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EXISTENCE AND TOPOLOGICAL STRUCTURE OF SOLUTION SETS FOR ϕ-LAPLACIAN IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In this article, we present results on the existence and the topological structure of the solution set for initial-value problems for the first-order impulsive differential equation

$$
\begin{gathered}
(\phi(y'))' = f(t, y(t)), \quad \text{a.e. } t \in [0, b], \\
y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \\
y'(t_k^+) - y'(t_k^-) = \bar{I}_k(y'(t_k^-)), \quad k = 1, \dots, m, \\
y(0) = A, \quad y'(0) = B,
\end{gathered}
$$

where $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = b$, $m \in \mathbb{N}$. The functions $I_k, \bar{I}_k$ characterize the jump in the solutions at impulse points t_k, $k = 1, \dots, m$. For the final result of the paper, the hypotheses are modified so that the nonlinearity f depends on y', but the impulsive conditions and initial conditions remain the same.

1. Introduction

The dynamics of many processes in physics, population dynamics, biology and medicine may be subject to abrupt changes such as shocks or perturbations (see for instance [3, 9] and the references therein). These perturbations may be seen as impulses. For instance, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment or a missing product. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs. Their models may be described by impulsive differential equations. The mathematical study of boundary value problems for differential equations with impulses was first considered in 1960 by Milman and Myshkis [11] and then followed by a period of active research which culminated in 1968 with the monograph by Halanay and Wexler [8].

Various mathematical results (existence, asymptotic behavior, and so on) have been obtained so far (see [1, 2, 4, 7, 10, 13, 12] and the references therein).

In this paper we shall establish an existence theory for initial-value problems with impulse effects. We will treat two cases. For the first case, the problem has

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the form

$$(\phi(y'(t)))' = f(t, y(t)), t \in J := [0, b], \quad t \neq t_k,\ k = 1, \dots, m, \tag{1.1}$$
$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{1.2}$$
$$y'(t_k^+) - y'(t_k^-) = \bar{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{1.3}$$
$$y(0) = A, \quad y'(0) = B, \tag{1.4}$$

where $f : [0, b] \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k, \bar{I}_k \in C(\mathbb{R}, \mathbb{R})$, $\phi : \mathbb{R} \to \mathbb{R}$ is a suitable monotone homeomorphism, and $A, B \in \mathbb{R}$. For this setting, the proofs of the two results presented, while involving some cases, are quite straight for word. The second case is when the second member f may depend on y', and the problem has the the form

$$(\phi(y'(t)))' = f(t, y(t), y'(t)), t \in J := [0, b], \quad t \neq t_k,\ k = 1, \dots, m, \tag{1.5}$$
$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{1.6}$$
$$y'(t_k^+) - y'(t_k^-) = \bar{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{1.7}$$
$$y(0) = A, \quad y'(0) = B, \tag{1.8}$$

where $f : [0, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k, \bar{I}_k, \phi$ and A, B are as in problem (1.1)–(1.4). Because of the dependency on y', the proof of the result presented is somewhat more involved. Of course, the second case also covers the first case when f is independent of y'.

The goals of this article are to provide some existence results and to establish the compactness of solution sets of the above problems.

2. Preliminaries

In this section, we recall from the literature some notation, definitions, and auxiliary results which will be used throughout this paper. Let $J = [0, b]$ be an interval of $\mathbb{R}$. $C([0, b], \mathbb{R})$ is the Banach space of all continuous functions from $[0, b]$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup_{t \in [0,b]} |y(t)|.$$

$L^1([0, b], \mathbb{R})$ denotes the Banach space of Lebesgue integrable functions, with the norm

$$\|y\|_{L^1} = \int_0^b |y(s)| ds.$$

Definition 2.1. A map $f : [p, q] \times \mathbb{R} \to \mathbb{R}$ is said to be L^1-Carathéodory if

- (i) $t \to f(t, y)$ is measurable for all $y \in \mathbb{R}$,
- (ii) $y \to f(t, y)$ is continuous for almost each $t \in [p, q]$,
- (iii) for each $r > 0$, there exists $h_r \in L^1([p, q], \mathbb{R}_+)$ such that $|f(t, y)| \le h_r(t)$ for almost each $t \in [p, q]$ and for all $|y| \le r$.

Lemma 2.2 (Grönwall-Bihari [5]). *Let $I = [p, q]$ and let $u, g : I \to \mathbb{R}$ be positive continuous functions. Assume there exist $c > 0$ and a continuous nondecreasing function $h : [0, \infty) \to (0, +\infty)$ such that*

$$u(t) \le c + g(s)h(u(s))ds, \quad \forall t \in I.$$

Then

$$u(t) \le H^{-1}\Big(\int_p^t g(s)ds\Big), \quad \forall t \in I,$$

provided

$$\int_c^{+\infty} \frac{dy}{h(y)} > \int_p^q g(s)ds,$$

where H^{-1} *refers to inverse of the function* $H(u) = \int_c^u \frac{dy}{h(y)}$ *for* $u \ge c$.

3. Main results

Let $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, \dots, m$, and let y_k be the restriction of a function y to J_k. To define solutions for (1.1) – (1.4), consider the space

$$\begin{aligned} PC = \big\{ & y : [0,b] \to \mathbb{R},\ y_k \in C(J_k, \mathbb{R}),\ k = 0, \dots, m, \text{ such that} \\ & y(t_k^-) \text{ and } y(t_k^+) \text{ exist and satisfy } y(t_k^-) = y(t_k) \text{ for } k = 1, \dots, m \big\}. \end{aligned}$$

Endowed with the norm

$$\|y\|_{PC} = \max\{\|y_k\|_\infty,\ k = 0, \dots, m\}, \quad \|y_k\|_\infty = \sup_{t \in J_k} |y(t)|,$$

PC is a Banach space.

$$\begin{aligned} PC^1 = \big\{ & y \in PC : y_k' \in C(J_k, \mathbb{R}),\ k = 0, \dots, m, \text{ such that} \\ & y'(t_k^-) \text{ and } y'(t_k^+) \text{ exist and satisfy } y'(t_k^-) = y'(t_k) \text{ for } k = 1, \dots, m \big\}. \end{aligned}$$

is a Banach space with the norm

$$\|y\|_{PC^1} = \max(\|y\|_{PC}, \|y'\|_{PC}), \quad \text{or} \quad \|y\|_{PC^1} = \|y\|_{PC} + \|y'\|_{PC}.$$

Theorem 3.1 (Nonlinear Alternative [6])**.** *Let* X *be a Banach space with* $C \subset X$ *closed and convex. Assume* U *is a relatively open subset of* C *with* $0 \in U$ *and* $G : \overline{U} \to C$ *is a compact map. Then either,*

(i) G *has a fixed point in* $\overline{U}$*; or*
(ii) *there is a point* $u \in \partial U$ *and* $\lambda \in (0, 1)$ *with* $u = \lambda G(u)$.

Theorem 3.2. *Suppose that:*

(H1) $f : [0, b] \times \mathbb{R} \to \mathbb{R}$ *is an Carathéodory function and* $I_k, \bar{I}_k \in C(\mathbb{R}, \mathbb{R})$.
(H2) *There exist* $p \in L^1(J, \mathbb{R}_+)$ *such that* $|f(t, u)| \le p(t)$ *for a.e.* $t \in J$

are satisfied. Then (1.1)-(1.4) *has at least one solution and the solutions set*

$$S = \{y \in PC([0, b], \mathbb{R}) : y \text{ is a solution of } (1.1)\text{-}(1.4)\}$$

is compact.

Proof. The proof involves several steps.

Step 1: Consider the problem

$$\begin{gathered} (\phi(y'))' = f(t, y) \quad t \in [0, t_1], \\ y(0) = A, \quad y'(0) = B, \end{gathered} \tag{3.1}$$

and the map $N_1 : C([0, t_1], \mathbb{R}) \to C([0, t_1], \mathbb{R})$,

$$y \mapsto (N_1 y)(t) = A + \int_0^t \phi^{-1}[\phi(B) + \int_0^s f(\tau, y)d\tau]ds.$$

Clearly the fixed points of N_1 are solutions of the problem (3.1).

To apply the nonlinear alternative of Leray-Schauder type, we first show that N_1 is completely continuous. The proof will be given in several steps.

Claim 1: N sends bounded sets into bounded sets in $C([0,t_1],\mathbb{R})$. Let

$$y \in D = \{y \in C([0,t_1],\mathbb{R}) : \|y\|_\infty \leq q\}.$$

Then for each $t \in [0,t_1]$, we have

$$|(N_1y)(t)| \leq |A| + \int_0^t |\phi^{-1}[\phi(B) + \int_0^s f(\tau,y)]|d\tau,$$

since

$$\begin{aligned} |\phi(B) + \int_0^s f(\tau,y)d\tau| &\leq |\phi(B)| + \int_0^s |f(\tau,y)|d\tau \\ &\leq |\phi(B)| + \int_0^s |p(\tau)|d\tau \\ &\leq |\phi(B)| + (\|p\|_{L^1})t_1, \end{aligned}$$

it follows that

$$[\phi(B) + \int_0^s f(\tau,y)d\tau] \in \overline{B}(0,l_1),$$

where $l_1 = |\phi(B)| + (\|p\|_{L^1})t_1$. Since ϕ^{-1} is continuous,

$$\sup_{x\in\overline{B}(0,l_1)} |\phi^{-1}(x)| < \infty.$$

Thus

$$\|N_1(y)\|_\infty \leq |A| + t_1 \sup_{x\in\overline{B}(0,l_1)} |\phi^{-1}(x)| := r$$

Claim 2: N_1 maps bounded sets into equicontinuous sets. Let $l_1,l_2 \in [0,t_1]$, $l_1 < l_2$ and D be a bounded set of $C([0,t_1],\mathbb{R})$ as in Claim 1. Let $y \in D$. Then

$$\begin{aligned} |(N_1y)'(t)| &= |\phi^{-1}[\phi(B) + \int_0^t f(s,y)ds] - \phi^{-1}(\phi(B))| \\ &\leq |\phi^{-1}[\phi(B) + \int_0^t f(s,y)ds]| + |B| \\ &\leq \sup_{x\in\overline{B}(0,l_1)} |\phi^{-1}(x)| + |B| := r'. \end{aligned}$$

By the mean value theorem, we obtain

$$|(N_1y)(l_2) - (N_1y)(l_1)| = |(N_1y)'(\xi)(l_2-l_1)| \leq r'|l_2-l_1|.$$

As $l_2 \to l_1$ the right-hand side of the above inequality tends to zero.

Claim 3: N_1 is continuous. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence such that $y_n \to y$ in $C([0,t_1],\mathbb{R})$. Then there is an integer q such that $\|y_n\|_\infty \leq q$ for all $n \in \mathbb{N}$ and $\|y\|_\infty \leq q$, $y_n \in D$ and $y \in D$. We have

$$\begin{aligned} &|(N_1y_n)(t) - (N_1y)(t)| \\ &\leq \int_0^t |\phi^{-1}[\phi(B) + \int_0^s f(\tau,y_n)d\tau] - \phi^{-1}[\phi(B) + \int_0^s f(\tau,y)d\tau]|ds. \end{aligned}$$

By the dominated convergence theorem, we have

$$|\phi(B) + \int_0^s f(\tau,y_n)d\tau - \phi(B) - \int_0^s f(\tau,y)d\tau| \to 0 \quad \text{as } n \to \infty,$$

and since ϕ^{-1} is continuous. Then by the dominated convergence theorem, we have

$$\begin{aligned}&\|N_1(y_n) - N_1(y)\|_\infty \\ &\leq \int_0^{t_1} |\phi^{-1}[\phi(B) + \int_0^s f(\tau, y_n)d\tau] - \phi^{-1}[\phi(B) + \int_0^s f(\tau, y)d\tau]|ds \to 0,\end{aligned}$$

as $n \to \infty$. Thus N_1 is continuous.

Claim 4: *A priori* estimate. Now we show that there exists a constant M_0 such that $\|y\|_\infty \leq M_0$ where y is a solution if the problem (3.1). Let y a solution of (3.1):

$$y(t) = A + \int_0^t \phi^{-1}[\phi(B) + \int_0^s f(\tau, y(\tau)d\tau)]ds.$$

Then

$$\begin{aligned}|y(t)| &\leq |A| + \int_0^t |\phi^{-1}[\phi(B) + \int_0^s f(\tau, y(\tau))d\tau]|ds \\ &\leq |A| + \int_0^t |\phi^{-1}[\phi(B) + \int_0^s p(\tau)d\tau]|ds \\ &\leq |A| + \int_0^t |\phi^{-1}[\phi(B) + \|p\|_1 t_1]|ds \\ &\leq |A| + \sup_{x\in\overline{B}(0,l_1)} \int_0^t ds \\ &\leq |A| + t_1 \sup_{x\in\overline{B}(0,l_1)} =: M_0.\end{aligned}$$

Thus, $\|y\|_\infty = \sup_{t\in[0,t_1]} |y(t)| \leq M_0$. Set

$$U = \{y \in C([0, t_1], \mathbb{R}) : \|y\|_\infty < M_0 + 1\}.$$

As a consequence of Claims 1–4 and the Ascoli-Arzela theorem, we can conclude that the map $N_1 : \overline{U} \to C([0, t_1], \mathbb{R})$ is compact. From the choice of U there is no $y \in \partial U$ such that $y = \lambda N_1 y$ for any $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder we deduce that N_1 has a fixed point denoted by $y_0 \in \overline{U}$ which is solution of the problem (3.1).

Step 2: Consider the problem

$$\begin{gathered}(\phi(y'))' = f(t, y) \quad t \in (t_1, t_2], \\ y(t_1^+) = y_0(t_1^-) + I_1(y_0(t_1^-)), \\ y'(t_1^+) = y_0'(t_1^-) + \bar{I}_1(y_0(t_1^-)).\end{gathered} \tag{3.2}$$

It is clear that all solutions of (3.2) are fixed points of the multi-valued operator $N_2 : C^* \to C^*$, defined by

$$(N_2 y)(t) = A_1 + \int_{t_1}^t \phi^{-1}[\phi(B_1) + \int_{t_1}^s f(\tau, y)d\tau]ds,$$

where

$$C^* = \{y \in C((t_1, t_2]) : y(t_1^+), y'(t_1^+) \text{ exist}\}$$

and

$$A_1 = y_1(t_1) + I_1(y_1(t_1)), \quad B_1 = y_1'(t_1) + \bar{I}(y_1(t_1)).$$

As in Step 1, we can prove that N_2 at least one fixed point which is a solution of (3.2).

Step 3: We continue this process taking into account that $y_m := y\big|_{(t_m,b]}$ is a solution of the problem

$$\begin{gathered}(\phi(y'))' = f(t,y) \quad t \in (t_m, b],\\ y(t_m^+) = y_{m-1}(t_m^-) + I_m(y_{m-1}(t_m^-)),\\ y'(t_m^+) = y'_{m-1}(t_m^-) + \bar{I}_m(y_{m-1}(t_m^-)).\end{gathered} \tag{3.3}$$

A solution y of problem (1.1)-(1.4) is ultimately defined by

$$y(t) = \begin{cases} y_0(t), & \text{if } t \in [0, t_1],\\ y_2(t), & \text{if } t \in (t_1, t_2],\\ \dots \\ y_m(t), & \text{if } t \in (t_m, t_{m+1}]. \end{cases}$$

Step 3: Now we show that the set

$$S = \{y \in PC([0, b], \mathbb{R}) : y \text{ is a solution of (1.1)-(1.4)}\}$$

is compact. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in S. We put $B = \{y_n : n \in \mathbb{N}\} \subseteq PC([0, b], \mathbb{R})$. Then from earlier parts of the proof of this theorem, we conclude that B is bounded and equicontinuous. Then from the Ascoli-Arzela theorem, we can conclude that B is compact.

Recall that $J_0 = [0, t_1]$ and $J_k = (t_k, t_{k+1}]$, $k = 1, \dots, m$. Hence:

- $y_n|_{J_0}$ has a subsequence

$$(y_{n_m})_{n_m\in\mathbb{N}} \subset S_1 = \{y \in C([0, t_1], \mathbb{R}) : y \text{ is a solution of (3.1)}\}$$

such that y_{n_m} converges to y. Let

$$z_0(t) = A + \int_0^t \phi^{-1}[\phi(B) + \int_0^s f(\tau, y)d\tau]ds,$$

and

$$\begin{aligned}&|y_{n_m}(t) - z_0(t)|\\ &\le \int_0^t |\phi^{-1}[\phi(B) + \int_0^s f(\tau, y_{n_m})d\tau] - \phi^{-1}[\phi(B) + \int_0^s f(\tau, y)d\tau]|ds.\end{aligned}$$

As $n_m \to +\infty$, $y_{n_m}(t) \to z_0(t)$, and then

$$y(t) = A + \int_0^t \phi^{-1}[\phi(B) + \int_0^s f(\tau, y)d\tau]ds.$$

- $y_n|_{J_1}$ has a subsequence relabeled as $(y_{n_m}) \subset S_2$ converging to y in C^* where

$$S_2 = \{y \in C^* : y \text{ is a solution of (3.2)}\}.$$

Let

$$z_1(t) = A_1 + \int_{t_1}^t \phi^{-1}[\phi(B_1) + \int_{t_1}^s f(\tau, y)d\tau]ds,$$

$$\begin{aligned}&|y_{n_m}(t) - z_1(t)|\\ &\le \int_{t_1}^t |\phi^{-1}[\phi(B_1) + \int_{t_1}^s f(\tau, y_{n_m})d\tau] - \phi^{-1}[\phi(B_1) + \int_{t_1}^s f(\tau, y)d\tau]|ds.\end{aligned}$$

As $n_m \to +\infty$, $y_{n_m}(t) \to z_1(t)$, and then

$$y(t) = A_1 + \int_{t_1}^{t} \phi^{-1}[\phi(B_1) + \int_{t_1}^{s} f(\tau, y)d\tau]ds.$$

• We continue this process, and we conclude that $\{y_n \mid n \in \mathbb{N}\}$ has subsequence converging to

$$z_m(t) = A_m + \int_{t_m}^{t} \phi^{-1}[\phi(B_m) + \int_{t_m}^{s} f(\tau, y)d\tau]ds, \quad t \in (t_m, b].$$

Hence S is compact. □

Next we replace (H2) in Theorem 3.2 by

(H3) There exists a continuous nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that

$$|f(t, u)| \le p(t)\psi(|u|) \quad \text{a.e. t} \in J \text{ and } u \in \mathbb{R}.$$

Theorem 3.3. *Under assumption* (H3)*, problem* (1.1)-(1.4) *has at least one solution and the solution set is compact.*

Proof. As in the proof of Theorem 3.2 we can show that (1.1)-(1.4) has at least one solution by an application of the nonlinear alternative of Leray-Schauder. We show only the estimation of a solution y of (1.1)-(1.4).

• For $t \in [0, t_1]$, we have

$$y(t) = A + \int_{0}^{t} \phi^{-1}[\phi(B) + \int_{0}^{s} f(\tau, y)d\tau]ds.$$

We put $m(r) = \max\{|y(r)| : r \in [0, t_1]\}$, and

$$\begin{aligned} |y(t)| &\le |A| + \int_0^t |\phi^{-1}[\phi(B) + \int_0^s f(\tau, y)d\tau]|ds \\ &\le |A| + \int_0^t |\phi^{-1}[\phi(B) + \int_0^s p(\tau)\psi(|y(\tau)|)d\tau]|ds \\ &\le |A| + \int_0^t |\phi^{-1}[\phi(B) + \int_0^s p(\tau)\psi(m(r))d\tau]|ds \\ &\le |A| + \int_0^t |\phi^{-1}[\phi(B) + t_1\|p\|_{L^1}\psi(m(r))]|ds. \end{aligned}$$

Then

$$m(t) \le |A| + \int_0^t \psi_1(m(s))ds, \quad t \in [0, t_1],$$

where $\psi_1 = (\phi^{-1} \circ \widetilde{\psi})$ and $\widetilde{\psi}(u) = \phi(B) + t_1(\|p\|_{L^1})\psi(u)$. By the nonlinear Grönwall-Bihari inequality (Lemma 2.2), we infer the bound

$$m(t) \le H^{-1}(t) \le M_0,$$

where $H(t) = \int_{|A|}^{t} \frac{d\tau}{(\phi^{-1} \circ \widetilde{\psi})(\tau)}$.

• For $t \in (t_1, t_2]$, we have

$$y(t) = A_1 + \int_{t_1}^{t} \phi^{-1}[\phi(B_1) + \int_{t_1}^{s} f(\tau, y)d\tau]ds.$$

We put $m(r) = \max\{|y(r)| : r \in (t_1, t_2]\}$, and

$$\begin{aligned}|y(t)| &\le |A_1| + \int_{t_1}^{t} |\phi^{-1}[\phi(B_1) + \int_{t_1}^{s} f(\tau, y)d\tau]|ds \\ &\le |A_1| + \int_{t_1}^{t} |\phi^{-1}[\phi(B_1) + \int_{t_1}^{s} p(\tau)\psi(|y(\tau)|)d\tau]|ds \\ &\le |A_1| + \int_{t_1}^{t} |\phi^{-1}[\phi(B_1) + \int_{t_1}^{s} p(\tau)\psi(m(r))d\tau]|ds \\ &\le |A_1| + \int_{t_1}^{t} |\phi^{-1}[\phi(B_1) + t_2(\|p\|_{L^1})\psi(m(r))]|ds.\end{aligned}$$

Then

$$m(t) \le |A_1| + \int_{t_1}^{t} \psi_1(m(s))ds, \quad t \in [t_1, t_2],$$

where $\psi_1 = (\phi^{-1} \circ \widetilde{\psi})$ and $\widetilde{\psi}(u) = \phi(B_1) + t_2(\|p\|_{L^1})\psi(u)$.

By the nonlinear Grönwall-Bihari inequality (Lemma 2.2), we infer the bound

$$m(t) \le H^{-1}(t) \le M_1,$$

where $H(t) = \int_{|A_1|}^{t} \frac{d\tau}{(\phi^{-1}\circ\widetilde{\psi})(\tau)}$.

- For $t \in (t_m, b]$, we have

$$y(t) = A_m + \int_{t_m}^{t} \phi^{-1}[\phi(B_m) + \int_{t_m}^{s} f(\tau, y)d\tau]ds.$$

As in the pattern, there exists $M_m > 0$ such that

$$m(t) \le H^{-1}(t) \le M_m,$$

where $H(t) = \int_{|A_m|}^{t} \frac{d\tau}{(\phi^{-1}\circ\widetilde{\psi})(\tau)}$. Hence

$$\|y\|_{PC} \quad \le \max(M_0, M_1, \dots, M_m) = M.$$

The proof is complete. □

For the next theorem we use the assumptions:

(H4) $f : [0, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

(H5) There exist a continuous nondecreasing function $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to (0, \infty)$ and $p \in L^1(J, \mathbb{R})$ such that

$$|f(t, x, y)| \le p(t)\psi(|x|, |y|) \quad for \quad all \quad x, y \in \mathbb{R}, \quad t \in J$$

with

$$\int_0^b p(s)ds < \int_{|A|+c|B|}^{\infty} \frac{du}{(\phi^{-1} \circ \psi)(u, u)}.$$

Theorem 3.4. *Under assumptions* (H4), (H5)*, problem* (1.5)-(1.8) *has at least one solution.*

Prior to the proof of Theorem 3.4, we present a useful lemma.

Lemma 3.5. *The operator $L : D \to PC(J, \mathbb{R})$ defined by $L(y) = (\phi(y'))'$ where $D = \{y \in PC^1(J, \mathbb{R}) : y(t_k^+) = y(t_k) + I_k(y(t_k)), y'(t_k^+) = y'(t_k) + \bar{I}_k(y(t_k)), y(t_k) = y(t_k^-), y'(t_k) = y'(t_k^-), k = 1, \dots, m, y(0) = A, y'(0) = B\}$. Assume that L is well defined. Then L is bijective and L^{-1} is completely continuous.*

Proof. **Step 1:** L is bijective.

- L is injective. Let $y_1, y_2 \in D$ be such that $L(y_1) = L(y_2)$. Then

$$(\phi(y_1'(t)))' = (\phi(y_2'(t)))', t \in [0, t_1],$$

and thus

$$\begin{gathered}\phi(y_1'(t)) - \phi(y_1'(0)) = \phi(y_2'(t)) - \phi(y_2'(0)), \quad t \in [0, t_1], \\ \phi(y_1'(t)) - \phi(B) = \phi(y_2'(t)) - \phi(B), \quad t \in [0, t_1].\end{gathered}$$

Hence $y_1'(t) = y_2'(t)$ for $t \in [0, t_1]$. By integration of this equality, we obtain

$$\int_0^t y_1'(s)ds = \int_0^t y_2'(s)ds, \quad t \in [0, t_1]$$

which implies $y_1(t)-y_1(0) = y_2(t)-y_2(0)$, $t \in [0, t_1]$. This implies that $y_1(t) = y_2(t)$, $t \in [0, t_1]$.

Next,

$$\phi(y_1'(t)) - \phi(y_1'(t_1) + \bar{I}_1(y_1(t_1))) = \phi(y_2'(t)) - \phi(y_2'(t_1) + \bar{I}_1(y_2(t_1))), \quad t \in (t_1, t_2]$$

implies $y_1'(t) = y_2'(t)$, $t \in (t_1, t_2]$, and so

$$\int_{t_1}^t y_1'(s)ds = \int_{t_1}^t y_2'(s)ds, \quad t \in (t_1, t_2]$$

implies $y_1(t) - (y_1(t_1) + I_1(y_1(t_1))) = y_2(t) - (y_2(t_1) + I_1(y_2(t_1)))$, $t \in (t_1, t_2]$, and then

$$y_1(t) = y_2(t), \quad t \in (t_1, t_2].$$

Continuing this pattern,

$$\phi(y_1'(t)) - \phi(y_1'(t_m) + \bar{I}_m(y_1(t_m))) = \phi(y_2'(t)) - \phi(y_2'(t_m) + \bar{I}_m(y_2(t_m))), \quad t \in (t_m, b]$$

implies $y_1'(t) = y_2'(t)$, $t \in (t_m, b]$, and so

$$\int_{t_m}^t y_1'(s)ds = \int_{t_m}^t y_2'(s)ds, \quad t \in (t_m, b]$$

implies $y_1(t) - (y_1(t_m) + I_m(y_1(t_m))) = y_2(t) - (y_2(t_m) + I_m(y_2(t_m)))$, $t \in (t_m, b]$, and hence $y_1(t) = y_2(t)$, $t \in (t_m, b]$. This implies that $y_1 = y_2$.

- L is surjective. Let $h \in PC(J, \mathbb{R})$, then we define

$$y(t) = \begin{cases} L_0(h)(t), & \text{if } t \in [0, t_1], \\ L_1(h)(t), & \text{if } t \in (t_1, t_2], \\ \dots \\ L_{m-1}(h)(t), & \text{if } t \in (t_m, b], \end{cases} \tag{3.4}$$

where

$$L_0(h)(t) = A + \int_0^t \phi^{-1}\Big[\phi(B) + \int_0^s h(\tau)d\tau\Big]ds, \quad t \in [0, t_1],$$

$$\begin{aligned} L_1(h)(t) = {} & L_0(h)(t_1) + I_1(L_0(h)(t_1)) \\ & + \int_{t_1}^t \phi^{-1}\Big[\phi(L_0'(h)(t_1) + \bar{I}_1(L_0(h)(t_1))) + \int_{t_1}^s h(\tau)d\tau\Big]ds, \quad t \in (t_1, t_2], \end{aligned}$$

$$\begin{aligned} L_2(h)(t) &= L_1(h)(t_2) + I_2(L_1(h)(t_2)) \\ &\quad + \int_{t_2}^t \phi^{-1}\Big[\phi(L_1'(h)(t_2) + \bar{I}_2(L_1(h)(t_2))) + \int_{t_2}^s h(\tau)d\tau\Big]ds, \quad t \in (t_2, t_3], \end{aligned}$$

$$\dots$$

$$\begin{aligned} L_m(h)(t) = L_{m-1}(h)(t_m) + I_m(L_{m-1}(h)(t_m)) + \int_{t_m}^t \phi^{-1}\Big[\phi(L_{m-1}'(h)(t_m) \\ + \bar{I}_m(L_{m-1}(h)(t_m))) + \int_{t_m}^s h(\tau)d\tau\Big]ds, \quad t \in (t_m, b]. \end{aligned}$$

From (3.4) we can easily check that

$$\begin{aligned} y(t) &= A + \sum_{0<t_k<t} I_k(L_{k-1}(h)(t_k)) \\ &\quad + \int_0^t \phi^{-1}\Big[\phi(B + \sum_{0<t_k<t} \bar{I}_k(L_{k-1}(h)(t_k))) + \int_0^s h(\tau)d\tau\Big]ds, \quad t \in J. \end{aligned}$$

Hence

$$y'(t) = \phi^{-1}\Big[\phi(B + \sum_{0<t_2<t} \bar{I}_k(L_{k-1}(h)(t_k))) + \int_0^s h(s)ds\Big].$$

From the definition of y and y' we can prove that $y(0) = A$, $y'(0) = B$, $y(t_k^+) = y(t_k) + I_k(y(t_k))$, $y'(t_k^+) = y'(t_k) + \bar{I}_k(y(t_k))$ and $y(t_k) = y_{t_k^-}$, $k = j, \dots, m$ and by using the fact that $I_k, \bar{I}_k$ are continuous we can easily prove that $y, y' \in PC(J, \mathbb{R})$.

Step 2: L^{-1} is completely continuous.

Claim 1: L^{-1} is continuous. Let $h_n \in PC(J, \mathbb{R})$ be such that h_n converges to h in $PC(J, \mathbb{R})$ as $n \to \infty$. We show that $L^{-1}(h_n)$ converges to $L^{-1}(h)$. Let $\{y_n\}_{n\in\mathbb{N}} \subset D$ such that $\{L(y_n)\}_{n\in\mathbb{N}} = \{h_n\}_{n\in\mathbb{N}}$. Then:

• For $t \in [0, t_1]$, we have

$$y_n'(t) = \phi^{-1}\Big[\phi(B) + \int_0^t h_n(s)ds\Big],$$

and

$$y_n(t) = A + \int_0^t \phi^{-1}\Big[\phi(B) + \int_0^s h_n(\tau)d\tau\Big]ds = A + \int_0^t y_n'(s)ds.$$

Hence

$$|y_n'(t)| \le \Big|\phi^{-1}\Big[\phi(B) + \int_0^t h_n(s)ds\Big]\Big|,$$

since

$$\begin{aligned} |\phi(B) + \int_0^t h_n(s)ds| &\le |\phi(B)| + \int_0^t |h_n(s)|ds \\ &\le |\phi(B)| + t_1\|h_n\|_{PC} \\ &\le |\phi(B)| + t_1 M_* = K, \end{aligned}$$

where $\|h_n\|_{PC} \le M_*$ for all $n \in \mathbb{N}$. Then $[\phi(B) + \int_0^t h_n(s)ds] \in \overline{B}(0, K)$. Since ϕ^{-1} is continuous and $\overline{B}(0, K)$ is compact,

$$\sup_{x\in\overline{B}(0,K)} |\phi^{-1}(x)| < \infty.$$

Then

$$\|y_n'\|_\infty \le \sup_{x\in \overline{B}(0,K)} |\phi^{-1}(x)| := M_0,$$

and

$$\|y_n\|_\infty \le |A| + M_0 t_1 := \overline{M_0}.$$

So, $\{y_n\}_{n\in\mathbb{N}}$ and $\{y_n'\}_{n\in\mathbb{N}}$ are bounded uniformly in $C([0,t_1],\mathbb{R})$.

We put $C = \{y_n : n \in \mathbb{N}\} \subseteq C([0,t_1],\mathbb{R})$. We can easily show that C is bounded and equicontinuous, and then from the Ascoli-Arzela theorem we conclude that C is compact. Then y_n has a subsequence (y_{n_m}) converging to y. Let

$$z(t) = A + \int_0^t \phi^{-1}\Big[\phi(B) + \int_0^s h(\tau)d\tau\Big]ds$$

so that

$$|y_{n_m}(t) - z(t)| \le \int_0^{t_1} |\phi^{-1}\Big[\phi(B) + \int_0^s h_{n_m}(\tau)d\tau\Big] - \phi^{-1}\Big[\phi(B) + \int_0^s h(\tau)d\tau\Big]|ds.$$

Since ϕ^{-1} is continuous and as $n_m \to \infty$, $y_{n_m} \to z(t)$, then

$$y(t) = A + \int_0^t \phi^{-1}\Big[\phi(B) + \int_0^s h(\tau)d\tau\Big]ds.$$

By the same technique, we can prove that $\{y_n'\}$ converges to $y'(t)$ for $t \in [0,t_1]$.

- For $t \in (t_1,t_2]$, we have

$$y_n'(t) = y_n'(t_1) + \phi^{-1}\Big[\phi(y_n'(t_1) + \bar{I}_1(y_n(t_1))) + \int_{t_1}^t h_n(s)ds\Big]$$

and

$$\begin{aligned} y_n(t) &= y_n(t_1) + I_1(y_n(t_1)) + \int_{t_1}^t \phi^{-1}\Big[\phi(y_n'(t_1) + \bar{I}_1(y_n(t_1))) + \int_{t_1}^s h_n(\tau)d\tau\Big] \\ &= y_n(t_1) + I_1(y_n(t_1)) + \int_{t_1}^t y_n'(s)ds. \end{aligned}$$

Hence

$$\begin{aligned} |y_n'(t)| &\le |y_n'(t_1)| + \Big|\phi^{-1}\Big[\phi(y_n'(t_1) + \bar{I}_1(y_n(t_1))) + \int_{t_1}^t h_n(s)ds\Big]\Big| \\ &\le M_0 + \Big|\phi^{-1}\Big[\phi(y_n'(t_1) + \bar{I}_1(y_n(t_1))) + \int_{t_1}^t h_n(s)ds\Big]\Big|. \end{aligned}$$

Since

$$|\phi(y_n'(t_1) + \bar{I}_1(y_n(t_1))) + \int_{t_1}^t h_n(s)ds| \le |\phi(M_0 + \sup_{x\in\overline{B}(0,M_0)} |\bar{I}_1(x)|)| + t_2 K = K_*,$$

then

$$|\phi(y_n'(t_1) + \bar{I}_1(y_n(t_1))) + \int_{t_1}^t h_n(s)ds| \in \overline{B}(0, |\phi(M_0 + \sup_{x\in\overline{B}(0,M_0)} |\bar{I}_1(x)|)| + t_2 K).$$

Since ϕ^{-1} is continuous and $\overline{B}(0,|\phi(M_0 + \sup_{x\in\overline{B}(0,M_0)} |\bar{I}_1(x)|)| + t_2K)$ is compact,

$$\|y_n'\|_\infty \le M_0 + \sup_{x\in\overline{B}(0,|\phi(M_0+\sup_{x\in\overline{B}(0,M_0)} |\bar{I}_1(x)|)|+t_2K)} |\phi^{-1}(x)| := M_1,$$

and

$$\|y_n\|_\infty \leq \overline{M_0} + \sup_{x\in\overline{B}(0,M_0)} |I_1(z)| + M_1 t_2 := \overline{M_1}.$$

Then $\{y_n\}_{n\in\mathbb{N}}$ and $\{y_n'\}_{n\in\mathbb{N}}$ are bounded uniformly in $C((t_1,t_2],\mathbb{R})$. We put $C = \{y_n : n \in \mathbb{N}\} \subseteq C((t_1,t_2],\mathbb{R})$. We can easily show again that C is bounded and equicontinuous, and then from the Ascoli-Arzela theorem we conclude that C is compact. Then y_n has a subsequence (y_{n_m}) converging to y.

Now, let

$$z(t) = y(t_1) + I_1(y(t_1)) + \int_{t_1}^t \phi^{-1}\Big[\phi(y'(t_1) + \bar{I}_1(y(t_1))) + \int_{t_1}^s h(\tau)d\tau\Big]ds.$$

Then

$$\begin{aligned} |y_{n_m}(t) - z(t)| \leq |y_{n_m}(t_1) - y(t_1)| + |I_1(y_{n_m}(t_1)) - I_1(y(t_1))| \\ \int_{t_1}^t |\phi^{-1}\Big[\phi(y'_{n_m}(t_1) + \bar{I}_1(y_{n_m}(t_1))) + \int_{t_1}^s h_{n_m}(\tau)d\tau\Big] \\ - \phi^{-1}\Big[\phi(y'(t_1) + \bar{I}_1(y(t_1))) + \int_{t_1}^s h(\tau)d\tau\Big]|ds, \end{aligned}$$

Since ϕ^{-1} is continuous, $\{y_n\}_{n\in\mathbb{N}}$ and $\{y_n'\}_{n\in\mathbb{N}}$ converge to y and y', respectively, for $t \in [0,t_1]$, and as $n_m \to \infty$, $y_{n_m} \to z(t)$, then

$$y(t) = y(t_1) + I_1(y(t_1)) + \int_{t_1}^t \phi^{-1}\Big[\phi(y'(t_1) + \bar{I}_1(y(t_1))) + \int_{t_1}^s h(\tau)d\tau\Big]ds.$$

By the same technique, we can prove that $\{y_n'\}$ converges to $y'(t)$ for $t \in (t_1,t_2]$.

• We continue this process until we get, for every $t \in (t_m,b]$, that $y_n(t)$ converges to $y(t)$ and $y_n'(t)$ converges to $y'(t)$. We conclude that $L(y) = h$, and this implies that L^{-1} is continuous.

Claim 2: L^{-1} is compact. Let $\mathcal{D}$ be a bounded set of $PC(J,\mathbb{R})$ and $\{y_n\}_{n\in\mathbb{N}} \subset L^{-1}(\mathcal{D})$. Then there exists $\{h_n\}_{n\in\mathbb{N}} \subset \mathcal{D}$ such that $L(y_n) = h_n$, for all $n \in \mathbb{N}$.

We show that $|L_0^{-1}(h_n)(l_2) - L^{-1}(h_n)(l_1)|$ tends to zero as $l_2 \to l_1$. Since $L_0(y_n)(t) = h_n(t),\ t \in [0,t_1]$, it follows that

$$y_n'(t) = \phi^{-1}\Big[\phi(B) + \int_0^t h_n(s)ds\Big], \quad t \in [0,t_1].$$

Using the fact that h_n is bounded, thus there exist $M_0 > 0$ such that

$$\|y_n\|_\infty, \|y_n'\|_\infty \leq M_0, \quad \text{for all } n \in \mathbb{N}.$$

• Let $l_1, l_2 \in [0,t_1]$, $l_1 < l_2$. Then, by the mean value theorem,

$$|y_n(l_2) - y_n(l_1)| = |y_n'(\xi_n)(l_2 - l_1)| \leq M_0|l_2 - l_1|,$$

and

$$|\phi(y_n')(l_2) - \phi(y_n')(l_1)| = \Big|\int_{l_1}^{l_2} h_n(s)ds\Big| \leq \int_{l_1}^{l_2} |h_n(s)|ds \leq |l_2 - l_1|M_*,$$

where $\|h_n\|_{PC} \leq M_*$ for all $n \in \mathbb{N}$. As $l_2 \to l_1$ the right hand side of the above inequality tends to zero. Then $\{y_n(\cdot)\}_{n\in\mathbb{N}}$ is equicontinuous and $\{y_n\}_{n\in\mathbb{N}}$ is bounded. By the Ascoli-Arzela theorem there exist $y_0, z_0 \in C([0,t_1],\mathbb{R})$ such that y_n and $\phi(y_n')$ converge, respectively, to y_0 and z_0. Since ϕ^{-1} is a continuous function,

$$y_n'(t) \to \phi^{-1}(z_0)(t),\ n \to \infty.$$

Set

$$y_*(t) = A + \int_0^t \phi^{-1}(z_0(s))ds,\ t \in [0, t_1],$$

and from

$$y_n(t) = A + \int_0^t y_n'(s)ds,\ t \in [0, t_1],$$

we have

$$\|y_n - y_*\|_\infty \le \int_0^{t_1} |y_n'(s) - \phi^{-1}(z_0)(s)|ds.$$

From the Lebesgue dominated convergence theorem, we deduce that $\{y_n\}_{n\in\mathbb{N}}$ converges to y_* in $C([0, t_1], \mathbb{R})$, and this implies that

$$y_0(t) = A + \int_0^t \phi^{-1}(z_0(s))ds,\ t \in [0, t_1] \Rightarrow y_0'(t) = \phi^{-1}(z_0(t)), \quad t \in [0, t_1].$$

Then y_n converges to y_0 in $C^1([0, t_1], \mathbb{R})$.

For $t \in (t_1, t_2]$,

$$y_n'(t) = \phi^{-1}\Big[\phi(y_n'(t_1) + \bar{I}_1(y_n(t_1))) + \int_{t_1}^t h_n(s)ds\Big], \quad t \in (t_1, t_2].$$

Using the fact that h_n is bounded, there exists $M_1 > 0$ such that

$$\|y_n\|_\infty, \|y_n'\|_\infty \le M_1,\ \text{for all } n \in \mathbb{N}.$$

- Let $l_1, l_2 \in (t_1, t_2]$, $l_1 < l_2$. Then

$$|y_n(l_2) - y_n(l_1)| = |y_n'(\xi_n)(l_2 - l_1)| \le M_1|l_2 - l_1|,$$

and

$$|\phi(y_n')(l_2) - \phi(y_n')(l_1)| \le M_*|l_2 - l_1|.$$

As $l_2 \to l_1$ the right hand side of the above inequality tends to zero, then $\{y_n(\cdot)\}$ and $\{\phi(y_n')(\cdot)\}$ are equicontinuous. By Ascoli-Arzela theorem there exist $y_1, z_1 \in C([0, t_1], \mathbb{R})$ such that y_n and $\phi(y_n')$ converge, respectively, to y_1, z_1. Since ϕ^{-1} and I_1 are a continuous functions, then $y_n'(t) \to \phi^{-1}(z_1)(t)$ as $n \to \infty$. Set

$$y_{**}(t) = y_0(t_1) + I_1(y_0(t_1)) + \int_{t_1}^t \phi^{-1}(z_1(s))ds, \quad t \in (t_1, t_2].$$

From

$$y_n(t) = y_0(t_1) + I_1(y_0(t_1)) + \int_{t_1}^t y_n'(s)ds,\ t \in (t_1, t_2],$$

we have

$$\|y_n - y_{**}\|_\infty \le \int_0^{t_1} |y_n'(s) - \phi^{-1}(z_1)(s)|ds.$$

From the Lebesgue dominated convergence theorem, we deduce that $\{y_n\}_{n\in\mathbb{N}}$ converges to y_{**} in $C_1 = \{y \in C^1((t_1, t_2], \mathbb{R}) : y(t_1^+), y'(t_1^+) \text{ exist}\}$. This implies that

$$y_1(t) = y_0(t_1) + I_1(y_0(t_1)) + \int_{t_1}^t \phi^{-1}(z_0(s))ds, \quad t \in [0, t_1]$$

implies $y_1'(t) = \phi^{-1}(z_1(t)), \quad t \in (t_1, t_2]$. Then y_n converges to y_1 in C_1.

• We continue this process until we have that there exists $y_m \in C^1((t_m,b],\mathbb{R})$ such that y_n converge to y_m in $C_m = \{y \in C^1((t_m,b],\mathbb{R}) | y(t_m^+), y'(t_m^+) \text{ exist}\}$. We define

$$y(t) = \begin{cases} y_0(t), & \text{if } t \in [0,t_1], \\ y_1(t), & \text{if } t \in (t_1,t_2], \\ \dots \\ y_m(t), & \text{if } t \in (t_m,b]. \end{cases} \tag{3.5}$$

It is clear that $\{y_n\}_{n\in\mathbb{N}} \subset PC$ and y_n converges to y in PC. Using the fact that L^{-1} is continuous, thus

$$h_n = L^{-1}(y_n) \to L^{-1}(y) = h \quad \text{as } n \to \infty$$

in PC^1. Then $L(y) = h$. Hence L^{-1} is compact. This completes the proof. □

Proof of Theorem 3.4. We consider the following fixed point problem that is equivalent to problem (1.5)–(1.8),

$$y = (L^{-1} \circ F)(y),$$

where $L^{-1} : PC(J,\mathbb{R}) \to D$ and F is the Nemystki operator given by $F(t,y) = f(t,y,y')$. From Lemma 3.5 we can prove that $(L^{-1}\circ F)$ is compact. Now we show that $y \neq \lambda(L^{-1}\circ F)(y)$.

For $t \in [0,t_1]$, we have

$$y'(t) = \phi^{-1}\Big[\phi(B) + \int_0^t f(s,y(s),y'(s))ds\Big],$$
$$y(t) = A + \int_0^t y'(s)ds.$$

Thus

$$|y'(t)| \le \Big|\phi^{-1}\Big[\phi(B) + \int_0^s p(s)\psi(|y(s)|,|y'(s)|)ds\Big]\Big|,$$

$$\begin{aligned} |y(t)| &\le |A| + \int_0^t |y'(s)|ds \\ &\le |A| + \int_0^t \Big|\phi^{-1}\Big[\phi(B) + \int_0^s p(\tau)\psi(|y(\tau)|,|y'(\tau)|)d\tau\Big]\Big|ds. \end{aligned}$$

Let $m(r) = \max(\sup_{t\in[0,t_1]}|y(t)|, \sup_{t\in[0,t_1]}|y'(t)|)$, then

$$\begin{aligned} |y(t)| &\le |A| + \int_0^t \Big|\phi^{-1}\Big[\phi(B) + \int_0^s p(\tau)\psi(m(r),m(r))d\tau\Big]\Big|ds \\ &\le |A| + \int_0^t \Big|\phi^{-1}[\phi(B) + t_1\|p\|_{L^1}\psi(m(r),m(r))]\Big|ds. \end{aligned}$$

Then

$$m(t) \le |A| + \int_0^t \psi_1(m(s))ds,$$

with $\psi_1 = \phi^{-1}\circ\widetilde{\psi}$ and $\widetilde{\psi}(u) = \phi(B) + t_1(\|p\|_{L^1})\psi(u)$. By the nonlinear Grönwall-Bihari inequality (Lemma 2.2), we infer the bound

$$m(t) \le H^{-1}(t) \le M_0,$$

where

$$H(t)=\int_{|A|}^{t}\frac{d\tau}{(\phi^{-1}\circ\widetilde{\psi})(\tau)}.$$

For $t\in(t_1,t_2]$, we have

$$y'(t)=y'(t_1)+\phi^{-1}\Big[\phi(y'(t_1)+\bar{I}_1(y(t_1)))+\int_{t_1}^{t}f(s,y(s),y'(s))ds\Big],$$
$$y(t)=y(t_1)+I_1(y(t_1))+\int_{t_1}^{t}y'(s)ds.$$

Thus

$$|y'(t)|\le|y'(t_1)|+\Big|\phi^{-1}\Big[\phi(y'(t_1)+\bar{I}_1(y(t_1)))+\int_{t_1}^{t}f(s,y(s),y'(s))ds\Big]\Big|.$$

Let

$$m(r)=\max(\sup_{t\in[0,t_1]}|y(t)|,\sup_{t\in[0,t_1]}|y'(t)|),$$

and then

$$\begin{aligned}|y(t)|&\le|y(t_1)|+|I_1(y(t_1))|+t_2|y'(t_1)|\\&\quad+\int_{t_1}^{t}\Big|\phi^{-1}\Big[\phi(y'(t_1)+\bar{I}_1(y(t_1)))+\int_{t_1}^{s}p(\tau)\psi(|y(\tau)|,|y'(\tau)|)d\tau\Big]\Big|ds\\&\le M_0+|I_1(y(t_1))|+t_2|y'(t_1)|\\&\quad\int_{t_1}^{t}\Big|\phi^{-1}\Big[\phi(y'(t_1)+\bar{I}_1(y(t_1)))+\int_{t_1}^{s}p(\tau)\psi(m(r),m(r))d\tau\Big]\Big|ds.\end{aligned}$$

Then $m(t)\le M^*+\int_{t_1}^{t}\psi_1(m(s))ds$, where

$$M^*=(1+t_2)M_0+\sup_{z\in\overline{B}(0,M_0)}|I_1(z)|,$$
$$\psi_1=\phi^{-1}\circ\widetilde{\psi},$$
$$\widetilde{\psi}(u)=\phi(y'(t_1)+\bar{I}_1(y(t_1)))+t_2(\|p\|_{L^1})\psi(u,u).$$

By the nonlinear Grönwall-Bihari inequality (Lemma 2.2), we infer the bound

$$m(t)\le H^{-1}(t)\le M_1,$$

where

$$H(t)=\int_{M^*}^{t}\frac{d\tau}{(\phi^{-1}\circ\widetilde{\psi})(\tau)}.$$

For $t\in(t_m,b]$, we have

$$\begin{aligned}y'(t)&=y'(t_m)+\phi^{-1}\Big[\phi(y'(t_m)+\bar{I}_m(y(t_m)))+\int_{t_m}^{t}f(s,y(s),y'(s))ds\Big],\\y(t)&=y(t_m)+I_m(y(t_m))\\&\quad+\int_{t_m}^{t}\phi^{-1}\Big[\phi(y'(t_m)+\bar{I}_m(y(t_m)))+\int_{t_m}^{s}f(\tau,y(\tau),y'(\tau))d\tau\Big]ds.\end{aligned}$$

So, there exists $M_m>0$ such that $m(t)\le H^{-1}(t)\le M_m$, where

$$H(t)=\int_{M^{**}}^{t}\frac{d\tau}{(\phi^{-1}\circ\widetilde{\psi})(\tau)}$$

and
$$M^{**} = (1+b)M_{m-1} + \sup_{z\in\overline{B}(0,M_{m-1})} |I_m(z)|.$$
Hence
$$\|y\|_\infty \le \max(M_0, M_1, \dots, M_m) := M.$$
Let
$$U = \{y \in PC^1(J,\mathbb{R}) : \|y\|_{PC^1} < M+1\}.$$
Then $L^{-1} \circ F : \overline{U} \to PC^1(J,\mathbb{R}$ is relatively compact. Assume that there exists $\lambda \in (0,1)$ and $y \in \partial U$ such that $y = \lambda(L^{-1} \circ F)(y)$. Then $\|y\|_{PC^1} = M+1$, but $\|y\|_{PC^1} \le M$. Thus by the nonlinear alternative of Leray-Schauder, we conclude that $L^{-1} \circ F$ has a fixed point which is a solution of (1.5)-(1.8). □

References

[1] R. P. Agarwal, M. Benchohra, D. O'Regan, A. Ouahab; Second order impulsive dynamic equations on time scales, *Functional Differential Equations* **11** (2004), no. 3-4, 223-234.

[2] R. P. Agarwal, D. O'Regan; Multiple nonnegative solutions for second order impulsive differential equations, *Applied Mathematics and Computation*, **114** (2000), no. 1, 51-59.

[3] Z. Agur, L. Cojocaru, G. Mazaur, R. M. Anderson, Y. L. Danon; Pulse mass measles vaccination across age cohorts, *Proc. Nat. Acad. Sci. USA* **90** (1993), 11698–11702.

[4] D. D. Bainov, P. S. Simeonov; *Systems with Impulse Effect*, Ellis Horwood Ltd., Chichister, 1989.

[5] D. D. Bainov, P. S. Simeonov; *Integral Inequalities and Applications, in: Mathematics and its Applications*, vol. 57, Kluwer Academic Publishers, Dordrecht, 1992

[6] A. Granas, J. Dugundji; *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[7] S. Djebali, L. Gorniewicz, A. Ouahab; *Existence and Structure of Solution Sets for Impulsive Differential Inclusions*, Lecture Notes,in Nonlinear Analysis, Nicolaus Copernicus University, Toruń, No 13 , 2012.

[8] A. Halanay, D. Wexler; *Teoria Calitativa a systeme cu Impulduri,* Editura Republicii Socialiste Romania, Bucharest, 1968.

[9] E. Kruger-Thiemer; Formal theory of drug dosage regiments, *J. Theo. Biol.* **13** (1966), 212–235.

[10] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov; *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[11] V. D. Milman, A. A. Myshkis; On the stability of motion in the presence of impulses (in Russian), *Sib. Math. J.* **1** (1960), 233–237.

[12] N. A. Perestyuk, V. A. Plotnikov, A. M. Samoilenko, N. V. Skripnik; *Differential Equations with Impulse Effects Multivalued Right-hand Sides with Discontinuities,* Walter de Gruyter & Co., Berlin, 2011.

[13] A. M. Samoilenko, N. A. Perestyuk; *Impulsive Differential Equations,* World Scientific, Singapore, 1995.

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