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MULTIPLE POSITIVE PERIODIC SOLUTIONS FOR A FOOD-LIMITED TWO-SPECIES RATIO-DEPENDENT PREDATOR-PREY PATCH SYSTEM WITH DELAY AND HARVESTING

HUI FANG

Abstract. By using Mawhin's coincidence degree theory, this paper establishes some sufficient conditions on the existence of four positive periodic solutions for a food-limited two-species ratio-dependent predator-prey patch system with delay and harvesting. Some novel techniques are employed to obtain the appropriate *a priori* estimates. An example is given to illustrate our results.

1. Introduction

Since the exploitation of biological resources and the harvest of population species are related to the optimal management of renewable resources (see [5]), many researchers pay much attention to the study of population dynamics with harvesting in mathematical bioeconomics. For example, Brauer and Soudack [1, 2] analyzed the global behaviour of some predator-prey systems under constant rate harvesting and/or stocking of both species. Xiao and Jennings [15] studied the Bogdanov-Takens bifurcations in predator-prey systems with constant rate harvesting. But all the coefficients in the system they studied are constants. Recently, some researchers studied the existence of multiple positive periodic solutions for some predator-prey systems under the assumption of periodicity of the parameters by using Mawhin's coincidence degree theory (see [4, 7, 14, 17]). These papers only focused on predator-prey systems without diffusion. However, due to the spatial heterogeneity and unbalanced food resources, the migration phenomena of biological species can often occur between heterogeneous spatial environments or patches. On the other hand, as pointed out by Huusko and Hyvarinen in [12]: "the dynamics of exploited populations are clearly affected by recruitment and harvesting, and the changes in harvesting induced a tendency to generation cycling in the dynamics of a freshwater fish population". So, it is very important to describe the potential role of harvesting as an external factor in producing and maintaining the

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periodic fluctuation of population of species. Mathematically, this is equivalent to the investigation of harvesting induced periodic solutions.

In this paper, we consider the following food-limited two-species ratio-dependent predator-prey patch system with delay and harvesting:

$$
\begin{aligned}
x_1'(t) &= \frac{x_1(t)}{k_1(t)+c_1(t)x_1(t)}\Big[a_1(t)-a_{11}(t)x_1(t)-\frac{a_{13}(t)x_1(t)y(t)}{m(t)y^2(t)+x_1^2(t)}\Big] \\
&\quad + D_1(t)[x_2(t)-x_1(t)]-H_1(t), \\
x_2'(t) &= \frac{x_2(t)}{k_2(t)+c_2(t)x_2(t)}\big[a_2(t)-a_{22}(t)x_2(t)\big]+D_2(t)[x_1(t)-x_2(t)]-H_2(t), \\
y'(t) &= y(t)\Big[-a_3(t)+\frac{a_{31}(t)x_1^2(t-\tau)}{m(t)y^2(t-\tau)+x_1^2(t-\tau)}\Big],
\end{aligned}
\tag{1.1}
$$

where x_1 and y are the population numbers of prey species x and predator species y in patch 1, and x_2 is the population number of species x in patch 2. Prey dispersion is included in the model to allow for prey refuge from predation. So, the predator species y is confined to patch 1, while the prey species x can diffuse between two patches. For biological relevance of allowing for prey refuge from predation, see Yakubu [16]. $a_1(t)(a_2(t))$ is the natural growth rate of prey species x in patch 1 (patch 2) in the absence of predation, $a_3(t)$ is the natural death rate of predator species y in the absence of food, $a_{13}(t)$ is the death rate per encounter of prey species x due to predation, $a_{31}(t)$ is the efficiency of turning predated prey species x into predator species y. $k_i(t)$ $(i=1,2)$ are the population numbers of prey species in patch 1 (patch 2) at saturation, respectively.

When $c_i(t)\neq 0 (i=1,2)$, $\frac{a_i(t)}{k_i(t)c_i(t)}$ $(i=1,2)$ are the rate of replacement of mass in the population at saturation (including the replacement of metabolic loss and of dead organisms). The effect of delay τ refers to the dynamics of a predator being related to the predation in the past. $m(t)$ is the half capturing saturation coefficient. $a_{ii}(t)$ $(i=1,2)$ are the density-dependent coefficients. $D_i(t)$ $(i=1,2)$ are diffusion coefficients of species x. $H_i(t)$ $(i=1,2)$ denote the harvesting rates.

$$
\frac{a_{13}(t)x_1(t)y(t)}{m(t)y^2(t)+x_1^2(t)}, \quad \frac{a_{31}(t)x_1^2(t-\tau)}{m(t)y^2(t-\tau)+x_1^2(t-\tau)}
$$

are ratio-dependent Holling type III functional responses. When $c_i(t)\equiv 0$ $(i=1,2)$, $H_i(t)\equiv 0$ $(i=1,2)$, Dong and Ge in [6] showed that (1.1) has at least one positive periodic solution under the appropriate conditions. When $c_i(t)\neq 0$ $(i=1,2)$, system (1.1) is a food-limited population model. For other food-limited population models, we refer to [9, 10, 11, 13] and the references cited therein.

To our knowledge, a few papers have been published on the existence of multiple periodic solutions for (1.1). In this paper, we study the existence of multiple positive periodic solutions of (1.1) by using Mawhin's coincidence degree. Since system (1.1) involves the diffusion terms and the rates of replacement, the methods used in [4, 7, 14, 17] are not available to system (1.1). Our method of defining the operator $N(u,\lambda)$ facilitates the computation of Brouwer degree $\deg_B(JQN(\cdot,0)|_{\ker L},\Omega\cap\ker L,0)$.

2. Existence of four positive periodic solutions

To give the proof of the main results, we first make the following preparations (see [8]).

Let X, Z be normed vector spaces, $L : \operatorname{dom} L \subset X \to Z$ a linear mapping, $N : X \times [0,1] \to Z$ is a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $dim \ker L = codim \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero, there then exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $ImP = \ker L, \operatorname{Im} L = \ker Q = Im(I - Q)$. If we define $L_P : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$ as the restriction $L|_{\operatorname{dom} L \cap \ker P}$ of L to $\operatorname{dom} L \cap \ker P$, then L_P is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega} \times [0,1]$ if $QN(\bar{\Omega} \times [0,1])$ is bounded and $K_P(I-Q)N : \bar{\Omega} \times [0,1] \to X$ is compact; i.e., continuous and such that $K_P(I - Q)N(\bar{\Omega} \times [0,1])$ is relatively compact. Since $\operatorname{Im} Q$ is isomorphic to $\ker L$, there exists isomorphism $J : \operatorname{Im} Q \to \ker L$.

For convenience, we introduce Mawhin's continuation theorem [8, p.29] as follows.

Lemma 2.1. *Let L be a Fredholm mapping of index zero and let $N : \bar{\Omega} \times [0,1] \to Z$ be L-compact on $\bar{\Omega} \times [0,1]$. Suppose*

(a) *$Lu \neq \lambda N(u, \lambda)$ for every $u \in \operatorname{dom} L \cap \partial\Omega$ and every $\lambda \in (0,1)$;*
(b) *$QN(u,0) \neq 0$ for every $u \in \partial\Omega \cap \ker L$;*
(c) *Brouwer degree $\deg_B(JQN(\cdot,0)|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$.*

Then $Lu = N(u,1)$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$.

For the sake of convenience and simplicity, we denote

$$\bar{g} = \frac{1}{T}\int_0^T g(t)dt, \quad g^l = \min_{t\in[0,T]} g(t), \quad g^u = \max_{t\in[0,T]} g(t),$$

where g is a nonnegative continuous T-periodic function. Set

$$N_1 = \max\big\{(\frac{a_1}{a_{11}})^u, (\frac{a_2}{a_{22}})^u\big\}, \quad b_1 = \frac{(\frac{a_{13}}{k_1})^u}{2\sqrt{m^l}}, \quad b_2 = 0.$$

For the rest of this article, we assume the following:

(A1) $\tau > 0$, $k_i(t)$ $(i = 1,2)$, $a_i(t)$ $(i = 1,2,3)$, $a_{ii}(t)$ $(i = 1,2)$, $a_{13}(t)$, $a_{31}(t)$, $m(t)$, $D_i(t)$ $(i = 1,2)$, $H_i(t)$ $(i = 1,2)$ are positive continuous T-periodic functions, $c_i(t)$ $(i = 1,2)$ are nonnegative continuous T-periodic functions.
(A2) $a_{31}^l - \bar{a}_3 > 0$.
(A3) $\frac{k_i^l}{k_i^l + c_i^u N_1}(\frac{a_i}{k_i})^l > D_i^u + b_i + 2\sqrt{(\frac{a_{ii}}{k_i})^u H_i^u}$ $(i = 1,2)$.
(A4) $H_i^l > D_i^u N_1$ $(i = 1,2)$.

For further convenience, we introduce the 12 positive numbers:

$$l_i^{\pm} = \frac{(\frac{a_i}{k_i})^u \pm \sqrt{[(\frac{a_i}{k_i})^u]^2 - \frac{4k_i^l}{k_i^l + c_i^u N_1}(\frac{a_{ii}}{k_i})^l (H_i^l - D_i^u N_1)}}{\frac{2k_i^l}{k_i^l + c_i^u N_1}(\frac{a_{ii}}{k_i})^l},$$

$$u_i^{\pm} = \frac{[\frac{k_i^l}{k_i^l + c_i^u N_1}(\frac{a_i}{k_i})^l - D_i^u - b_i] \pm \sqrt{[\frac{k_i^l}{k_i^l + c_i^u N_1}(\frac{a_i}{k_i})^l - D_i^u - b_i]^2 - 4(\frac{a_{ii}}{k_i})^u H_i^u}}{2(\frac{a_{ii}}{k_i})^u},$$

$$x_i^{\pm} = \frac{\overline{(\frac{a_i}{k_i})} \pm \sqrt{\overline{[(\frac{a_i}{k_i})]^2} - 4\overline{(\frac{a_{ii}}{k_i})}\bar{H}_i}}{2\overline{(\frac{a_{ii}}{k_i})}}, i = 1, 2.$$

It is not difficult to show that

$$l_i^- < x_i^- < u_i^- < u_i^+ < x_i^+ < l_i^+ \quad (i = 1, 2) \tag{2.1}$$

Now, we are ready to state the main result of this article.

Theorem 2.2. *Assume that* (A1)-(A4) *hold, then* (1.1) *has at least four positive T-periodic solutions.*

Proof. Since we are concerning with positive solutions of (1.1), we make the change of variables,

$$x_j(t) = e^{u_j(t)} \ (j = 1, 2), \quad y(t) = e^{u_3(t)}.$$

Then (1.1) is rewritten as

$$\begin{aligned} u_1'(t) &= \frac{1}{k_1(t) + c_1(t)e^{u_1(t)}}\Big[a_1(t) - a_{11}(t)e^{u_1(t)} - \frac{a_{13}(t)e^{u_1(t)}e^{u_3(t)}}{m(t)e^{2u_3(t)} + e^{2u_1(t)}}\Big] \\ &\quad + D_1(t)[\frac{e^{u_2(t)}}{e^{u_1(t)}} - 1] - \frac{H_1(t)}{e^{u_1(t)}}, \\ u_2'(t) &= \frac{1}{k_2(t) + c_2(t)e^{u_2(t)}}[a_2(t) - a_{22}(t)e^{u_2(t)}] + D_2(t)[\frac{e^{u_1(t)}}{e^{u_2(t)}} - 1] - \frac{H_2(t)}{e^{u_2(t)}}, \\ u_3'(t) &= -a_3(t) + \frac{a_{31}(t)e^{2u_1(t-\tau)}}{m(t)e^{2u_3(t-\tau)} + e^{2u_1(t-\tau)}}. \end{aligned} \tag{2.2}$$

Take

$$X = Z = \{u = (u_1, u_2, u_3)^T \in C(R, R^3) : u_i(t + T) = u_i(t), i = 1, 2, 3\}$$

and define

$$\|u\| = \max_{t\in[0,T]} |u_1(t)| + \max_{t\in[0,T]} |u_2(t)| + \max_{t\in[0,T]} |u_3(t)|,$$

for $u = (u_1, u_2, u_3)^T$ in X or in Z. Equipped with the above norm $\|\cdot\|$, it is easy to verify that X and Z are Banach spaces.

Set

$$\begin{aligned} \Delta_1(u, t, \lambda) &= \Big[\frac{k_1(t) + (1-\lambda)c_1(t)e^{u_1(t)}}{k_1(t) + c_1(t)e^{u_1(t)}}\Big]\Big[\frac{a_1(t)}{k_1(t)} - \frac{a_{11}(t)e^{u_1(t)}}{k_1(t)} \\ &\quad - \frac{\lambda k_1(t)^{-1}a_{13}(t)e^{u_1(t)}e^{u_3(t)}}{m(t)e^{2u_3(t)} + e^{2u_1(t)}}\Big] + \lambda D_1(t)\big[\frac{e^{u_2(t)}}{e^{u_1(t)}} - 1\big] - \frac{H_1(t)}{e^{u_1(t)}}, \\ \Delta_2(u, t, \lambda) &= \Big[\frac{k_2(t) + (1-\lambda)c_2(t)e^{u_2(t)}}{k_2(t) + c_2(t)e^{u_2(t)}}\Big]\Big[\frac{a_2(t)}{k_2(t)} - \frac{a_{22}(t)e^{u_2(t)}}{k_2(t)}\Big] \\ &\quad + \lambda D_2(t)\big[\frac{e^{u_1(t)}}{e^{u_2(t)}} - 1\big] - \frac{H_2(t)}{e^{u_2(t)}}, \\ \Delta_3(u, t, \lambda) &= -a_3(t) + \frac{a_{31}(t)e^{2u_1(t-\tau)}}{m(t)e^{2u_3(t-\tau)} + \lambda e^{2u_1(t-\tau)}}. \end{aligned}$$

For any $u \in X$, because of the periodicity, we can easily check that $\Delta_i(u, t, \lambda) \in C(R^2, R)$ $(i = 1, 2, 3)$ are T-periodic in t.

Let

$$L : \operatorname{dom} L = \{u \in X : u \in C(R, R^3)\} \ni u \mapsto u' \in Z,$$

$$P : X \ni u \mapsto \frac{1}{T}\int_0^T u(t)dt \in X,$$
$$Q : Z \ni u \mapsto \frac{1}{T}\int_0^T u(t)dt \in Z,$$
$$N : X \times [0,1] \ni (u,\lambda) \mapsto (\Delta_1(u,t,\lambda), \Delta_2(u,t,\lambda), \Delta_3(u,t,\lambda))^T \in Z.$$

Here, for any $k \in R^3$, we also identify it as the constant function in X or Z with the constant value k. It is easy to see that

$$\ker L = R^3, \quad \operatorname{Im} L = \{u \in X : \int_0^T u_i(t)dt = 0, i = 1,2,3\}$$

is closed in Z, $\dim\ker L = \operatorname{codim}\operatorname{Im} L = 3$, and P, Q are continuous projectors such that

$$ImP = \ker L, \operatorname{Im} L = \ker Q = Im(I-Q).$$

Therefore, L is a Fredholm mapping of index zero. On the other hand, $K_p : \operatorname{Im} L \mapsto \operatorname{dom} L \cap KerP$ has the form

$$K_p(u) = \int_0^t u(s)ds - \frac{1}{T}\int_0^T \int_0^t u(s)\, ds\, dt.$$

Thus,

$$QN(u,\lambda) = \Big(\frac{1}{T}\int_0^T \Delta_1(u,t,\lambda)dt, \frac{1}{T}\int_0^T \Delta_2(u,t,\lambda)dt, \frac{1}{T}\int_0^T \Delta_3(u,t,\lambda)dt\Big)^T,$$
$$K_p(I-Q)N(u,\lambda) = (\Phi_1(u,t,\lambda), \Phi_2(u,t,\lambda),\ \Phi_3(u,t,\lambda))^T,$$

where

$$\begin{aligned}\Phi_j(u,t,\lambda) &= \int_0^t \Delta_j(u,s,\lambda)ds - \frac{1}{T}\int_0^T \int_0^t \Delta_j(u,s,\lambda)\, ds\, dt \\ &\quad - (\frac{t}{T} - \frac{1}{2})\int_0^T \Delta_j(u,s,\lambda)ds, \quad j = 1,2,3.\end{aligned}$$

Obviously, QN and $K_p(I-Q)N$ are continuous. By the Arzela-Ascoli theorem, it is not difficult to show that the closure of $K_p(I-Q)N(\overline{\Omega} \times [0,1])$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\bar{\Omega} \times [0,1])$ is bounded. Thus N is L-compact on $\bar{\Omega} \times [0,1]$ with any open bounded set $\Omega \subset X$.

To apply Lemma 2.1, we need to find at least four appropriate open, bounded subsets $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ in X.

Corresponding to the operator equation $Lu = \lambda N(u,\lambda)$, $\lambda \in (0,1)$, we have

$$\begin{aligned}u_1'(t) = \lambda\Big[&\frac{k_1(t) + (1-\lambda)c_1(t)e^{u_1(t)}}{k_1(t) + c_1(t)e^{u_1(t)}}\Big]\Big[\frac{a_1(t)}{k_1(t)} - \frac{a_{11}(t)e^{u_1(t)}}{k_1(t)} \\ &- \frac{\lambda k_1(t)^{-1}a_{13}(t)e^{u_1(t)}e^{u_3(t)}}{m(t)e^{2u_3(t)} + e^{2u_1(t)}}\Big] + \lambda^2 D_1(t)\Big[\frac{e^{u_2(t)}}{e^{u_1(t)}} - 1\Big] - \frac{\lambda H_1(t)}{e^{u_1(t)}},\end{aligned} \tag{2.3}$$

$$\begin{aligned}u_2'(t) = \lambda\Big[&\frac{k_2(t) + (1-\lambda)c_2(t)e^{u_2(t)}}{k_2(t) + c_2(t)e^{u_2(t)}}\Big]\Big[\frac{a_2(t)}{k_2(t)} - \frac{a_{22}(t)e^{u_2(t)}}{k_2(t)}\Big] \\ &+ \lambda^2 D_2(t)\Big[\frac{e^{u_1(t)}}{e^{u_2(t)}} - 1\Big] - \frac{\lambda H_2(t)}{e^{u_2(t)}},\end{aligned} \tag{2.4}$$

$$u_3'(t) = \lambda\Big[-a_3(t) + \frac{a_{31}(t)e^{2u_1(t-\tau)}}{m(t)e^{2u_3(t-\tau)} + \lambda e^{2u_1(t-\tau)}}\Big]. \tag{2.5}$$

Suppose that $(u_1(t), u_2(t), u_3(t))^T$ is a T-periodic solution of (2.3), (2.4) and (2.5) for some $\lambda \in (0,1)$. Choose $t_i^M, t_i^m \in [0,T]$, $i = 1,2,3$, such that

$$u_i(t_i^M) = \max_{t\in[0,T]} u_i(t), \quad u_i(t_i^m) = \min_{t\in[0,T]} u_i(t), \quad i = 1,2,3.$$

Then, it is clear that

$$u_i'(t_i^M) = 0, \quad u_i'(t_i^m) = 0, \quad i = 1,2,3.$$

From this and (2.3), (2.4), we obtain that

$$\begin{aligned} 0 = \Big[\frac{k_1(t_1^M) + (1-\lambda)c_1(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M) + c_1(t_1^M)e^{u_1(t_1^M)}}\Big]\Big[\frac{a_1(t_1^M) - a_{11}(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M)} \\ - \frac{\lambda k_1(t_1^M)^{-1}a_{13}(t_1^M)e^{u_1(t_1^M)}e^{u_3(t_1^M)}}{m(t_1^M)e^{2u_3(t_1^M)} + e^{2u_1(t_1^M)}}\Big] + \lambda D_1(t_1^M)\Big[\frac{e^{u_2(t_1^M)}}{e^{u_1(t_1^M)}} - 1\Big] - \frac{H_1(t_1^M)}{e^{u_1(t_1^M)}}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} 0 = \Big[\frac{k_2(t_2^M) + (1-\lambda)c_2(t_2^M)e^{u_2(t_2^M)}}{k_2(t_2^M) + c_2(t_2^M)e^{u_2(t_2^M)}}\Big]\Big[\frac{a_2(t_2^M)}{k_2(t_2^M)} - \frac{a_{22}(t_2^M)e^{u_2(t_2^M)}}{k_2(t_2^M)}\Big] \\ + \lambda D_2(t_2^M)\Big[\frac{e^{u_1(t_2^M)}}{e^{u_2(t_2^M)}} - 1\Big] - \frac{H_2(t_2^M)}{e^{u_2(t_2^M)}}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} 0 = \Big[\frac{k_1(t_1^m) + (1-\lambda)c_1(t_1^m)e^{u_1(t_1^m)}}{k_1(t_1^m) + c_1(t_1^m)e^{u_1(t_1^m)}}\Big]\Big[\frac{a_1(t_1^m) - a_{11}(t_1^m)e^{u_1(t_1^m)}}{k_1(t_1^m)} \\ - \frac{\lambda k_1(t_1^m)^{-1}a_{13}(t_1^m)e^{u_1(t_1^m)}e^{u_3(t_1^m)}}{m(t_1^m)e^{2u_3(t_1^m)} + e^{2u_1(t_1^m)}}\Big] + \lambda D_1(t_1^m)\Big[\frac{e^{u_2(t_1^m)}}{e^{u_1(t_1^m)}} - 1\Big] - \frac{H_1(t_1^m)}{e^{u_1(t_1^m)}}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} 0 = \Big[\frac{k_2(t_2^m) + (1-\lambda)c_2(t_2^m)e^{u_2(t_2^m)}}{k_2(t_2^m) + c_2(t_2^m)e^{u_2(t_2^m)}}\Big]\Big[\frac{a_2(t_2^m)}{k_2(t_2^m)} - \frac{a_{22}(t_2^m)e^{u_2(t_2^m)}}{k_2(t_2^m)}\Big] \\ + \lambda D_2(t_2^m)\Big[\frac{e^{u_1(t_2^m)}}{e^{u_2(t_2^m)}} - 1\Big] - \frac{H_2(t_2^m)}{e^{u_2(t_2^m)}}. \end{aligned} \tag{2.9}$$

For $u(t_i^M)$ $(i = 1,2)$, there are two cases to consider.

Case 1. Assume that $u_1(t_1^M) \geq u_2(t_2^M)$, then $u_1(t_1^M) \geq u_2(t_1^M)$. From this and (2.6), we have

$$a_1(t_1^M) - a_{11}(t_1^M)e^{u_1(t_1^M)} > 0,$$

which implies

$$e^{u_1(t_1^M)} < \frac{a_1(t_1^M)}{a_{11}(t_1^M)} \leq \Big(\frac{a_1}{a_{11}}\Big)^u \leq N_1.$$

That is

$$u_2(t_2^M) \leq u_1(t_1^M) < \ln\Big(\frac{a_1}{a_{11}}\Big)^u \leq \ln N_1.$$

Case 2. Assume that $u_1(t_1^M) < u_2(t_2^M)$, then $u_2(t_2^M) > u_1(t_2^M)$. From this and (2.7), we have

$$a_2(t_2^M) - a_{22}(t_2^M)e^{u_2(t_2^M)} > 0,$$

which implies

$$e^{u_2(t_2^M)} < \frac{a_2(t_2^M)}{a_{22}(t_2^M)} \leq \Big(\frac{a_2}{a_{22}}\Big)^u \leq N_1.$$

That is,

$$u_1(t_1^M) < u_2(t_2^M) < \ln\Big(\frac{a_2}{a_{22}}\Big)^u \leq \ln N_1.$$

Therefore,

$$\max\{u_1(t_1^M), u_2(t_2^M)\} < \ln N_1. \tag{2.10}$$

It follows from (2.6) that

$$\begin{aligned}
&\Big[\frac{k_1(t_1^M) + (1-\lambda)c_1(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M) + c_1(t_1^M)e^{u_1(t_1^M)}}\Big]\Big[\frac{a_1(t_1^M)}{k_1(t_1^M)} - \frac{a_{11}(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M)}\Big] - \frac{H_1(t_1^M)}{e^{u_1(t_1^M)}}\\
&= \lambda\Big[\frac{k_1(t_1^M) + (1-\lambda)c_1(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M) + c_1(t_1^M)e^{u_1(t_1^M)}}\Big]\frac{k_1(t_1^M)^{-1}a_{13}(t_1^M)e^{u_1(t_1^M)}e^{u_3(t_1^M)}}{m(t_1^M)e^{2u_3(t_1^M)} + e^{2u_1(t_1^M)}}\\
&\quad - \lambda D_1(t_1^M)\frac{e^{u_2(t_1^M)}}{e^{u_1(t_1^M)}} + \lambda D_1(t_1^M).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Big[\frac{k_1(t_1^M) + (1-\lambda)c_1(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M) + c_1(t_1^M)e^{u_1(t_1^M)}}\Big]\Big[(\frac{a_1}{k_1})^l - (\frac{a_{11}}{k_1})^u e^{u_1(t_1^M)}\Big] - \frac{H_1^u}{e^{u_1(t_1^M)}}\\
&< \Big[\frac{k_1(t_1^M) + (1-\lambda)c_1(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M) + c_1(t_1^M)e^{u_1(t_1^M)}}\Big]\frac{a_{13}(t_1^M)}{2k_1(t_1^M)\sqrt{m(t_1^M)}} + D_1(t_1^M),
\end{aligned}$$

which implies

$$\begin{aligned}
&\Big[\frac{k_1(t_1^M) + (1-\lambda)c_1(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M) + c_1(t_1^M)e^{u_1(t_1^M)}}\Big]\Big[(\frac{a_1}{k_1})^l - (\frac{a_{11}}{k_1})^u e^{u_1(t_1^M)}\Big] - \frac{H_1^u}{e^{u_1(t_1^M)}}\\
&< \Big[\frac{k_1(t_1^M) + (1-\lambda)c_1(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M) + c_1(t_1^M)e^{u_1(t_1^M)}}\Big]\frac{(\frac{a_{13}}{k_1})^u}{2\sqrt{m^l}} + D_1^u.
\end{aligned}$$

From this and noticing that

$$\frac{k_1(t_1^M)}{k_1(t_1^M) + c_1(t_1^M)e^{u_1(t_1^M)}} \le \frac{k_1(t_1^M) + (1-\lambda)c_1(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M) + c_1(t_1^M)e^{u_1(t_1^M)}} \le 1,$$

we have

$$\frac{k_1(t_1^M)}{k_1(t_1^M) + c_1(t_1^M)e^{u_1(t_1^M)}}(\frac{a_1}{k_1})^l - (\frac{a_{11}}{k_1})^u e^{u_1(t_1^M)} - \frac{H_1^u}{e^{u_1(t_1^M)}} < \frac{(\frac{a_{13}}{k_1})^u}{2\sqrt{m^l}} + D_1^u.$$

Therefore, we have

$$\frac{k_1^l}{k_1^l + c_1^u N_1}(\frac{a_1}{k_1})^l - (\frac{a_{11}}{k_1})^u e^{u_1(t_1^M)} - \frac{H_1^u}{e^{u_1(t_1^M)}} < \frac{(\frac{a_{13}}{k_1})^u}{2\sqrt{m^l}} + D_1^u;$$

that is,

$$(\frac{a_{11}}{k_1})^u e^{2u_1(t_1^M)} - \Big[\frac{k_1^l}{k_1^l + c_1^u N_1}(\frac{a_1}{k_1})^l - D_1^u - b_1\Big]e^{u_1(t_1^M)} + H_1^u > 0.$$

From (A3) and the above inequality, we have

$$u_1(t_1^M) > \ln u_1^+ \quad\text{or}\quad u_1(t_1^M) < \ln u_1^- \tag{2.11}$$

Similarly, we have

$$u_1(t_1^m) > \ln u_1^+ \quad\text{or}\quad u_1(t_1^m) < \ln u_1^- \tag{2.12}$$

By a similar argument, from (2.7), it follows that

$$(\frac{a_{22}}{k_2})^u e^{2u_2(t_2^M)} - \Big[\frac{k_2^l}{k_2^l + c_2^u N_1}(\frac{a_2}{k_2})^l - D_2^u - b_2\Big]e^{u_2(t_2^M)} + H_2^u > 0.$$

By (A3) and the above inequality, we have

$$u_2(t_2^M) > \ln u_2^+ \quad \text{or} \quad u_2(t_2^M) < \ln u_2^-. \tag{2.13}$$

Similarly, we have

$$u_2(t_2^m) > \ln u_2^+ \quad \text{or} \quad u_2(t_2^m) < \ln u_2^-. \tag{2.14}$$

Again, from (2.6) it follows that

$$\begin{aligned}
&\Big[\frac{k_1(t_1^M)+(1-\lambda)c_1(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M)+c_1(t_1^M)e^{u_1(t_1^M)}}\Big]\Big[\frac{a_1(t_1^M)}{k_1(t_1^M)}-\frac{a_{11}(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M)}\Big]\\
&> \frac{H_1(t_1^M)}{e^{u_1(t_1^M)}} - D_1(t_1^M)\frac{e^{u_2(t_1^M)}}{e^{u_1(t_1^M)}}.
\end{aligned}$$

Hence, we have

$$\frac{k_1^l}{k_1^l+c_1^uN_1}\frac{a_{11}(t_1^M)}{k_1(t_1^M)}e^{2u_1(t_1^M)} - \frac{a_1(t_1^M)}{k_1(t_1^M)}e^{u_1(t_1^M)} + H_1(t_1^M) - D_1(t_1^M)e^{u_2(t_1^M)} < 0,$$

which implies

$$\frac{k_1^l}{k_1^l+c_1^uN_1}(\frac{a_{11}}{k_1})^l e^{2u_1(t_1^M)} - (\frac{a_1}{k_1})^u e^{u_1(t_1^M)} + H_1^l - D_1^uN_1 < 0.$$

From (A3), (A4) and the above inequality, we have

$$\ln l_1^- < u_1(t_1^M) < \ln l_1^+. \tag{2.15}$$

Similarly, we have

$$\ln l_1^- < u_1(t_1^m) < \ln l_1^+. \tag{2.16}$$

By a similar argument, it follows from (2.7) that

$$\frac{k_2^l}{k_2^l+c_2^uN_1}(\frac{a_{22}}{k_2})^l e^{2u_2(t_2^M)} - (\frac{a_2}{k_2})^u e^{u_2(t_2^M)} + H_2^l - D_2^uN_1 < 0.$$

From (A3), (A4) and the above inequality, we have

$$\ln l_2^- < u_2(t_2^M) < \ln l_2^+. \tag{2.17}$$

Similarly, we have

$$\ln l_2^- < u_2(t_2^m) < \ln l_2^+. \tag{2.18}$$

It follows from (2.11), (2.12), (2.15), (2.16) that

$$\begin{aligned}
u_1(t_1^M) &\in (\ln l_1^-, \ln u_1^-) \cup (\ln u_1^+, \ln l_1^+),\\
u_1(t_1^m) &\in (\ln l_1^-, \ln u_1^-) \cup (\ln u_1^+, \ln l_1^+).
\end{aligned}$$

It follows from (2.13), (2.14), (2.17), (2.18) that

$$\begin{aligned}
u_2(t_2^M) &\in (\ln l_2^-, \ln u_2^-) \cup (\ln u_2^+, \ln l_2^+),\\
u_2(t_2^m) &\in (\ln l_2^-, \ln u_2^-) \cup (\ln u_2^+, \ln l_2^+).
\end{aligned}$$

From (2.5), we have

$$\int_0^T a_3(t)dt = \int_0^T \frac{a_{31}(t)e^{2u_1(t-\tau)}}{m(t)e^{2u_3(t-\tau)}+\lambda e^{2u_1(t-\tau)}}dt. \tag{2.19}$$

Therefore,

$$\begin{aligned}\int_0^T |u_3'(t)|dt &< \int_0^T a_3(t)dt + \int_0^T \frac{a_{31}(t)e^{2u_1(t-\tau)}}{m(t)e^{2u_3(t-\tau)} + \lambda e^{2u_1(t-\tau)}}dt \\ &= \int_0^T a_3(t)dt + \int_0^T a_3(t)dt \\ &< 2T\bar{a}_3 := d_3.\end{aligned} \tag{2.20}$$

By (2.19), there must be $\eta \in (0, T)$ such that

$$\bar{a}_3 = \frac{a_{31}(\eta)e^{2u_1(\eta-\tau)}}{m(\eta)e^{2u_3(\eta-\tau)} + \lambda e^{2u_1(\eta-\tau)}}. \tag{2.21}$$

From (2.21), we have

$$e^{2u_3(\eta-\tau)} = \frac{a_{31}(\eta) - \lambda\bar{a}_3}{\bar{a}_3 m(\eta)} e^{2u_1(\eta-\tau)}.$$

Therefore,

$$e^{2u_3(\eta-\tau)} < \frac{a_{31}^u}{\bar{a}_3 m^l} N_1^2 := p, \tag{2.22}$$

$$e^{2u_3(\eta-\tau)} > \frac{a_{31}^l - \bar{a}_3}{\bar{a}_3 m^u} (l_1^-)^2 := q. \tag{2.23}$$

Since $u_3(t)$ is a T-periodic function, there exists $\eta_1 \in [0, T]$ such that

$$u_3(\eta_1) = u_3(\eta - \tau).$$

Again, it follows from (2.20), (2.22), (2.23) that for any $t \in [0, T]$, we have

$$u_3(t) = u_3(\eta_1) + \int_{\eta_1}^t u_3'(s)ds < \frac{1}{2}\ln p + d_3,$$

$$u_3(t) = u_3(\eta_1) + \int_{\eta_1}^t u_3'(s)ds > \frac{1}{2}\ln q - d_3.$$

Set

$$H = \max\{|\frac{1}{2}\ln p + d_3|, |\frac{1}{2}\ln q - d_3|\}.$$

Then

$$\max_{t\in[0,T]} |u_3(t)| < H. \tag{2.24}$$

Clearly, $l_i^\pm, u_i^\pm (i = 1, 2), H$ are independent of λ. Now, let us consider $QN(u, 0)$ with $u = (u_1, u_2, u_3)^T \in R^3$. Note that

$$QN(u,0) = \begin{pmatrix} \overline{(\frac{a_1}{k_1})} - \overline{(\frac{a_{11}}{k_1})}e^{u_1} - \frac{\bar{H}_1}{e^{u_1}} \\ \overline{(\frac{a_2}{k_2})} - \overline{(\frac{a_{22}}{k_2})}e^{u_2} - \frac{\bar{H}_2}{e^{u_2}} \\ -\bar{a}_3 + \overline{(\frac{a_{31}}{m})}\frac{e^{2u_1}}{e^{2u_3}} \end{pmatrix}.$$

Letting $QN(u, 0) = 0$, we have

$$\overline{(\frac{a_1}{k_1})} - \overline{(\frac{a_{11}}{k_1})}e^{u_1} - \frac{\bar{H}_1}{e^{u_1}} = 0, \tag{2.25}$$

$$\overline{(\frac{a_2}{k_2})} - \overline{(\frac{a_{22}}{k_2})}e^{u_2} - \frac{\bar{H}_2}{e^{u_2}} = 0, \tag{2.26}$$

$$-\bar{a}_3 + \overline{(\frac{a_{31}}{m})}\frac{e^{2u_1}}{e^{2u_3}} = 0. \tag{2.27}$$

From (2.25), we have

$$\overline{(\frac{a_{11}}{k_1})}e^{2u_1} - \overline{(\frac{a_1}{k_1})}e^{u_1} + \bar{H}_1 = 0, \tag{2.28}$$

which implies that (2.28) has two distinct roots $\ln x_1^{\pm}$. From (2.26), we have

$$\overline{(\frac{a_{22}}{k_2})}e^{2u_2} - \overline{(\frac{a_2}{k_2})}e^{u_2} + \bar{H}_2 = 0, \tag{2.29}$$

which implies that (2.29) has two distinct roots $\ln x_2^{\pm}$. Again, it follows from (2.27) that

$$x_3^{\pm} = \sqrt{\frac{\overline{(\frac{a_{31}}{m})}}{\bar{a}_3}}x_1^{\pm}.$$

Therefore, $QN(u,0) = 0$ has four distinct solutions.

$$\tilde{u}_1 = (\ln x_1^+, \ln x_2^+, \ln x_3^+)^T, \quad \tilde{u}_2 = (\ln x_1^+, \ln x_2^-, \ln x_3^+)^T,$$
$$\tilde{u}_3 = (\ln x_1^-, \ln x_2^+, \ln x_3^-)^T, \quad \tilde{u}_4 = (\ln x_1^-, \ln x_2^-, \ln x_3^-)^T.$$

Choose $C > 0$ such that

$$C > \max\Big\{|\ln\sqrt{\frac{\overline{(\frac{a_{31}}{m})}}{\bar{a}_3}}x_1^+|, |\ln\sqrt{\frac{\overline{(\frac{a_{31}}{m})}}{\bar{a}_3}}x_1^-|\Big\}. \tag{2.30}$$

Let

$$\begin{aligned}\Omega_1 = \Big\{u = (u_1,u_2,u_3)^T \in X : &\max_{t\in[0,T]} u_1(t) \in (\ln u_1^+, \ln l_1^+),\\ &\min_{t\in[0,T]} u_1(t) \in (\ln u_1^+, \ln l_1^+),\ \max_{t\in[0,T]} u_2(t) \in (\ln u_2^+, \ln l_2^+),\\ &\min_{t\in[0,T]} u_2(t) \in (\ln u_2^+, \ln l_2^+),\ \max_{t\in[0,T]} |u_3(t)| < H + C\Big\},\end{aligned}$$

$$\begin{aligned}\Omega_2 = \Big\{u = (u_1,u_2,u_3)^T \in X : &\max_{t\in[0,T]} u_1(t) \in (\ln u_1^+, \ln l_1^+),\\ &\min_{t\in[0,T]} u_1(t) \in (\ln u_1^+, \ln l_1^+),\ \max_{t\in[0,T]} u_2(t) \in (\ln l_2^-, \ln u_2^-),\\ &\min_{t\in[0,T]} u_2(t) \in (\ln l_2^-, \ln u_2^-),\ \max_{t\in[0,T]} |u_3(t)| < H + C\Big\},\end{aligned}$$

$$\begin{aligned}\Omega_3 = \Big\{u = (u_1,u_2,u_3)^T \in X : &\max_{t\in[0,T]} u_1(t) \in (\ln l_1^-, \ln u_1^-),\\ &\min_{t\in[0,T]} u_1(t) \in (\ln l_1^-, \ln u_1^-),\ \max_{t\in[0,T]} u_2(t) \in (\ln u_2^+, \ln l_2^+),\\ &\min_{t\in[0,T]} u_2(t) \in (\ln u_2^+, \ln l_2^+),\ \max_{t\in[0,T]} |u_3(t)| < H + C\Big\},\end{aligned}$$

$$\begin{aligned}\Omega_4 = \Big\{u = (u_1,u_2,u_3)^T \in X : &\max_{t\in[0,T]} u_1(t) \in (\ln l_1^-, \ln u_1^-),\\ &\min_{t\in[0,T]} u_1(t) \in (\ln l_1^-, \ln u_1^-),\ \max_{t\in[0,T]} u_2(t) \in (\ln l_2^-, \ln u_2^-),\\ &\min_{t\in[0,T]} u_2(t) \in (\ln l_2^-, \ln u_2^-),\ \max_{t\in[0,T]} |u_3(t)| < H + C\Big\}.\end{aligned}$$

Then $\Omega_1,\dots,\Omega_4$ are bounded open subsets of X. It follows from (2.1) and (2.30) that $\tilde{u}_i\in\Omega_i$ $(i=1,2,3,4)$. It is easy to see that $\bar{\Omega}_i\cap\bar{\Omega}_j=\emptyset$ $(i,j=1,2,3,4,i\neq j)$ and Ω_i satisfies (a) in Lemma 2.1 for $i=1,2,3,4$. Moreover, $QN(u,0)\neq 0$ for $u\in\partial\Omega_i\cap\ker L$.

Set

$$h(x)=b-ax-\frac{c}{x},\quad x\in(0,+\infty),$$

where a,b,c are positive constants. When $b>2\sqrt{ac}$, it is easy to see that $h(x)=0$ has exactly two positive solutions

$$x^-=\frac{b-\sqrt{b^2-4ac}}{2a},\quad x^+=\frac{b+\sqrt{b^2-4ac}}{2a}$$

such that $h'(x^-)>0$, $h'(x^+)<0$. From this, a direct computation gives

$$\begin{gathered}\deg\{JQN(\cdot,0),\Omega_1\cap\ker L,0\}=-1,\quad \deg\{JQN(\cdot,0),\Omega_2\cap\ker L,0\}=1,\\ \deg\{JQN(\cdot,0),\Omega_3\cap\ker L,0\}=1,\quad \deg\{JQN(\cdot,0),\Omega_4\cap\ker L,0\}=-1.\end{gathered}$$

Here, J is taken as the identity mapping since $\operatorname{Im}Q=\ker L$. So far we have proved that Ω_i satisfies all the assumptions in Lemma 2.1. Hence, (2.2) has at least four T-periodic solutions $(u_1^i(t),u_2^i(t),u_3^i(t))^T$ $(i=1,2,3,4)$ and $(u_1^i,u_2^i,u_3^i)^T\in\operatorname{dom}L\cap\bar{\Omega}_i$. Obviously, $(u_1^i,u_2^i,u_3^i)^T$ $(i=1,2,3,4)$ are different. Let $x_j^i(t)=e^{u_j^i(t)}$ $(j=1,2)$, $y^i(t)=e^{u_3^i(t)}$ $(i=1,2,3,4)$. Then $(x_1^i(t),x_2^i(t),y^i(t))^T$ $(i=1,2,3,4)$ are four different positive T-periodic solutions of (1.1). The proof is complete. □

Theorem 2.3. *In addition to* (A1), (A2), (A4), *assume further that* (1.1) *satisfies*

(A3)* $H_i^u<\frac{N_1}{N_1+1}\big[\frac{k_i^l}{k_i^l+c_i^uN_1}(\frac{a_i}{k_i})^l-(\frac{a_{ii}}{k_i})^u-b_i\big]$ $(i=1,2)$.

Then (1.1) *has at least four positive T-periodic solutions.*

Proof. It suffices to verify (A3) in Theorem 2.2. Indeed, it follows from (A3)* and (A4) that

$$D_i^u+b_i+2\sqrt{(\frac{a_{ii}}{k_i})^uH_i^u}<\frac{H_i^l}{N_1}+b_i+(\frac{a_{ii}}{k_i})^u+H_i^u<\frac{k_i^l}{k_i^l+c_i^uN_1}(\frac{a_i}{k_i})^l,$$

for $i=1,2$. □

Finally, we describe the biological meaning of (A1)–(A4) and Theorem 2.2. In realistic world, the environment is always varying periodically with time. This motivates us to consider system (1.1) under the condition (A1). (A2) indicates that the efficiency of turning predated prey species x into predator species y is higher than the natural death rate of predator species y, which is the necessary condition for the survival of predator species y. Note that $\sqrt{(\frac{a_{ii}}{k_i})^uH_i^u}$ is the geometric mean of $(\frac{a_{ii}}{k_i})^u$ and H_i^u $(i=1,2)$, which describes the mean effect of the intra-species competition and harvesting. Hence, (A3) indicates that the natural growth rate of prey species x with a food-limited supply in patch 1 is higher than the decay rate due to the intra-species competition, predation, dispersion and harvesting, and that the natural growth rate of prey species x with a limited food supply in patch 2 is higher than the decay rate due to the intra-species competition, the dispersion and harvesting. Since N_1 is the environmental carrying capacity of prey species x, (A4) indicates that the effect of harvesting on populations is stronger than the effect of the dispersion during each time period. From the ecological viewpoint, Theorem

2.2 indicates that a food-limited two-species ratio-dependent predator-prey patch systems with delay and harvesting can lead to four different periodic fluctuations in a periodic environment if the regulated harvesting on prey populations is made according to (A1)–(A4).

3. An Example

Take $\tau = 2$, $T = 2$, $k_1(t) = 4 + \sin(\pi t)$, $k_2(t) = 3 + \sin(\pi t)$, $c_i(t) = 0.2 + 0.05\sin(\pi t)$ $(i = 1, 2)$,

$$H_1(t) = \frac{1+\sin^2(\pi t)}{24}, \quad H_2(t) = \frac{1+\sin^2(\pi t)}{10}, \quad D_1(t) = D_2(t) = \frac{1+\sin^2(\pi t)}{200},$$
$$a_1(t) = (4+\sin(\pi t))^2, \quad a_{11}(t) = \frac{(4+\sin(\pi t))^2}{4}, \quad a_{13}(t) = \frac{(4+\sin(\pi t))^2}{4},$$
$$m(t) = 1+\sin^2(\pi t), \quad a_2(t) = (3+\sin(\pi t))^2, \quad a_{22}(t) = \frac{(3+\sin(\pi t))^2}{4},$$
$$a_{31}(t) = 4+\sin(\pi t), \quad a_3(t) = 2+\sin(\pi t).$$

Then we have $k_1^l = 3$, $k_2^l = 2$, $c_i^u = 0.25$ $(i = 1, 2)$,

$$H_1^u = \frac{1}{12}, \quad H_2^u = \frac{1}{5}, \quad H_1^l = \frac{1}{24}, \quad H_2^l = \frac{1}{10}, \quad D_1^u = D_2^u = \frac{1}{100},$$
$$N_1 = 4, \quad (\frac{a_1}{k_1})^l = 3, \quad (\frac{a_{11}}{k_1})^u = \frac{5}{4}, \quad (\frac{a_{13}}{k_1})^u = \frac{5}{4}, \quad m^l = 1,$$
$$b_1 = \frac{5}{8}, \quad (\frac{a_2}{k_2})^l = 2, \quad (\frac{a_{22}}{k_2})^u = 1, \quad a_{31}^l = 3, \quad \bar{a}_3 = 2.$$

It is easy to see that the conditions in Theorem 2.3 are satisfied. By Theorem 2.3, system (1.1) has at least four positive 2-periodic solutions.

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Hui Fang

Department of Mathematics, Kunming University of Science and Technology, Kunming, Yunnan 650500, China

E-mail address: kmustfanghui@hotmail.com