution. These large values are usually the primary difficulty in establishing local well-posedness. Hence, the precise degree of non-linearity is very important in scattering theory.

Because the Klein-Gordon and Schrödinger equations both have a $t^{-d/2}$ decay as $t \to \infty$, quadratic non-linearities are something of a critical case in two dimensions. For example, it has been proven that on $\mathbb{R}^2$, one can construct the wave operators for the Schrödinger equation and the Klein-Gordon equation with power type non-linearities of the form $|\phi|^{p-1}\phi$ if $p > 2$, but not if $1 < p \le 2$ (see, e.g., [3, 6, 14, 15]).

One explicit difficulty in dealing with quadratic-type non-linearities in two dimensions is our inability to use the Strichartz estimates. Perturbation methods that use the Strichartz estimates require the norm of the non-linearity $F(u)$ in some conjugate admissible pair space to be bounded by the norm of u in an admissible pair space. However, since quadratic non-linearities fall below two-dimensional Strichartz scaling, we cannot hope to control $F(u)$ in this way.

Previous results for (1.1) are mostly restricted to the Yukawa interaction and rely critically on conservation laws. Ozawa and Tsutsumi first studied this problem with the non-linearities uv and $-|u|^2$ in [8]. They proved the existence of wave operators under certain smallness conditions on the scattered states, as well as the assumption that the Fourier support of the scattered state of u was outside the unit disc. Shimomura[10, 11, 12] improved these results with the same non-linearities. In [10], the existence of wave operators was established without any smallness condition on v's scattered state, but the Fourier support of u's scattered state was still required to be outside the unit disc. In [11], the support condition on $\hat{u}_+$ was substituted for a smallness condition on v_+ and a controllability assumption on the supports of $\hat{u}_+$ and $\hat{v}_+$ on a single circle. Finally, in [12], wave operators were shown to exist without any smallness condition on v_+ and no support conditions other than the controllability assumption of [11]. All results rely on the energy method and first and second approximations to the asymptotic profiles of u and

v to construct solutions on the the interval $[T,\infty)$ for some large T. Global well-posedness results were then used to extend the solution to $[0,\infty)$. Again, these results relied critically on the precise form of the non-linearity.

The Cauchy problem for the KGS system was solved in [9] and [16] for the Yukawa coupling $-uv$ and $|u|^2$. In [9] the Fourier restriction norm method was used; in [16] the I-method was used. Both papers used Strichartz estimates on finite intervals and relied on energy and charge conservation to show existence of global solutions. The ability to use these conservation laws depends delicately on the non-linearities; hence, the specific choice of $-uv$ and $|u|^2$ was crucial in the result. In [5], Pecher was able to show local existence without the use of energy conservation, and thus for a wider variety of Yukawa interactions ($\pm uv$ and $\pm|u|^2$).

1.1. **Notation.** We now discuss notation and introduce some operators. We use the Fourier Transform defined by

$$\mathcal{F}(g)(\xi) = \int_{\mathbb{R}^2} e^{-2\pi i\xi\cdot x} g(x)dx.$$

Sometimes we will use the notation $\hat{g}$ to denote $\mathcal{F}(g)$. We use the operator

$$\Box = \partial_{tt} - \Delta.$$

The function $\langle\cdot\rangle$, known as the Japanese bracket, is defined for vectors by

$$\langle x\rangle = \sqrt{1+|x|^2},$$

while the operators $\langle\nabla\rangle$, $e^{it\Delta}$, and $e^{-it\langle\nabla\rangle}$ (known as Fourier multipliers) are defined by the identities

$$\mathcal{F}(\langle\nabla\rangle f) = \sqrt{1+|\xi|^2}\hat{f}, \quad \mathcal{F}(e^{it\Delta}f) = e^{-it|\xi|^2}\hat{f}, \quad \mathcal{F}(e^{-it\langle\nabla\rangle}f) = e^{-it\sqrt{1+|\xi|^2}}\hat{f}.$$

We define the Klein-Gordon linear propagators L and $\dot{L}$ by

$$\mathcal{F}(L(f,g)) = \cos(\langle\xi\rangle t)\hat{f} + \langle\xi\rangle^{-1}\sin(\langle\xi\rangle t)\hat{g}$$
$$\mathcal{F}(\dot{L}(f,g)) = -\langle\xi\rangle^{-1}\sin(\langle\xi\rangle t)\hat{f} + \cos(\langle\xi\rangle t)\hat{g}.$$

Let $\varphi(r)$ be a smooth cutoff function on $\mathbb{R}^+\cup\{0\}$ that is congruent to 1 for $r\le 1$ and supported on $[0,2]$. The Littlewood-Paley operators P_k are then defined for $k\in\mathbb{Z}$ by

$$\mathcal{F}(P_k f)(\xi) = (\varphi(|\xi|/2^k) - \varphi(|\xi|/2^{k-1}))\hat{f}(\xi).$$

The space $L^p = L^p(\mathbb{R}^n)$ for $1\le p<\infty$ is defined by its norm

$$\|g\|_{L^p} = \Big(\int_{\mathbb{R}^n} |g|^p\Big)^{1/p},$$

while L^∞ is the space of all essentially bounded functions. From the L^p spaces, we define the Sobolev spaces $W^{s,p}$ by the norm

$$\|g\|_{W^{s,p}} = \|\langle\nabla\rangle^s g\|_{L^p}$$

and denote $W^{s,2}$ as H^s.

We define the Besov Spaces $B^s_{p,q}$ by the norm

$$\|g\|_{B^s_{p,q}} = \Big(\sum_{k\in\mathbb{Z}} \|\langle 2^k\rangle^s g\|^q_{L^P}\Big)^{1/q}.$$

We also introduce the Y space, which will be where we require the scattered states u_+ and $(v_+, \dot{v}_+)$ to be small. Y is defined by the norm

$$\|g\|_Y = \sum_{|\alpha+\beta|\le 12} \|x^\alpha \partial^\beta g\|_{L^2} + \|g\|_{W^{16,1}} + \|g\|_{H^{16}} + \|g\|_{B^6_{1,1}}.$$

For small δ, we introduce the set A_δ, defined as

$$A_\delta = \{(\xi,\eta) \mid r_\xi - \delta \le |\xi| \le r_\xi + \delta,\ r_\eta - \delta \le |\eta| \le r_\eta + \delta\},$$

where the ordered pair (r_ξ, r_η) is the unique solution on $\mathbb{R}^+$ to the system

$$\frac{1}{2} r_\xi^2 - \frac{1}{2}(r_\xi - r_\eta)^2 - \langle r_\eta\rangle = 0$$
$$r_\xi - r_\eta\Big(1 + \frac{1}{\langle r_\eta\rangle}\Big) = 0.$$

This system corresponds to the resonance set of a phase function. We will describe the meaning of resonance set and the role A_δ plays in our proof later in Subsection 1.3. Roughly, we have that $(r_\xi, r_\eta) \approx (1.9002, 1.1466)$.

In this paper, we use the following dispersive estimates on $\mathbb{R}^2$ for $t > 0$:

$$\|e^{\frac{1}{2}i\Delta t} f\|_{L^\infty_x} \lesssim t^{-1}\|f\|_{L^1_x}$$
$$\|e^{-i\langle\nabla\rangle t} P_k g\|_{L^\infty_x} \lesssim t^{-1}\langle 2^k\rangle^2 \|P_k g\|_{L^1_x}.$$

We will also use the Coifman-Meyer theorem (see [1, 4]). The version of the theorem we will use is stated as follows:

Theorem 1.1 (Coifman-Meyer Theorem)**.** *Let $m(\xi,\eta)$ be a bounded function on $\mathbb{R}^2 \times \mathbb{R}^2$. Suppose that*

$$\sup_{k\in\mathbb{Z}} \|m(2^k\cdot)\hat{P}_0\|_{H^5_\zeta} < \infty,$$

where $\zeta = (\xi,\eta)$ and P_0 is the $k = 0$ Littlewood-Paley operator. Then the operator $T_m(\xi,\eta)$, defined by

$$\widehat{T_m(f,g)}(\xi) = \int_{\mathbb{R}^2} m(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta,$$

maps $L^p \times L^q \to L^r$, provided $2 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

1.2. **Main Results.**

We may now present our main result:

Theorem 1.2. *Suppose we have*

$$\operatorname{supp}(\hat{u}_+(\eta)) \cap \{\eta : .75 \le |\eta| \le .76\} = \emptyset,$$

and that for a sufficiently small $\delta > 0$,

$$\|u_+\|_Y + \|v_+\|_Y + \|\dot{v}_+\|_Y \le \delta.$$

Then there is a unique solution (u, v) to (1.1) such that

$$\|\langle t\rangle \int_t^\infty e^{\frac{1}{2}i(t-s)\Delta} u\cdot v ds\|_{L^\infty_t H^3_x} + \|\langle t\rangle \int_t^\infty e^{-i(t-s)\langle\nabla\rangle}\langle\nabla\rangle^{-1}(\nabla u\cdot\overline{\nabla u})ds\|_{L^\infty_t H^3_x} < \infty,$$

and as $t \to \infty$,

$$\|e^{\frac{1}{2}it\Delta}u_+ - u(t)\|_{H^3_x} + \|L(v_+, \dot{v}_+) - v(t)\|_{H^3_x} + \|\dot{L}(v_+, \dot{v}_+) - \partial_t v(t)\|_{H^2_x} \to 0.$$

In our method of proof, we may relax the support condition on $\hat{u}_+$ to the assumption

$$\hat{u}_+(\xi-\eta)[v_+(\eta)+i\langle\nabla\rangle^{-1}\dot{v}_+(\eta)]\equiv 0 \quad \text{on } A_\delta.$$

One should note that this theorem is established for non-linearities for which it is impossible to use the energy conservation laws on which previous results have relied critically.

1.3. **Overview of Method.** Our method follows the work in [7]. We briefly explain the steps for the proof as follows:

Step 1: Reformulation We transform the system into one that is first-order in time by the variable assignment $h = v + i\langle\nabla\rangle^{-1}\partial_t v$. In terms of h, the system (1.1) is then transformed to

$$\begin{gathered} i\partial_t u + \frac{1}{2}\Delta u = \pm u\,\mathrm{Re}(h) \\ -i\partial_t h + \langle\nabla\rangle h = \pm\langle\nabla\rangle^{-1}(\nabla u\cdot\overline{\nabla u}). \end{gathered}$$

We can then reformulate the problem using Duhamel's formula as:

$$\begin{pmatrix} u(t) \\ h(t)\end{pmatrix} = \begin{pmatrix} e^{i\frac{1}{2}t\Delta}u_+ \pm i\int_t^\infty e^{i\frac{1}{2}(t-s)\Delta}u\cdot\mathrm{Re}(h)ds \\ e^{-it\langle\nabla\rangle}h_+ \pm i\int_t^\infty e^{-i(t-s)\langle\nabla\rangle}\langle\nabla\rangle^{-1}(\nabla u\cdot\overline{\nabla u})ds\end{pmatrix}.$$

Step 2: First Iterate Analysis With the system reformulated, we next look to show that the natural first iterate of our contraction scheme has a $\langle t\rangle^{-1}$ decay in H^3. More specifically, we show the bilinear operators

$$\begin{gathered} B_1(f,g) = \int_t^\infty e^{i\frac{1}{2}(t-s)\Delta}f\cdot\mathrm{Re}(g)ds \\ B_2(f,g) = \int_t^\infty e^{-i(t-s)\langle\nabla\rangle}\langle\nabla\rangle^{-1}(\nabla f\cdot\overline{\nabla g})ds \end{gathered}$$

have the following property:

$$\Big\|\begin{matrix} B_1(e^{i\frac{1}{2}\Delta t}u_+, e^{-i\langle\nabla\rangle t}h_+) \\ B_2(e^{i\frac{1}{2}\Delta t}u_+, e^{i\frac{1}{2}\Delta t}u_+)\end{matrix}\Big\|_{H_x^3} \lesssim \frac{1}{\langle t\rangle}.$$

Establishing this decay on the first iterate comprises the majority of the paper. The decay estimates are achieved by going to the frequency domain and carefully analyzing the resonance points for the purpose of using stationary phase methods.

The phase functions to consider are:

$$\begin{gathered} \phi_0(\xi,\eta) = \frac{1}{2}|\xi|^2 - \frac{1}{2}|\xi-\eta|^2 + \langle\eta\rangle, \\ \phi_1(\xi,\eta) = \frac{1}{2}|\xi|^2 - \frac{1}{2}|\xi-\eta|^2 - \langle\eta\rangle, \\ \phi_2(\xi,\eta) = \langle\xi\rangle - \frac{1}{2}|\xi-\eta|^2 + \frac{1}{2}|\eta|^2. \end{gathered}$$

We call the sets where $\partial_\eta\phi_i = 0$ and $\phi_i = 0$ the space resonance and time resonance sets of ϕ_i, respectively. Roughly, the method is to integrate by parts in frequency space away from the space resonance set and to integrate by parts in time away from the time resonance set. Those points where both $\partial_\eta\phi_i = 0$ and $\phi_i = 0$ comprise the resonance set and cause the greatest difficulty as we cannot integrate by parts in either variable.

When $i = 0, 2$, the resonance set is empty. Thus, with carefully chosen cutoff functions, we can always integrate by parts in either frequency space or time. The phase function ϕ_1 is the most problematic and requires the most delicate treatment. This is due to the existence of a resonant set. We have

$$\partial_\eta \phi_1(\xi,\eta) = \xi - \eta\Big(1 + \frac{1}{\langle \eta \rangle}\Big).$$

Hence, $\partial_\eta \phi_1(\xi,\eta)$ and $\phi_1(\xi,\eta)$ are both zero if ξ and η are co-linear and the moduli of ξ and η solve the system

$$\begin{gathered}\frac{1}{2}|\xi|^2 - \frac{1}{2}(|\xi| - |\eta|)^2 - \langle |\eta| \rangle = 0\,,\\ |\xi| - |\eta|\Big(1 + \frac{1}{\langle |\eta| \rangle}\Big) = 0\,.\end{gathered}$$

Hence, the resonance set takes the form

$$\{(\xi,\eta):\ \xi \parallel \eta,\ |\xi| = r_\xi,\ |\eta| = r_\eta\}.$$

It is because of this set that we must assume

$$\hat{u}_+(\xi-\eta)\hat{h}_+(\eta) \equiv 0 \quad \text{on } A_\delta.$$

The method of stationary phase for oscillatory integrals is well known and many classical results can be found in [13]. The idea of carefully analyzing resonance points is presented in [2].

Step 3: Bilinear Estimates We introduce the space X where we construct our solutions' non-linear terms by its norm

$$\|f\|_X = \|\langle t \rangle f\|_{L_t^\infty H_x^3}.$$

Through the use of Sobolev embedding, we establish the following estimates on the above-defined bilinear operators:

$$\begin{gathered}\|B_i(f,g)\|_X \lesssim \|f\|_X \cdot \|g\|_X,\\ \|B_i(f,g)\|_X \lesssim \|\langle t \rangle f\|_{L_t^\infty W^{3,\infty}([0,\infty)\times\mathbb{R}^2)} \cdot \|g\|_X,\\ \|B_i(f,g)\|_X \lesssim \|f\|_X \cdot \|\langle t \rangle g\|_{L_t^\infty W^{3,\infty}([0,\infty)\times\mathbb{R}^2)}.\end{gathered}$$

These estimates are used to establish our iterative scheme as a contraction.

Step 4: Contraction in the X-Space Finally, we define two iteration schemes as follows:

$$\begin{gathered}u_1 = e^{i\frac{1}{2}\Delta t} u_+,\\ h_1 = e^{-i\langle \nabla \rangle t} h_+,\end{gathered}$$

and

$$\begin{gathered}u_{k+1} = u_1 + B_1(u_k, h_k),\\ h_{k+1} = h_1 + B_2(u_k, u_k).\end{gathered}$$

These are then shown to be contractions in X. This is done through induction. In the process, we use the dispersive estimates of the linear propagators, the first iterate estimates established in Step 2, and the bilinear estimates from Step 3.

Remark: One should note that because the linear operators for the Schrödinger and Klein-Gordon equations are isometries in H^3, the linear solutions to both equations do not exist in the space X. In this way, our scheme is very different from any perturbation methods.

The remainder of the paper is organized as follows:

In Section 2, we transform (1.1) into a system that is first-order in time. We then introduce the X space and establish the bilinear estimates on X. Finally, we prove the Duhamel operators are contractions on X using induction and the bilinear estimates. Part of this proof relies on the assumption that the first iterate of our scheme is sufficiently small in X.

In Section 3, which accounts for the bulk of the paper, we prove that the assumption on the smallness of the first iterate is valid if the final data (u_+, h_+) is sufficiently small in some suitable space. This is done by resonance analysis and stationary phase calculations.

2. Reformulation, bilinear estimates, and contraction in the X-space

2.1. **Reformulation.**

In this section, we will show the Duhamel operator maps are contractions. To this end, we rewrite the system (1.1) into one that is first-order in time. To do this, we introduce the variable $h = v + i\langle\nabla\rangle^{-1}\partial_t v$. (1.1) is then transformed to

$$\begin{gathered} i\partial_t u + \frac{1}{2}\Delta u = \pm u\,\mathrm{Re}(h) \\ -i\partial_t h + \langle\nabla\rangle h = \pm\langle\nabla\rangle^{-1}(\nabla u\cdot\overline{\nabla u}). \end{gathered} \tag{2.1}$$

If (u, h) are to scatter to free solutions (u_+, h_+) in H^3, then we need

$$\|u - e^{\frac{1}{2}it\Delta}u_+\|_{H^3} + \|h - e^{-it\langle\nabla\rangle}h_+\|_{H^3} \to 0 \quad \text{as } t\to\infty. \tag{2.2}$$

Because the linear operators $e^{\frac{1}{2}it\Delta}$ and $e^{-it\langle\nabla\rangle}$ are both isometries in H^3, (2.2) is equivalent to

$$\|e^{-\frac{1}{2}it\Delta}u - u_+\|_{H^3} + \|e^{it\langle\nabla\rangle}h - h_+\|_{H^3} \to 0 \quad \text{as } t\to\infty.$$

Combining this with the Duhamel formulas for u and h, we obtain

$$\begin{pmatrix} u(t) \\ h(t) \end{pmatrix} = \begin{pmatrix} e^{i\frac{1}{2}t\Delta}u_+ \pm i\int_t^\infty e^{i\frac{1}{2}(t-s)\Delta}u\cdot\mathrm{Re}(h)ds \\ e^{-it\langle\nabla\rangle}h_+ \pm i\int_t^\infty e^{-i(t-s)\langle\nabla\rangle}\langle\nabla\rangle^{-1}(\nabla u\cdot\overline{\nabla u})ds \end{pmatrix}. \tag{2.3}$$

Thus, we formally expect solutions to (1.1) to have the form of (2.3). We will find a pair $(u, h)^T$ that satisfies (2.3) and whose non-linear components are in X. The work of showing that $(u, h)^T$ satisfies the conditions of (1.2) is straightforward and omitted.

2.2. **Bilinear Estimates.**

Based on (2.3), we define the following bilinear operators:

$$\begin{gathered} B_1(f, g) = \int_t^\infty e^{i\frac{1}{2}(t-s)\Delta}f\cdot\mathrm{Re}(g)ds \\ B_2(f, g) = \int_t^\infty e^{-i(t-s)\langle\nabla\rangle}\langle\nabla\rangle^{-1}(\nabla f\cdot\overline{\nabla g})ds. \end{gathered}$$

Through these operators, we define our two iteration schemes:

$$\begin{gathered} u_1 = e^{i\frac{1}{2}\Delta t}u_+, \\ h_1 = e^{-i\langle\nabla\rangle t}h_+, \end{gathered}$$

and

$$u_{k+1} = u_1 + B_1(u_k, h_k),$$

$$h_{k+1} = h_1 + B_2(u_k, u_k).$$

To prove these schemes are contractions, we must show the bilinear operators have particular algebra estimates:

Lemma 2.1. *For $i = 1, 2$, we have*

$$\|B_i(f,g)\|_X \lesssim \|f\|_X \cdot \|g\|_X, \tag{2.4}$$

$$\|B_i(f,g)\|_X \lesssim \|\langle t\rangle f\|_{L_t^\infty W^{3,\infty}([0,\infty)\times\mathbb{R}^2)} \cdot \|g\|_X, \tag{2.5}$$

$$\|B_i(f,g)\|_X \lesssim \|f\|_X \cdot \|\langle t\rangle g\|_{L_t^\infty W^{3,\infty}([0,\infty)\times\mathbb{R}^2)}, \tag{2.6}$$

where the space X is defined by the norm

$$\|g\|_X = \|\langle t\rangle g\|_{L_t^\infty H_x^3}.$$

Proof. This is a matter of direct calculation. For $i = 2$, by the Minkowski inequality,

$$\|B_2(f,g)\|_X \lesssim \Big\|\langle t\rangle \int_t^\infty \|\nabla f \cdot \overline{\nabla g}(s)\|_{H^2} ds\Big\|_{L_t^\infty([0,\infty))}.$$

Next, by the Hölder inequality and Sobolev embedding,

$$\begin{aligned}\|\nabla f \cdot \overline{\nabla g}(s)\|_{H^2} &\lesssim \|\nabla^3 f\|_2 \|\nabla g\|_\infty + \|\nabla^2 f\|_4 \|\nabla^2 g\|_4 + \|\nabla f\|_\infty \|\nabla^3 g\|_2 \\ &\lesssim \langle s\rangle^{-2} \|f\|_X \|g\|_X.\end{aligned}$$

Combining these estimates, we have

$$\|B_2(f,g)\|_X \lesssim \sup_{t\geq 0} \langle t\rangle \int_t^\infty \langle s\rangle^{-2} ds \cdot \|f\|_X \cdot \|g\|_X \lesssim \|f\|_X \|g\|_X.$$

This completes the proof of (2.4). The proofs for (2.5) and (2.6) and the case $i = 1$ are similar and we omit them. □

2.3. **Contraction in the X-Space.** With Lemma (2.1), we may now show our iteration schemes are contractions in the X space with fixed points satisfying (2.3). We will use the estimate

$$\|e^{itD} f\|_{L^\infty} \lesssim \frac{1}{\langle t\rangle} \|f\|_{B^3_{1,1}},$$

where $D = \frac{1}{2} t\Delta$ or $-\langle\nabla\rangle$. For $t > 1$, this bound comes from the dispersive estimates and the inequality

$$\|f\|_{L^\infty} \lesssim \sum_{k\in\mathbb{Z}} \|P_k f\|_{L^\infty}.$$

For $t \leq 1$, we use Sobolev embedding. Denote $w_{k+1} = u_{k+1} - u_1$ and $z_{k+1} = h_{k+1} - h_1$ for $k \geq 1$. We are interested in the differences between successive values of w_k and z_k as, by construction, they are also the differences of successive values of u_k and h_k, respectively. For $k \geq 2$, we have

$$\begin{aligned}\|z_{k+2} - z_{k+1}\|_X &= \|B_2(u_1 + w_{k+1}, u_1 + w_{k+1}) - B_2(u_1 + w_k, u_1 + w_k)\|_X \\ &= \|B_2(w_{k+1} - w_k, u_1 + w_{k+1}) + B_2(u_1 + w_k, w_{k+1} - w_k)\|_X \\ &\lesssim \|w_{k+1} - w_k\|_X (\|w_k\|_X + \|w_{k+1}\|_X + \|\langle t\rangle u_1\|_{L_t^\infty W^{3,\infty}}) \\ &\lesssim \|w_{k+1} - w_k\|_X (\|w_k\|_X + \|w_{k+1}\|_X + \|u_+\|_{B^6_{1,1}}).\end{aligned} \tag{2.7}$$

Similarly,

$$\begin{aligned}\|w_{k+2}-w_{k+1}\|_X &= \|B_1(u_1+w_{k+1},h_1+z_{k+1})-B_1(u_1+w_k,h_1+z_k)\|_X\\ &= \|B_1(w_{k+1}-w_k,h_1+z_{k+1})+B_1(u_1+w_k,z_{k+1}-z_k)\|_X\\ &\lesssim \|w_{k+1}-w_k\|_X(\|z_{k+1}\|_X+\|h_+\|_{B^6_{1,1}})\\ &\quad+\|z_{k+1}-z_k\|_X(\|w_k\|_X+\|u_+\|_{B^6_{1,1}}).\end{aligned} \tag{2.8}$$

With these estimates, we prove by induction that under suitable conditions on the final data the iteration scheme is a contraction.

Lemma 2.2. *Let w_j, z_j, and the space X be as defined above. Assume*

$$\|u_+\|_{B^6_{1,1}},\ \|h_+\|_{B^6_{1,1}}\le\delta$$

and

$$\|B_1(u_1,h_1)\|_X,\ \|B_2(u_1,u_1)\|_X\le\delta$$

for some sufficiently small δ. Then for $n\ge 3$, we have

$$\|w_j\|_X,\ \|z_j\|_X\le\delta\cdot\Big(\sum_{j=0}^{n}\frac{1}{2^l}\Big),\quad \forall 2\le j\le n.$$

Proof. This is a proof by induction on n. For $n=3$, we have

$$\begin{aligned}\|w_3\|_X &= \|B_1(w_2+u_1,z_2+h_1)\|_X\\ &\le C_0\|w_2\|_X(\|z_2\|_X+\|h_+\|_{B^6_{1,1}})+\|z_2\|_X\|u_+\|_{B^6_{1,1}}+\|B_1(u_1,h_1)\|_X\\ &\le C_1\cdot\delta^2+\delta.\end{aligned}$$

$$\begin{aligned}\|z_3\|_X &= \|B_2(w_2+u_1,w_2+h_1)\|_X\\ &\le C_2\|w_2\|_X(\|u_+\|_{B^6_{1,1}}+\|h_+\|_{B^6_{1,1}})+\|B_2(u_1,h_1)\|_X\\ &\le C_3\cdot\delta^2+\delta.\end{aligned}$$

Choosing δ sufficiently small completes the base case. Now consider $j=n+1$. By equations (2.7) and (2.8), we have

$$\begin{aligned}\|w_{n+1}-w_n\|_X &\le C\cdot\|w_n-w_{n-1}\|_X\cdot(\|z_n\|_X+\|h_+\|_{B^6_{1,1}})\\ &\quad+C\cdot\|z_n-z_{n-1}\|_X\cdot(\|w_{n-1}\|_X+\|u_+\|_{B^6_{1,1}})\\ &\le (C\cdot 3\delta)^{n-2}\cdot(\|w_3-w_2\|_X+\|z_3-z_2\|_X)\\ &\le (C\cdot 3\delta)^{n-2}\cdot 6\delta.\end{aligned}$$

$$\begin{aligned}\|z_{n+1}-z_n\|_X &\le C'\cdot\|w_n-w_{n-1}\|_X(\|w_n\|_X+\|w_{n-1}\|_X+\|u_+\|_{B^6_{1,1}})\\ &\le (C'\cdot 5\delta)(C\cdot 6\delta)^{n-1}\cdot\|w_3-w_2\|_X\\ &\le (C'\cdot 5\delta)(C\cdot 6\delta)^{n-1}\cdot 3\delta.\end{aligned}$$

Therefore, for sufficiently small δ,

$$\|w_{n+1}\|_X\le\|w_n\|_X+\delta\cdot\frac{1}{2^{n+1}}\le\delta\cdot\Big(\sum_{l=0}^{n+1}\frac{1}{2^l}\Big),$$

$$\|z_{n+1}\|_X\le\|z_n\|_X+\delta\cdot\frac{1}{2^{n+1}}\le\delta\cdot\Big(\sum_{l=0}^{n+1}\frac{1}{2^l}\Big).$$

□

Corollary 2.3. *With the hypothesis of Lemma 2.2, w_k and z_k are Cauchy sequences in X and therefore have strong limits in X.*

Using this corollary, we may show the existence of solutions to (2.3) for positive time values if we assume small u_+ and h_+ data in $B^6_{1,1}$ and that the first iterate of our scheme is sufficiently small in the X-space. The remainder of the paper will demonstrate that the latter can be achieved if we assume some suitable norm of the data u_+ and h_+ is small.

3. Analysis of the first iterate

The purpose of this section is to prove that under suitable conditions on (u_+, h_+), $B_1(u_1, h_1)$ and $B_2(u_1, u_1)$ have a decay of $\langle t\rangle^{-1}$ in H^3.

By Plancherel's theorem, it is sufficient to measure an L^2 norm in the frequency domain. By definition, we have

$$\begin{pmatrix} B_1(u_1,h_1) \\ B_2(u_1,u_1)\end{pmatrix} = \begin{pmatrix} \int_t^\infty e^{i\frac12(t-s)\Delta}\big[\big(e^{i\frac12 s\Delta}u_+\big)\operatorname{Re}\big(e^{-is\langle\nabla\rangle}h_+\big)\big]ds \\ \int_t^\infty e^{-i(t-s)\langle\nabla\rangle}\langle\nabla\rangle^{-1}\big[\nabla\big(e^{i\frac12 s\Delta}u_+\big)\cdot\overline{\nabla\big(e^{i\frac12 s\Delta}u_+\big)}\big]ds\end{pmatrix}. \tag{3.1}$$

Hence, in the frequency domain,

$$\begin{aligned} \Big\|\begin{pmatrix} B_1(u_1,h_1) \\ B_2(u_1,u_1)\end{pmatrix}\Big\|_{H^3_x} &\lesssim \Big\|\begin{pmatrix} \langle\xi\rangle^3\mathcal{F}(B_1(u_1,h_1)) \\ \langle\xi\rangle^3\mathcal{F}(B_2(u_1,u_1))\end{pmatrix}\Big\|_{L^2_\xi} \\ &\lesssim \Big\|\begin{pmatrix} \langle\xi\rangle^3\int_t^\infty\int_{\mathbb{R}^2} e^{is\phi_j(\xi,\eta)}\hat u_+(\xi-\eta)\hat g_j(\eta)d\eta ds \\ \langle\xi\rangle^2\int_t^\infty\int_{\mathbb{R}^2} e^{is\phi_2(\xi,\eta)}(\xi-\eta)\hat u_+(\xi-\eta)\eta\bar{\hat u}_+(-\eta)d\eta ds\end{pmatrix}\Big\|_{L^2_\xi}. \end{aligned}$$

Where $j = 0, 1$; the phase functions ϕ_0, ϕ_1, and ϕ_2 take the form

$$\begin{aligned} \phi_0(\xi,\eta) &= \frac12|\xi|^2 - \frac12|\xi-\eta|^2 + \langle\eta\rangle, \\ \phi_1(\xi,\eta) &= \frac12|\xi|^2 - \frac12|\xi-\eta|^2 - \langle\eta\rangle, \\ \phi_2(\xi,\eta) &= \langle\xi\rangle - \frac12|\xi-\eta|^2 + \frac12|\eta|^2; \end{aligned}$$

and $g_0(\xi) = \bar{\hat h}_+(-\xi)$, $g_1(\xi) = \hat h_+(\xi)$.

For readability, we will divide the work of establishing the necessary frequency bounds into several lemmas and propositions.

3.1. **Preliminary lemmas.**

Lemma 3.1. *Let $h(\xi,\eta)$ be a bounded function, then for all $n \in \mathbb{N}$ and $f, g \in H^n$, we have*

$$\int_{\mathbb{R}^2} |h(\xi,\eta)\hat f(\xi-\eta)\hat g(\eta)|d\eta \lesssim \frac{1}{\langle\xi\rangle^n}\|f\|_{H^n_x}\|g\|_{H^n_x}.$$

Proof. First, by the triangle inequality, we have $\langle\xi\rangle \le \langle\xi-\eta\rangle + \langle\eta\rangle$. Hence, $\langle\xi\rangle^n \lesssim \langle\xi-\eta\rangle^n + \langle\eta\rangle^n$. Thus, since $h(\xi,\eta)$ is assumed to be bounded, we have

$$\begin{aligned} \int_{\mathbb{R}^2} |h(\xi,\eta)\hat f(\xi-\eta)\hat g(\eta)|d\eta &\lesssim \int_{\mathbb{R}^2}\Big|\frac{\langle\xi-\eta\rangle^n + \langle\eta\rangle^n}{\langle\xi\rangle^n}\hat f(\xi-\eta)\hat g(\eta)\Big|d\eta \\ &= \frac{1}{\langle\xi\rangle^n}\int_{\mathbb{R}^2}|\langle\xi-\eta\rangle^n\hat f(\xi-\eta)\hat g(\eta) + \langle\eta\rangle^n\hat g(\eta)\hat f(\xi-\eta)|d\eta \end{aligned}$$

$$\lesssim \frac{1}{\langle\xi\rangle^n}(\|\langle\xi-\cdot\rangle^n\hat{f}(\xi-\cdot)\hat{g}(\cdot)\|_{L^1_\eta}+\|\langle\cdot\rangle^n\hat{g}(\cdot)\hat{f}(\xi-\cdot)\|_{L^1_\eta}).$$

By Hölder, we then have

$$\begin{aligned}\int_{\mathbb{R}^2}|h(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)|d\eta &\lesssim \frac{1}{\langle\xi\rangle^n}(\|f\|_{H^n_x}\|g\|_{L^2_x}+\|g\|_{H^n_x}\|f\|_{L^2_x})\\ &\lesssim \frac{1}{\langle\xi\rangle^n}\|f\|_{H^n_x}\|g\|_{H^n_x}.\end{aligned}$$

□

Lemma 3.2 (Short Time Control). *Suppose* $0\le t\le 1$ *and* $f,g\in H^5_x$, *then for* $j=0,1,2$;

$$\Big\|\langle\xi\rangle^3\int_t^1\int_{\mathbb{R}^2}e^{is\phi_j(\xi,\eta)}\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta ds\Big\|_{L^2_\xi}\lesssim\frac{1}{\langle t\rangle}\|f\|_{H^5}\|g\|_{H^5}.$$

Proof. Using Lemma 3.1, we have

$$\begin{aligned}|\int_t^1\int_{\mathbb{R}^2}e^{is\phi_j(\xi,\eta)}\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta ds| &\lesssim \int_t^1\frac{1}{\langle\xi\rangle^5}\|f\|_{H^5_x}\|g\|_{H^5_x}ds\\ &\lesssim \frac{1}{\langle\xi\rangle^5}\|f\|_{H^5_x}\|g\|_{H^5_x}.\end{aligned}$$

Hence,

$$\begin{aligned}\Big\|\langle\xi\rangle^3\int_t^1\int_{\mathbb{R}^2}e^{is\phi_j(\xi,\eta)}\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta ds\Big\|_{L^2_\xi} &\lesssim \|f\|_{H^5_x}\|g\|_{H^5_x}\|\frac{1}{\langle\xi\rangle^2}\|_{L^2_\xi}\\ &\lesssim \frac{1}{\langle t\rangle}\|f\|_{H^5_x}\|g\|_{H^5_x}.\end{aligned}$$

□

Using Lemma 3.2, it is now sufficient to estimate the first iterate just for the case $t\ge 1$. From henceforth, we will assume $t\ge 1$.

In the following lemma, we show how, away from the space resonance set, one may integrate by parts in time to obtain a $\langle t\rangle^{-1}$ decay in H^3.

Lemma 3.3 (Decay Away from Space Resonance). *Let* $\phi:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ *such that* $|\partial_\eta\phi(\xi,\eta)|\gtrsim s^{-\alpha_1}$ *and* $|\partial^k_\eta\phi(\xi,\eta)|\lesssim 1$ *for all* $k\ge 2$. *Further, suppose* $h(\xi,\eta)$ *is a smooth function with* $|\partial^j_\eta h(\xi,\eta)|\lesssim s^{j\cdot\alpha_2}$ *for all* $j\ge 1$. *If* $2\alpha_1+\alpha_2\le\frac{2}{3}$, *then*

$$\Big\|\langle\xi\rangle^3\int_t^\infty\int_{\mathbb{R}^2}e^{is\phi(\xi,\eta)}h(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta ds\Big\|_{L^2_\xi}\lesssim\frac{1}{t}\|f\|_Y\|g\|_Y.$$

Proof. Define the operator D_ϕ on a sufficiently smooth scalar-valued function $f(\xi,\eta)$ by

$$D_\phi f(\xi,\eta)=\frac{\nabla_\eta\phi(\xi,\eta)}{is|\nabla_\eta\phi(\xi,\eta)|^2}\cdot\nabla_\eta f(\xi,\eta).$$

By the inequality $2\alpha_1+\alpha_2\le 2/3$, we have that there exists a natural number $N\le 6$ such that $N(1-2\alpha_1-\alpha_2)\ge 2$. Observe that

$$D^N_\phi(e^{is\phi})=e^{is\phi},$$

so we may integrate by parts N times using D_ϕ. This yields one principal term and N boundary terms. The boundary terms each vanish due to the decay assumptions on f and g and the assumption that $\partial_\eta \phi(\xi,\eta)$ is bounded below. Thus, we are only left to consider the principal term.

$$\begin{aligned} &\Big|\int_t^\infty \int_{\mathbb{R}^2} e^{is\phi(\xi,\eta)} h(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta ds\Big| \\ &= \Big|\int_t^\infty \int_{R^2} e^{is\phi(\xi,\eta)} (D^t)^N [h(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)]d\eta ds\Big|. \end{aligned} \tag{3.2}$$

Let X represent the right-hand side of (3.2). Inductively, one can establish the bound

$$\begin{aligned} X \lesssim \int_t^\infty \int_{\mathbb{R}^2} &\frac{1}{s^N}\Big(\frac{1}{|\nabla_\eta \phi(\xi,\eta)|^{2N}} + \frac{1}{|\nabla_\eta \phi(\xi,\eta)|^N}\Big) \\ &\times \sum_{\beta_i \le N} |\partial_\eta^{\beta_1} h(\xi,\eta)\partial_\eta^{\beta_2}\hat{f}(\xi-\eta)\partial^{\beta_3}\hat{g}(\eta)|d\eta ds. \end{aligned}$$

Hence,

$$\begin{aligned} X &\lesssim \int_t^\infty \int_{\mathbb{R}^2} \frac{s^{N\cdot\alpha_2}}{s^{N(1-2\alpha_1)}} \sum_{\beta_2,\beta_3\le N} |\partial_\eta^{\beta_2}\hat{f}(\xi-\eta)\partial_\eta^{\beta_3}\hat{g}(\eta)|d\eta ds \\ &\lesssim \int_t^\infty \frac{1}{s^2}\frac{1}{\langle\xi\rangle^5} \sum_{\beta_2,\beta_3\le N} (\|\mathcal{F}^{-1}(\partial_\eta^{\beta_2}\hat{f})\|_{H_x^5} \|\mathcal{F}^{-1}(\partial_\eta^{\beta_3}\hat{g})\|_{H_x^5}) \\ &\lesssim \frac{1}{\langle\xi\rangle^5 t} \sum_{\beta_2,\beta_3\le N} (\|\mathcal{F}^{-1}(\partial_\eta^{\beta_2}\hat{f})\|_{H_x^5} \|\mathcal{F}^{-1}(\partial_\eta^{\beta_3}\hat{g})\|_{H_x^5}). \end{aligned}$$

□

We may preform the same calculations with $(\xi-\eta)\hat{f}(\xi-\eta)$ and $\eta\hat{g}(\eta)$ replacing $\hat{f}(\xi-\eta)$ and $\hat{g}(\eta)$, respectively, to obtain the following corollary:

Corollary 3.4. *Let $\phi(\xi,\eta)$ and $h(\xi,\eta)$ satisfy the conditions of Lemma 3.3. Then*

$$\Big\|\langle\xi\rangle^2 \int_t^\infty \int_{\mathbb{R}^2} e^{is\phi(\xi,\eta)} h(\xi,\eta)(\xi-\eta)\hat{f}(\xi-\eta)\eta\hat{g}(\eta)d\eta ds\Big\|_{L_\xi^2} \lesssim \frac{1}{t}\|f\|_Y\|g\|_Y.$$

Now that we have established a $\langle t\rangle^{-1}$ decay away from the frequency space resonance set, we will move on to establishing this decay away from the time resonance set. This will be accomplished by the use of integration in time and the following lemma:

Lemma 3.5. *Let $h(\xi,\eta)$ be a smooth, bounded function. Suppose that for $0 \le \gamma_1,\gamma_2 \le 5$ we have*

$$|\partial_\xi^{\gamma_1}\partial_\eta^{\gamma_2} h(\xi,\eta)| \lesssim \langle\xi\rangle^5 + \langle\eta\rangle^5.$$

Then for $j = 0,1,2$;

$$\Big\|\langle\xi\rangle^3 \int_{\mathbb{R}^2} e^{it\phi_j(\xi,\eta)} h(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta\Big\|_{L_\xi^2} \lesssim \frac{1}{t}\|f\|_{W_x^{15,1}}\|g\|_{H_x^{15}}. \tag{3.3}$$

Proof. Let K denote the left-hand side of (3.3). Then by Plancherel, we see

$$K = \|T_h(e^{\frac{1}{2}it\Delta}f, e^{itD_j}g)\|_{H_x^3}.$$

Here, T_h is a bilinear operator with symbol $h(\xi,\eta)$; D_0, D_1 and D_2 are the operators $\langle\nabla\rangle$, $-\langle\nabla\rangle$ and $-\Delta/2$, respectively. More precisely,

$$\widehat{T_h(f_1,f_2)}(\xi) = \int h(\xi,\eta)\hat{f}_1(\xi-\eta)\hat{f}_2(\eta)d\eta.$$

Rewriting, we have

$$\begin{aligned} T_h(e^{\frac{1}{2}it\Delta}f, e^{itD_j}g) &= \int \frac{h(\xi,\eta)}{\langle\eta\rangle^{12}\langle\xi-\eta\rangle^{12}} \widehat{e^{\frac{1}{2}it\Delta}\langle\nabla\rangle^{12}f}(\xi-\eta)\widehat{e^{itD_j}\langle\nabla\rangle^{12}g}(\eta)d\eta \\ &= T_{\tilde{h}}(e^{\frac{1}{2}it\Delta}\langle\nabla\rangle^{12}f, e^{itD_j}\langle\nabla\rangle^{12}g), \end{aligned}$$

where

$$\tilde{h}(\xi,\eta) = \frac{h(\xi,\eta)}{\langle\eta\rangle^{12}\langle\xi-\eta\rangle^{12}}.$$

Therefore,

$$\begin{aligned} K &= \|T_h(e^{\frac{1}{2}it\Delta}f, e^{itD_j}g)\|_{H^3_x} \\ &\lesssim \sum_{0\le\alpha\le3}\Big\|\xi^\alpha\int \tilde{h}(\xi,\eta)\widehat{e^{\frac{1}{2}it\Delta}\langle\nabla\rangle^{12}f}(\xi-\eta)\widehat{e^{itD_j}\langle\nabla\rangle^{12}g}(\eta)d\eta\Big\|_{L^2_\xi} \\ &\lesssim \sum_{|\alpha'|\le3}\sum_{(\alpha_1,\alpha_2)=\alpha'}\Big\|(\xi-\eta)^{\alpha_1}\eta^{\alpha_2} \\ &\quad\times\int \tilde{h}(\xi,\eta)\langle\xi-\eta\rangle^{12}\widehat{e^{\frac{1}{2}it\Delta}f}(\xi-\eta)\langle\eta\rangle^{12}\widehat{e^{itD_j}g}(\eta)d\eta\Big\|_{L^2_\xi} \\ &\lesssim \sum_{|\alpha'|\le3}\sum_{(\alpha_1,\alpha_2)=\alpha'}\big\|T_{\tilde{h}}(e^{\frac{1}{2}it\Delta}\langle\nabla\rangle^{12}\partial^{\alpha_1}f, e^{itD_j}\langle\nabla\rangle^{12}\partial^{\alpha_2}g)\big\|_{L^2_x}. \end{aligned}$$

Because we have assumed that for $0<\gamma_1,\gamma_2\le5$,

$$|\partial_\xi^{\gamma_1}\partial_\eta^{\gamma_2}h(\xi,\eta)| \lesssim \langle\xi\rangle^5+\langle\eta\rangle^5,$$

$\tilde{h}$ is a Coifman-Meyer multiplier. Hence, using the Coifman-Meyer multiplier theorem and the dispersive estimate for the Schrödinger's fundamental solution, we have

$$\begin{aligned} &\sum_{0\le\alpha_1,\alpha_2\le3}\big\|T_{\tilde{h}}(e^{\frac{1}{2}it\Delta}\langle\nabla\rangle^{12}\partial^{\alpha_1}f, e^{itD_j}\langle\nabla\rangle^{12}\partial^{\alpha_2}g)\big\|_{L^2_x} \\ &\lesssim \sum_{0\le\alpha_1\le3}\big\|e^{\frac{1}{2}it\Delta}\langle\nabla\rangle^{12}\partial^{\alpha_1}f\big\|_\infty\|g\|_{H^{15}} \\ &\lesssim \frac{1}{t}\|\langle\nabla\rangle^{15}f\|_{L^1_x}\|g\|_{H^{15}}. \end{aligned}$$

□

With Lemma 3.5, we may now show how, away from the time resonance set, one may use integration in time to obtain a $\langle t\rangle^{-1}$ decay in H^3.

Lemma 3.6 (Decay Away Time Resonance)**.** *Let $h(\xi,\eta)$ satisfy the bounds*

$$|\partial_\xi^\alpha\partial_\eta^\beta h(\xi,\eta)| \lesssim 1$$

for $0 \le \alpha, \beta \le 5$. *Furthermore, suppose* $\phi_j(\xi,\eta) \gtrsim 1$ *for* $j = 0,1,2$. *Then for* $f, g \in Y$, *we have*

$$\Big\| \langle\xi\rangle^3 \int_t^\infty \int_{\mathbb{R}^2} e^{is\phi_j(\xi,\eta)} h(\xi,\eta) \hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta\,ds \Big\|_{L^2_\xi} \lesssim \frac{1}{t}\|f\|_Y \|g\|_Y.$$

Remark: For our purposes, $h(\xi,\eta)$ will typically be a smooth cutoff function.

Proof. Integration in time yields

$$\begin{aligned} &\Big| \int_t^\infty \int_{\mathbb{R}^2} e^{is\phi_j(\xi,\eta)} h(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta\,ds \Big| \\ &\lesssim \lim_{M\to\infty} \Big| \int_{\mathbb{R}^2} e^{iM\phi_j(\xi,\eta)} \frac{h(\xi,\eta)}{\phi_j(\xi,\eta)} \hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta \Big| \quad (3.4)\\ &\quad + \Big| \int_{\mathbb{R}^2} e^{it\phi_j(\xi,\eta)} \frac{h(\xi,\eta)}{\phi_j(\xi,\eta)} \hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta \Big|. \quad (3.5) \end{aligned}$$

We wish to control both of these terms by using Lemma 3.5. Hence, we must verify that for $0 \le \gamma_1, \gamma_2 \le 5$,

$$\partial_\xi^{\gamma_1}\partial_\eta^{\gamma_2}\Big(\frac{h(\xi,\eta)}{\phi_j(\xi,\eta)}\Big) \lesssim \langle\xi\rangle^5 + \langle\eta\rangle^5.$$

By the bounds on $h(\xi,\eta)$ and $\phi_j(\xi,\eta)$, we have

$$|\partial_\xi^{\gamma_1}\partial_\eta^{\gamma_2}(h(\xi,\eta)\cdot\phi_j^{-1}(\xi,\eta))| \lesssim \sum_{0\le\gamma_1,\gamma_2\le 5} |\partial_\xi^{\gamma_1}\partial_\eta^{\gamma_2}(\phi_j^{-1}(\xi,\eta))|.$$

One can verify that for a multi-index γ with $|\gamma| \ge 2$, $|\partial^\gamma \phi_j(\xi,\eta)| \lesssim 1$. Hence,

$$\begin{aligned} \sum_{0\le\gamma_1,\gamma_2\le 5} |\partial_\xi^{\gamma_1}\partial_\eta^{\gamma_2}(\phi_j^{-1}(\xi,\eta))| &\lesssim 1 + |\partial_\xi\phi_j(\xi,\eta)|^5 + |\partial_\eta\phi_j(\xi,\eta)|^5 \\ &\lesssim 1 + |\eta|^5 + |\xi|^5. \end{aligned}$$

Thus, we may use Lemma 3.5 to conclude

$$\|\langle\xi\rangle^3(3.5)\|_{L^2_\xi} \lesssim \frac{1}{t}\|f\|_{W_x^{15,1}}\|g\|_{H_x^{15}}.$$

Similarly,

$$\|(3.4)\|_{L^2_\xi} \lesssim \lim_{M\to\infty}\frac{1}{M}\|f\|_{W_x^{12,1}}\|g\|_{H_x^{12}} = 0.$$

Furthermore, one can verify that

$$\|(3.4)\|_{H^2_\xi} \lesssim \sum_{|\alpha|\le 2}\|x^\alpha f\|_{H_x^5}\|g\|_{H_x^5} \lesssim \|f\|_Y\|g\|_Y.$$

By Sobolev embedding, (3.4) is then an L^∞_ξ function whose L^2_ξ-norm is zero. Hence, it is identically zero for all ξ. □

With these lemmas in place, we are now ready to establish the necessary decay on $B_1(u_1,h_1)$ and $B_2(u_1,u_1)$. Recall that u_1 and h_1 are the linear evolutions of u and h respectively and that in the frequency space, $B_1(u_1,h_1)$ and $B_2(u_1,u_1)$ have the form

$$B_1(u_1,h_1) = \int_t^\infty \int_{\mathbb{R}^2} e^{is\phi_j(\xi,\eta)}\hat{u}_+(\xi-\eta)\hat{g}_j(\eta)\,d\eta\,ds$$

$$B_2(u_1,u_1)=\int_t^\infty\int_{\mathbb{R}^2} e^{is\phi_2(\xi,\eta)}(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta ds,$$

where $g_0(\xi)=\bar{\hat{h}}_+(-\xi)$ and $g_1(\xi)=\hat{h}_+(\xi)$.

The main idea is to identify the space and time resonance sets of each phase function ϕ_j. Away from the space resonance set, we can establish the $\langle t\rangle^{-1}$ decay using Lemma 3.3; away from the time resonance set, we use Lemma 3.6. Points where both ϕ_j and $\partial_\eta\phi_j$ are identically zero are in the resonance set. We avoid difficulties at these points by placing proper assumptions on the support of $\hat{u}_+$ and $\hat{h}_+$.

For the purposes of separating the integration regime based on resonance sets, we will use the function φ, which was defined in Subsection 1.1 and has the property that $\varphi(r)=1$ for $0\le r$ and $\varphi(r)=0$ for $r>2$.

Recall that by Lemma 3.2, we need only establish decay for $t\ge 1$. Hence, it is enough to show the H^3 norms of $B_1(u_1,h_1)$ and $B_2(u_1,u_1)$ have a t^{-1} decay.

3.2. **Analysis of ϕ_0.** We first estimate $\|\langle\xi\rangle^3\mathcal{F}(B_1(u_1,h_1)\|_{L^2_\xi}$. We begin by considering the case when the phase function is ϕ_0. In this case, one can verify that the set of (ξ,η) for which $\phi_0(\xi,\eta)=0$ is disjoint from the set for which $\partial_\eta\phi_0(\xi,\eta)=0$. Thus, with carefully chosen cutoff functions, we can divide the space into several regimes and integrate by parts on every regime in either time or frequency space.

Proposition 3.7. *Let*

$$\phi_0(\xi,\eta)=\frac{1}{2}|\xi|^2-\frac{1}{2}|\xi-\eta|^2+\langle\eta\rangle,$$

then we have

$$\Big\|\langle\xi\rangle^3\int_t^\infty\int_{\mathbb{R}^2} e^{is\phi_0(\xi,\eta)}\hat{u}_+(\xi-\eta)\bar{\hat{h}}_+(-\eta)d\eta ds\Big\|_{L^2_\xi}\lesssim\frac{1}{t}\|u_+\|_Y\|h_+\|_Y.$$

Proof. Dividing the integration regime, we have

$$\begin{aligned}
&\int_t^\infty\int_{\mathbb{R}^2} e^{is\phi_0(\xi,\eta)}\hat{u}_+(\xi-\eta)\bar{\hat{h}}_+(-\eta)d\eta ds && (3.6)\\
&=\varphi(A|\xi|)\int_t^\infty\int_{\mathbb{R}^2}\varphi(2|\eta|)e^{is\phi_0(\xi,\eta)}\hat{u}_+(\xi-\eta)\bar{\hat{h}}_+(-\eta)d\eta ds && (3.7)\\
&\quad+\varphi(A|\xi|)\int_t^\infty\int_{\mathbb{R}^2}[1-\varphi(2|\eta|)]e^{is\phi_0(\xi,\eta)}\hat{u}_+(\xi-\eta)\bar{\hat{h}}_+(-\eta)d\eta ds && (3.8)\\
&\quad+[1-\varphi(A|\xi|)]\int_t^\infty\int_{\mathbb{R}^2}\varphi(B|\eta-\eta_0|)e^{is\phi_0(\xi,\eta)}\hat{u}_+(\xi-\eta)\bar{\hat{h}}_+(-\eta)d\eta ds && (3.9)\\
&\quad+[1-\varphi(A|\xi|)]\int_t^\infty\int_{\mathbb{R}^2}[1-\varphi(B|\eta-\eta_0|)]e^{is\phi_0(\xi,\eta)}\hat{u}_+(\xi-\eta)\bar{\hat{h}}_+(-\eta)d\eta ds, && (3.10)
\end{aligned}$$

where A and B are large constants and for each ξ, the point η_0 is the unique point in $\mathbb{R}^2$ such that

$$\xi=\eta_0\Big(1-\frac{1}{\langle\eta_0\rangle}\Big).$$

Note that since

$$\partial_\eta\phi_0(\xi,\eta)=\xi-\eta\Big(1-\frac{1}{\langle\eta\rangle}\Big),$$

we have that for a fixed ξ, η_0 is the unique point for which $\partial_\eta\phi_0(\xi,\eta)=0$.

We begin our estimate with (3.7), where $|\xi| \leq 2A^{-1}$. On this regime, we have that $|\eta| \leq 1$. Hence,

$$|\phi_0(\xi,\eta)| \geq \langle\eta\rangle - |\xi||\eta| - \frac{1}{2}|\eta|^2 \geq 1 - \frac{2}{A} - \frac{1}{2} \geq \frac{1}{4}.$$

We therefore apply Lemma 3.6 to obtain

$$\|\langle\xi\rangle^3(3.7)\|_{L^2_\xi} \lesssim \frac{1}{t}\|u_+\|_Y\|h_+\|_Y.$$

We next consider (3.8). For this term, we have that $|\xi| \leq 2A^{-1}$ and we are integrating over a regime where $|\eta| \geq 1/2$. Thus,

$$\begin{aligned}
|\partial_\eta\phi_0(\xi,\eta)| &= \Big|\xi - \eta\Big(1 - \frac{1}{\langle\eta\rangle}\Big)\Big| \\
&\geq |\eta|\Big(1 - \frac{1}{\langle\eta\rangle}\Big) - |\xi| \\
&\geq \frac{1}{2}\Big(1 - \frac{1}{\langle\frac{1}{2}\rangle}\Big) - \frac{2}{A} \\
&\geq \frac{1}{100}.
\end{aligned}$$

Using Lemma 3.3 with $\alpha_1, \alpha_2 = 0$ and $N = 2$ yields

$$\|\langle\xi\rangle^3(3.8)\|_{L^2_\xi} \lesssim \frac{1}{t}\|u_+\|_Y\|h_+\|_Y.$$

For (3.9), we seek an absolute lower bound on $|\phi_0(\xi,\eta)|$ for the purpose of using Lemma 3.6. Let θ be the angle between η and η_0. By making B large relative to A, we may assume $0 \leq \cos\theta$. Now we consider two cases:

Case 1: $|\eta| \leq 2$. Then we have

$$|\phi_0(\xi,\eta)| \geq \langle\eta\rangle - \frac{1}{2}|\eta|^2 \geq \langle\eta\rangle - |\eta|.$$

On the closed ball $B(0,2)$, $\langle\eta\rangle - |\eta|$ is a strictly positive function. Thus, it attains an absolute minimum greater than zero. Hence, $|\phi_0(\xi,\eta)| \geq \delta$.

Case 2: $|\eta| > 2$. Enlarging B we may assume $\eta_0 \geq 1.99$ and $|\eta_0| \geq .99|\eta|$. This gives

$$\begin{aligned}
|\phi_0(\xi,\eta)| &\geq |\eta_0|\Big(1 - \frac{1}{\langle\eta_0\rangle}\Big)|\eta|\cos\theta - \frac{1}{2}|\eta|^2 \\
&\geq |\eta_0|\Big(1 - \frac{1}{\langle 1.99\rangle}\Big)|\eta|\cos\theta - \frac{1}{2}|\eta|^2 \\
&\geq .53|\eta_0||\eta|\cos\theta - \frac{1}{2}|\eta|^2 \\
&\geq |\eta|(.52\cos\theta - .5).
\end{aligned}$$

Because $|\eta|$ has a lower bound, we may, if necessary, again enlarge B so that $\cos\theta \geq .99$. This gives $|\phi_0(\xi,\eta)| \geq 1/5$. With an absolute lower bound on $|\phi_0(\xi,\eta)|$, we may now use Lemma 3.6 to conclude

$$\|\langle\xi\rangle^3(3.9)\|_{L^2_\xi} \lesssim \frac{1}{t}\|u_+\|_Y\|h_+\|_Y.$$

For (3.10), we wish to show $|\partial_\eta\phi_0(\xi,\eta)| \gtrsim 1$, with the intent of using Lemma 3.3. For this purpose, set θ as the angle between η_0 and η. We consider several cases:

Case 1: $|\theta| \geq \pi/2$. Then

$$|\partial_\eta \phi_0(\xi,\eta)| \geq |\xi| \geq \frac{1}{A}.$$

Case 2: $|\eta| \leq \frac{1}{2}|\xi|$. Then

$$|\partial_\eta \phi_0(\xi,\eta)| \geq |\xi| - |\eta| \geq \frac{1}{2A}.$$

Case 3: $|\eta| \geq \frac{1}{2}|\xi|$, $0 \leq \theta < \pi/2$. By our assumption on the distance between η and η_0, we have

$$B^{-2} \leq |\eta_0|^2 + |\eta|^2 - 2|\eta_0||\eta|\cos\theta.$$

Thus, we can bound $\cos\theta$ as

$$0 \leq \cos\theta \leq \frac{|\eta_0|^2 + |\eta|^2 - \epsilon}{2|\eta_0||\eta|},$$

for $\epsilon = B^{-2}$. Hence,

$$\begin{aligned}
&|\partial_\eta \phi_0(\xi,\eta)|^2 \\
&= |\eta_0|^2\Big(1 - \frac{1}{\langle\eta_0\rangle}\Big)^2 + |\eta|^2\Big(1 - \frac{1}{\langle\eta\rangle}\Big)^2 - 2|\eta_0||\eta|\Big(1 - \frac{1}{\langle\eta_0\rangle}\Big)\Big(1 - \frac{1}{\langle\eta\rangle}\Big)\cos\theta \\
&\gtrsim |\eta_0|^2\Big(1 - \frac{1}{\langle\eta_0\rangle}\Big)^2 + |\eta|^2\Big(1 - \frac{1}{\langle\eta\rangle}\Big)^2 - \Big(1 - \frac{1}{\langle\eta_0\rangle}\Big)\Big(1 - \frac{1}{\langle\eta\rangle}\Big)(|\eta_0|^2 + |\eta|^2 - \epsilon) \\
&= \Big(\frac{1}{\langle\eta\rangle} - \frac{1}{\langle\eta_0\rangle}\Big)\Big[|\eta_0|^2\Big(1 - \frac{1}{\langle\eta_0\rangle}\Big) - |\eta|^2\Big(1 - \frac{1}{\langle\eta\rangle}\Big)\Big] \qquad (3.11) \\
&\quad + \epsilon\Big(1 - \frac{1}{\langle\eta_0\rangle}\Big)\Big(1 - \frac{1}{\langle\eta\rangle}\Big). \qquad (3.12)
\end{aligned}$$

Since $|x| \geq |y|$ implies $\langle y\rangle^{-1} \geq \langle x\rangle^{-1}$, the term (3.11) is a product of two numbers with the same sign. Thus,

$$\begin{aligned}
|\partial_\eta \phi_0(\xi,\eta)| &\gtrsim \epsilon\Big(1 - \frac{1}{\langle\eta_0\rangle}\Big)\Big(1 - \frac{1}{\langle\eta\rangle}\Big) \\
&\gtrsim \epsilon\Big(1 - \frac{1}{\langle A^{-1}\rangle}\Big)\Big(1 - \frac{1}{\langle\frac{1}{2A}\rangle}\Big) \gtrsim 1.
\end{aligned}$$

Hence, we may use Lemma 3.3 with $\alpha_1, \alpha_2 = 0$ and $N = 2$ to obtain

$$\|\langle\xi\rangle^3 (3.10)\|_{L^2_\xi} \lesssim \frac{1}{t}\|u_+\|_Y \|h_+\|_Y.$$

□

3.3. **Analysis of ϕ_1.** We now consider the case where the phase function is ϕ_1. In this case, there is a set $A \subset \mathbb{R}^2 \times \mathbb{R}^2$ on which $\phi_1(\xi,\eta)$ and $\partial_\eta\phi_1(\xi,\eta)$ are both zero. On this set, we can not integrate by parts in either frequency space or time. Instead, we place appropriate assumptions on the final data so that $B_1(u_1,h_1)$ is identically zero around A.

Proposition 3.8. *Let*

$$\phi_1(\xi,\eta) = \frac{1}{2}|\xi|^2 - \frac{1}{2}|\xi - \eta|^2 - \langle\eta\rangle,$$

then the resonance set of ϕ_1 is non-empty and contained in the set A_δ defined in Subsection 1.1. If

$$\hat{u}_+(\xi-\eta)\hat{h}_+(\eta) \equiv 0 \quad \text{on } A_\delta,$$

then

$$\Big\|\langle\xi\rangle^3 \int_t^\infty \int_{\mathbb{R}^2} e^{is\phi_1(\xi,\eta)}\hat{u}_+(\xi-\eta)\hat{h}_+(\eta)d\eta ds\Big\|_{L^2_\xi} \lesssim \frac{1}{t}\|u_+\|_Y\|h_+\|_Y.$$

Proof. Note that

$$\partial_\eta\phi_1(\xi,\eta) = \xi - \eta\Big(1+\frac{1}{\langle\eta\rangle}\Big).$$

Recall that A_δ is defined as

$$A_\delta = \{(\xi,\eta) : r_\xi - \delta \le |\xi| \le r_\xi + \delta,\ r_\eta - \delta \le |\eta| \le r_\eta + \delta\},$$

where the ordered pair (r_ξ, r_η) is the unique solution on $\mathbb{R}^+$ to the system

$$\begin{gathered}\frac{1}{2}r_\xi^2 - \frac{1}{2}(r_\xi - r_\eta)^2 - \langle r_\eta\rangle = 0\\ r_\xi - r_\eta\Big(1+\frac{1}{\langle r_\eta\rangle}\Big) = 0.\end{gathered}$$

From the form of $\partial_\eta\phi_1(\xi,\eta)$ and the definition of r_ξ and r_η, one can see that if (ξ_R,η_R) is a resonance point of $\phi_1(\xi,\eta)$, then ξ is co-linear with η, $|\xi_R| = r_\xi$, and $|\eta_R| = r_\eta$.

Let ψ_1 be a smooth cutoff function on $\mathbb{R}$ supported on $[r_\xi - \delta/2, r_\xi + \delta/2]$ and congruent to 1 on $[r_\xi - \delta/4, r_\xi + \delta/4]$. Similarly, let ψ_2 be a smooth cutoff function on $\mathbb{R}$ supported on $[r_\eta - \delta/2, r_\eta + \delta/2]$ and congruent to 1 on $[r_\eta - \delta/4, r_\eta + \delta/4]$.

For each ξ, we also define η_0 as the unique point in $\mathbb{R}^2$ such that

$$\xi = \eta_0\Big(1+\frac{1}{\langle\eta_0\rangle}\Big).$$

Decomposing, we have

$$\begin{aligned}
&\int_t^\infty \int_{\mathbb{R}^2} e^{is\phi_1(\xi,\eta)}\hat{u}_+(\xi-\eta)\hat{h}_+(\eta)d\eta ds \qquad (3.13)\\
&= \int_t^\infty \int_{\mathbb{R}^2} [1-\varphi(B|\eta-\eta_0|)]e^{is\phi_1(\xi,\eta)}\hat{u}_+(\xi-\eta)\hat{h}_+(\eta)d\eta ds \qquad (3.14)\\
&\quad + \int_t^\infty \int_{\mathbb{R}^2} \varphi(B|\eta-\eta_0|)[1-\psi_1(|\xi|)\psi_2(|\eta|)]e^{is\phi_1(\xi,\eta)}\hat{u}_+(\xi-\eta)\hat{h}_+(\eta)d\eta ds \qquad (3.15)\\
&\quad + \int_t^\infty \int_{\mathbb{R}^2} \varphi(B|\eta-\eta_0|)\psi_1(|\xi|)\psi_2(|\eta|)e^{is\phi_1(\xi,\eta)}\hat{u}_+(\xi-\eta)\hat{h}_+(\eta)d\eta ds. \qquad (3.16)
\end{aligned}$$

We begin our estimates with (3.14). On its regime, we have

$$\begin{aligned}
|\partial_\eta\phi_1(\xi,\eta)| &= \Big|\eta_0\Big(1+\frac{1}{\langle\eta_0\rangle}\Big) - \eta\Big(1+\frac{1}{\langle\eta\rangle}\Big)\Big|\\
&\ge |\eta-\eta_0|\\
&\ge \frac{1}{B}.
\end{aligned}$$

Hence, we may use Lemma 3.3 with $\alpha_1,\alpha_2 = 0$ and $N = 2$ to establish

$$\|\langle\xi\rangle^3(3.14)\|_{L^2_\xi} \lesssim \frac{1}{t}\|u_+\|_Y\|h_+\|_Y.$$

We now bound (3.15). To this end, we first consider the values of ϕ_1 at the stationary phase points. More precisely, we will first establish a lower bound on $\phi_1(\xi,\eta_0)$. If $|\xi| > 100$, then we have

$$\begin{aligned}|\phi_1(\xi,\eta_0)| &= |\xi\cdot\eta_0 - \frac{1}{2}|\eta_0| - \langle\eta_0\rangle| \\ &= |\eta_0|^2\Big(1+\frac{1}{\langle\eta_0\rangle}\Big) - \frac{1}{2}|\eta_0|^2 - \langle\eta_0\rangle \\ &\geq \frac{1}{2}|\eta_0|^2 - \langle\eta_0\rangle \geq 1.\end{aligned}$$

In the case of $|\xi| \leq 100$, we claim that by enlarging B relative to δ, we may assume

$$0 \leq |\xi| \leq r_\xi - \delta/100,$$

or

$$r_\xi + \delta/100 \leq |\xi| \leq 100.$$

To see this, assume for contradiction that some ξ with

$$|\xi| \in [r_\xi + \delta/100, r_\xi - \delta/100]$$

is in our integration regime. Then we also have that

$$|\eta_0| \in [r_\eta - \delta/20, r_\eta + \delta/20].$$

Hence, by enlarging B, we may use the term $\varphi(B|\eta-\eta_0|)$ to assume that

$$|\eta| \in [r_\eta - \delta/10, r_\eta + \delta/10].$$

But this point must then lie outside our integration regime as $1-\psi_1(|\xi|)\psi_2(|\eta|) = 0$. Thus, if $|\xi| \leq 100$, then we may also assume that

$$|\xi| \notin [r_\xi - \delta/100, r_\xi + \delta/100].$$

Furthermore, we know that either $\phi_1(\xi,\eta)$ or $\partial_\eta\phi_1(\xi,\eta)$ is non-zero at a given point in the regime of (3.15). Thus, $|\phi_1(\xi,\eta_0)| > 0$ for all relevant ξ. We then have that $\phi_1(\xi,\eta_0)$ is a function in ξ that is continuous and non-zero. This implies that on the compact set

$$[0, r_\xi - \delta/100] \cup [r_\xi + \delta/100, 100],$$

$\phi_1(\xi,\eta_0)$ must attain some absolute lower bound δ_0. We also have that

$$|\partial_\eta\phi_1(\xi,\eta)| \leq |\eta_0 - \eta| + \Big|\frac{\eta_0}{\langle\eta_0\rangle} - \frac{\eta}{\langle\eta\rangle}\Big| \leq 3.$$

Then, by the Fundamental Theorem of Calculus,

$$|\phi_1(\xi,\eta)| \geq \delta_0 - 3|\eta-\eta_0|.$$

Enlarging B, we obtain $|\phi_1(\xi,\eta)| \geq \delta_0/2$. Now we may use Lemma 3.6 to obtain

$$\|\langle\xi\rangle^3(3.15)\|_{L^2_\xi} \lesssim \frac{1}{t}\|u_+\|_Y\|h_+\|_Y.$$

Finally, (3.16) is identically zero by the assumption that $\hat{u}_+(\xi-\eta)\hat{h}_+(\eta) \equiv 0$ on A_δ. □

3.4. **Analysis of ϕ_2.** Finally, we estimate $\|\langle\xi\rangle^3\mathcal{F}(B_2(u_1,u_1))\|_{L^2_\xi}$.

Proposition 3.9. *Let*

$$\phi_2(\xi,\eta)=\langle\xi\rangle-\frac{1}{2}|\xi-\eta|^2+\frac{1}{2}|\eta|^2,$$

then we have

$$\Big\|\langle\xi\rangle^2\int_t^\infty\int_{\mathbb{R}^2}e^{is\phi_2(\xi,\eta)}(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta ds\Big\|_{L^2_\xi}\lesssim\frac{1}{t}\|u_+\|_Y^2.$$

Proof. We decompose as follows:

$$\Big|\int_t^\infty\int_{\mathbb{R}^2}e^{is\phi_2(\xi,\eta)}(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta ds\Big| \tag{3.17}$$

$$\lesssim\Big|\int_t^\infty\int_{\mathbb{R}^2}e^{is\phi_2(\xi,\eta)}(1-\varphi(As^{1/3}|\xi|))(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta ds\Big| \tag{3.18}$$

$$+\Big|\int_t^\infty\varphi(As^{1/3}|\xi|)\int_{\mathbb{R}^2}\varphi\Big(\frac{|\eta|}{s^{1/3}}\Big)e^{is\phi_2(\xi,\eta)}(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta ds\Big| \tag{3.19}$$

$$+\Big|\int_t^\infty\varphi(As^{1/3}|\xi|)\int_{\mathbb{R}^2}\Big(1-\varphi\Big(\frac{|\eta|}{s^{1/3}}\Big)\Big)e^{is\phi_2(\xi,\eta)}(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta ds\Big|. \tag{3.20}$$

We first consider (3.18), where

$$|\xi|\geq\frac{1}{As^{1/3}}.$$

Since $\partial_\eta\phi_2(\xi,\eta)=\xi$, we may use Corollary 3.4 with $\alpha_1=1/3$, $\alpha_2=0$, and index $N=6$ to obtain

$$\|\langle\xi\rangle^2(3.18)\|_{L^2_\xi}\lesssim\frac{1}{t}\|u_+\|_Y^2.$$

We now move on to estimating (3.19), where we have

$$|\xi|\leq\frac{2}{As^{1/3}}$$

and we are integrating over a regime on which

$$|\eta|\leq 2s^{1/3}.$$

Therefore, we have

$$\begin{aligned}|\phi_2(\xi,\eta)|&\geq\langle\xi\rangle-\frac{1}{2}|\xi|^2-|\xi\cdot\eta|\\&\geq 1-\frac{1}{4}-\frac{4}{A}\\&\geq\frac{1}{2}.\end{aligned}$$

Though $\phi_2(\xi,\eta)$ has an absolute lower bound, we may not directly use Lemma 3.6, since our symbol h is a function of time. Instead, we integrate by parts in time to obtain

$$(3.19)\leq\Big|\varphi(At^{1/3}|\xi|)\int_{\mathbb{R}^2}e^{it\phi_2(\xi,\eta)}\frac{\varphi(|\eta|t^{-1/3})}{\phi_2(\xi,\eta)}(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta\Big| \tag{3.21}$$

$$+\Big|\int_{\mathbb{R}^2}(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)$$

$$\times \int_t^\infty \frac{e^{is\phi_2(\xi,\eta)}}{\phi_2(\xi,\eta)} \partial_s\Big(\varphi(As^{1/3}|\xi|)\varphi(|\eta|s^{-1/3})\Big) ds d\eta\Big|. \tag{3.22}$$

Let $\psi(r)$ be a smooth function defined for $r \geq 0$ such that $\psi(r) = 0$ for $r < 1/4$ and $\psi(r) = 1$ for $r \geq 1/2$. Then we have

$$(3.21) \leq \Big| \int_{\mathbb{R}^2} e^{it\phi_2(\xi,\eta)} \frac{\psi(|\phi_2(\xi,\eta)|)\varphi(|\eta|t^{-1/3})}{\phi_2(\xi,\eta)} (\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta) d\eta\Big|.$$

Using Lemma 3.5, we have

$$\begin{aligned} &\Big| \int_{\mathbb{R}^2} e^{it\phi_2(\xi,\eta)} \frac{\psi(|\phi_2(\xi,\eta)|)\varphi(|\eta|t^{-1/3})}{\phi_2(\xi,\eta)} (\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta) d\eta\Big| \\ &\lesssim \frac{1}{t} \|\nabla u_+\|_{W^{15,1}} \|\nabla u_+\|_{H_x^{15}} \\ &\lesssim \frac{1}{t} \|u_+\|_Y^2. \end{aligned}$$

For (3.22), integrating by parts again yields

$$\begin{aligned} (3.22) \lesssim &\Big| \int_{\mathbb{R}^2} \frac{e^{it\phi_2(\xi,\eta)}}{(\phi_2(\xi,\eta))^2} \partial_t\Big(\varphi(A|\xi|t^{1/3})\varphi(|\eta|t^{-1/3})\Big) \\ &\times (\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta) d\eta\Big| \quad (3.23) \\ &+ \Big| \int_{\mathbb{R}^2} (\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta) \\ &\times \int_t^\infty \frac{e^{is\phi_2(\xi,\eta)}}{(\phi_2(\xi,\eta))^2} \partial_s^2\Big(\varphi(As^{1/3}|\xi|)\varphi(|\eta|s^{-1/3})\Big) ds d\eta\Big|. \quad (3.24) \end{aligned}$$

Calculating, we have

$$\begin{aligned} &(3.23) \\ &\lesssim \Big| \int_{\mathbb{R}^2} \frac{e^{it\phi_2(\xi,\eta)}}{(\phi_2(\xi,\eta))^2} |\xi| t^{-2/3} \varphi'(A|\xi|t^{1/3})\varphi(|\eta|t^{-1/3}))(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta) d\eta\Big| \\ &\quad + \Big| \int_{\mathbb{R}^2} \frac{e^{it\phi_2(\xi,\eta)}}{(\phi_2(\xi,\eta))^2} \varphi(A|\xi|t^{1/3})|\eta|t^{-4/3}\varphi'(|\eta|t^{-1/3}))(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta) d\eta\Big| \\ &\lesssim \int_{\mathbb{R}^2} \Big(|\xi|t^{-2/3}\varphi'(A|\xi|t^{1/3}) + |\eta|t^{-4/3}\varphi'(|\eta|t^{-1/3})\Big) |(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)| d\eta \\ &\lesssim \frac{1}{t} \int_{\mathbb{R}^2} |(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)| d\eta \\ &\lesssim \frac{1}{t\langle\xi\rangle^4} \|u_+\|_{H_x^5}^2. \end{aligned}$$

Similarly,

$$\begin{aligned} (3.24) &\lesssim \int_{\mathbb{R}^2} |(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)| \int_t^\infty \frac{1}{s^2} ds d\eta \\ &\lesssim \frac{1}{t\langle\xi\rangle^4} \|u_+\|_{H_x^5}^2. \end{aligned}$$

Finally, we bound (3.20), where

$$|\xi| \leq \frac{2}{As^{1/3}}$$

and we are integrating over a regime on which

$$|\eta| \geq s^{1/3}.$$

For this term, we can get the necessary decay by assuming high regularity on u_+. More precisely,

$$\begin{aligned}
&\|\langle\xi\rangle^3(3.20)\|_{L^2_\xi}\\
&\lesssim \Big\|\langle\xi\rangle^2\int_t^\infty\int_{\mathbb{R}^2}\Big(1-\varphi\Big(\frac{|\eta|}{s^{1/3}}\Big)\Big)e^{is\phi_2(\xi,\eta)}\frac{\langle\eta\rangle^6}{\langle\eta\rangle^6}(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta ds\Big\|\\
&\lesssim \Big\|\langle\xi\rangle^2\int_t^\infty\frac{1}{s^2}\int_{\mathbb{R}^2}\langle\eta\rangle^7|(\xi-\eta)\hat{u}_+(\xi-\eta)\bar{\hat{u}}_+(-\eta)|d\eta ds\Big\|_{L^2_\xi}\\
&\lesssim \Big\|\langle\xi\rangle^2\int_t^\infty\frac{1}{s^2\langle\xi\rangle^4}\|u_+\|_{H^{11}_x}\|u_+\|_{H^5_x}ds\Big\|_{L^2_\xi}\\
&\lesssim \frac{1}{t}\|u_+\|^2_{H^{11}_x}.
\end{aligned}$$

□

Acknowledgements. I would like to thank Professors Xiaoyi Zhang and Dong Li for their suggestions, and Professor Zhang also for her help in proofreading this manuscript. The research resulting in this paper was partially supported by grant DMS-0602242 from the NSF VIGRE.

References

[1] R. Coifman, Y. Meyer; *Au delà des opérateurs pseudo-différentiels*. Astérisque, **57** (1978), 1-185.
[2] P. Germain; *Global existence for coupled Klein-Gordon equations with different velocities*. Annales de l'Institut Fourier (to appear), 2011.
[3] R. Glassey; *On the asymptotic behavior of nonlinear wave equations*. Trans. Amer. Math. Soc., **182** (1973), 187-200.
[4] L. Grafakos, Z. Si; *The Hörmander multiplier theorem for multilinear operators*. Crelles Journal (to appear).
[5] A. Grünrock, H. Pecher; *Bounds in time for the Klein-Gordon-Schrödinger and the Zakharov system*. Hokkaido Math. J. **35** (2006), no. 1, 139-153.
[6] N. Hayashi, P. Naumkin; *Scattering Operator for nonlinear Klein-Gordon equations*. Commun. Contemp. Math, **11** (2009), no. 5, 771-781.
[7] J. Jang, D. Li, and X. Zhang; *Smooth global solutions for the two dimensional Euler Poisson system*. Preprint arXiv:1109.3882v2.
[8] T. Ozawa, Y. Tsutusmi; *Asymptotic behavior of solutions for the coupled Klein-Gordon-Schrödinger equations*. Adv. Stud. Pure Math., **23** (1994), 295-305.
[9] H. Pecher; *Some new well-posedness results for the Klein-Gordon-Schrödinger System*. Preprint arXiv: 1106.2116v3.
[10] A. Shimomura; *Scattering Theory for the coupled Klein-Gordon-Schrödinger equations in two space dimensions*. J. Math. Sci. Univ. Tokyo. **10** (2003), no. 4, 661-685.
[11] A. Shimomura; *Wave operators for the coupled Klein-Gordon-Schrödinger equations in two space dimensions*. Funkcial. Ekvac. **47** (2004), no. 1, 63-82.
[12] A. Shimomura; *Scattering Theory for the coupled Klein-Gordon-Schrödinger equations in two space dimensions II*. Hokkaido Math. J. **34** (2005), no. 2, 405-433.
[13] E. Stein; *Harmonic Analysis*. Princeton mathematical series, **43**, Princeton University Press, 1993.
[14] W. A. Strauss; *Nonlinear Wave equations*. CBMS Regional Conference Series in Math., **73**, AMS, 1989.
[15] Y. Tsutsumi and K. Yajima; *The asymptotic behavior of nonlinear Schrödinger equations*. Bull. Amer. Math. Soc., **11** (1984), no. 1, 186-188.

[16] N. Tzirakis; *The Cauchy problem for the Klein-Gordon-Schrödinger system in low dimensions below the energy space*. Comm. Partial Differential Equations, **30** (2005), no. 4-6, 605-641.

[17] H. Yukawa; *On the interaction of elementary particles*. Proc. Phys.-Math. Soc. Japan, **17** (1935), 48-57.

Kai Tsuruta

Department of Applied Mathematical and Computational Sciences, University of Iowa, 14 MacLean Hall, Iowa City, IA 52245, USA

E-mail address: kai-tsuruta@uiowa.edu