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# EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR QUASI-LINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Abstract. We prove the existence and uniqueness of a local solution to a quasi-linear differential equation of parabolic type with deviated argument in an arbitrary Banach space. The results are obtained by applying the Sobolevskiĭ-Tanabe theory of parabolic equations, fractional powers of operators, and the Banach fixed point theorem. We include an example that illustrates the theory.

## 1. Introduction

Differential equations with a deviating argument are differential equations in which the unknown function and its derivative appear under different values of the argument. Differential equations with a deviating argument have many applications in science and technology. These includes the theory of automatic control, the theory of self-oscillating systems, the problems of long-term planning in economics, the study of problems related with combustion in rocket motion, a series of biological problems, and many other areas of science and technology, the number which is steadily expanding, for more details we refer to [3, 6, 7, 8, 12, 16] and references cited therein.

We shall study the existence and uniqueness of a local solution for the following differential equation in a Banach space $(X, \|\cdot\|)$,

$$\begin{gathered} \frac{du}{dt} + A(t, u(t))u(t) = f(t, u(t), u(h(t, u(t)))), \quad t > 0; \\ u(0) = u_0. \end{gathered} \tag{1.1}$$

Here, we assume that $-A(t, x)$, for each $t \geq 0$ and $x \in X$, generates an analytic semigroup of bounded linear operators on $X$. The nonlinear $X$-valued functions $f$ and $h$ satisfy suitable growth conditions in their arguments stated in Section 2.

The existence and uniqueness of solutions for a quasi-linear differential equation in Banach spaces have been studied by many authors (see e.g. [1, 2, 13, 14, 15, 18, 19, 21, 22, 23]). Using fixed point argument, Pazy [19] obtained the mild and

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classical solution to the following homogeneous quasi-linear differential equation in a Banach space $(X, \|\cdot\|)$,

$$\frac{du}{dt} + A(t, u(t))u(t) = 0, \quad 0 < t \le T;$$
$$u(0) = u_0,$$

for some $T$ (See Pazy [19]).

Consider the following inhomogeneous quasi-linear differential equation in a Banach space $(X, \|\cdot\|)$,

$$\begin{aligned} \frac{du}{dt} + A(t, u(t))u(t) &= f(t, u(t)), \quad 0 < t \le T; \\ u(0) &= u_0, \end{aligned} \tag{1.2}$$

where $-A(t, x)$, for each $T \ge t \ge 0$ and $x \in X$, generates an analytic semigroup of bounded linear operators on $X$ and the nonlinear function $f$ is uniformly locally Hölder continuous in $t$ and uniformly locally Lipschitz continuous in $x$. The existence and uniqueness of a classical solution of Equation (1.2) had been obtained by Sobolevskiĭ [23]. For more detail, we refer to Friedman [4] and Sobolevskiĭ [23].

Our objective is to establish the existence and uniqueness of a local solution to (1.1) that will generalize the results of Sobolevskiĭ [23].

The article is organized as follows. In Section 2, we will provide preliminaries, assumptions and Lemmas that will be needed for proving our main results. We shall prove the local existence and uniqueness of a solution to (1.1) in Section 3. Finally, we shall provide an example to illustrate the application of the abstract results.

## 2. Preliminaries and Assumptions

In this section, we will introduce assumptions, preliminaries and Lemmas that will be used in the sequel. We briefly outline the facts concerning analytic semigroups, fractional powers of operators, and the homogeneous and inhomogeneous linear Cauchy initial value problem. The material presented here is covered in more detail by Friedman [4] and Tanabe [24].

Let X be a complex Banach space with norm $\|\cdot\|$. Let $T \in [0, \infty)$ and $\{A(t) : 0 \le t \le T\}$ be a family of closed linear operators on the Banach space $X$. Let the following assumptions hold:

- (A1) The domain $D(A)$ of $A(t)$ is dense in $X$ and independent of $t$.
- (A2) For each $t \in [0, T]$, the resolvent $R(\lambda; A(t))$ exists for all $\mathrm{Re}\lambda \le 0$ and there is a constant $C > 0$ (independent of $t$ and $\lambda$) such that
$$\|R(\lambda; A(t))\| \le \frac{C}{|\lambda| + 1}, \quad \mathrm{Re}\,\lambda \le 0,\ t \in [0, T].$$
- (A3) There are constants $C > 0$ and $\rho \in (0, 1]$, such that
$$\|[A(t) - A(\tau)]A^{-1}(s)\| \le C|t - \tau|^{\rho},$$
for $t, s, \tau \in [0, T]$. Here, $C$ and $\rho$ are independent of $t, \tau$ and $s$.

It is well known that assumption (A2) implies that for each $s \in [0, T]$, $-A(s)$ generates a strongly continuous analytic semigroup $\{e^{-tA(s)} : t \ge 0\}$ in $\mathcal{L}(X)$, where $\mathcal{L}(X)$ denotes the Banach algebra of all bounded linear operators on $X$. Then there exist positive constants $C$ and $d$ such that

$$\|e^{-tA(s)}\| \le Ce^{-dt}, \quad t \ge 0; \tag{2.1}$$

$$\|A(s)e^{-tA(s)}\| \le Ce^{-dt}/t, \quad t > 0, \tag{2.2}$$

for all $s \in [0, T]$. It is to be noted that the assumption (A3) implies that there exists a constant $C > 0$ such that

$$\|A(t)A^{-1}(s)\| \le C, \tag{2.3}$$

for all $0 \le s, t \le T$. Hence, for each $t$, the functional $y \to \|A(t)y\|$ defines an equivalent norm on $D(A) = D(A(0))$ and the mapping $t \to A(t)$ from $[0, T]$ into $\mathcal{L}(X_1, X)$ is uniformly Hölder continuous.

Consider the homogeneous Cauchy problem

$$\frac{du}{dt} + A(t)u = 0; \quad u(t_0) = u_0. \tag{2.4}$$

Then the solution to this problem is given by the following Theorem.

**Theorem 2.1** ([4, 23]). *Let the Assumptions* (A1)–(A3) *hold. Then there exists a unique fundamental solution* $\{U(t,s) : 0 \le s \le t \le T\}$ *to* (2.4) *that possesses the following properties:*

- (i) $U(t,s) \in \mathcal{L}(X)$ *and* $U(t,s)$ *is strongly continuous in* $t, s$ *for all* $0 \le s \le t \le T$.
- (ii) $U(t,s)x \in D(A)$ *for each* $x \in X$, *for all* $0 \le s \le t \le T$.
- (iii) $U(t,r)U(r,s) = U(t,s)$ *for all* $0 \le s \le r \le t \le T$.
- (iv) *the derivative* $\partial U(t,s)/\partial t$ *exists in the strong operator topology and belongs to* $\mathcal{L}(X)$ *for all* $0 \le s < t \le T$, *and strongly continuous in* $s$ *and* $t$, *where* $s < t \le T$.
- (v) $\frac{\partial U(t,s)}{\partial t} + A(t)U(t,s) = 0$ *and* $U(s,s) = I$ *for all* $0 \le s < t \le T$.

For $t_0 \ge 0$, let $C^\beta([t_0, T]; X)$ denote the space of all $X$-valued functions $h(t)$, that are uniformly Hölder continuous on $[0, T]$ with exponent $\beta$, where $0 < \beta \le 1$. Define

$$[h]_\beta = \sup_{t_0 \le t, s \le T} \|h(t) - h(s)\|/|t - s|^\beta.$$

Then $C^\beta([t_0, T]; X)$ is a Banach space with respect to the norm

$$\|h\|_{C^\beta([t_0,T];X)} = \sup_{t_0 \le t \le T} \|h(t)\| + [h]_\beta.$$

Consider the inhomogeneous Cauchy problem

$$\frac{du}{dt} + A(t)u = f(t); \quad u(t_0) = u_0. \tag{2.5}$$

**Theorem 2.2** ([4, 23]). *Let the assumptions* (A1)-(A3) *hold. If* $f \in C^\beta([t_0, T]; X)$, *then there exists a unique solution of* (2.5). *Furthermore, the solution can be written as*

$$u(t) = U(t, t_0)u_0 + \int_{t_0}^t U(t,s)f(s)ds, \quad t_0 \le t \le T,$$

*and* $u : [t_0, T] \to X$ *is continuously differentiable on* $(t_0, T]$.

We shall use the following assumption to establish the existence and uniqueness of a local solution to (1.1).

(F1) The operator $A_0 = A(0, u_0)$ is closed operator with domain $D_0$ ($D_0$ denote domain of $A_0$) dense in $X$ and there exists a constant $C > 0$ independent of $\lambda$ such that

$$\|(\lambda I - A_0)^{-1}\| \leq \frac{C}{1 + |\lambda|}; \quad \text{for all } \lambda \text{ with } \operatorname{Re} \lambda \leq 0. \tag{2.6}$$

Assumption (F1) allows us to define negative fractional powers of the operator $A_0$. For $\alpha > 0$, define negative fractional powers $A_0^{-\alpha}$ by the formula

$$A_0^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-tA_0} t^{\alpha-1} dt.$$

Then $A_0^{-\alpha}$ is one-to-one and bounded linear operator on $X$. Thus, there exists an inverse of the operator $A_0^{-\alpha}$. We define the positive fractional powers of $A_0$ by $A_0^{\alpha} \equiv [A_0^{-\alpha}]^{-1}$. Then $A_0^{\alpha}$ is closed linear operator with dense domain $D(A_0^{\alpha})$ in $X$ and $D(A_0^{\alpha}) \subset D(A_0^{\beta})$ if $\alpha > \beta$. For $0 < \alpha \leq 1$, let $X_\alpha = D(A_0^{\alpha})$ and equip this space with the graph norm

$$\|x\|_\alpha = \|A_0^{\alpha} x\|.$$

Then $X_\alpha$ is a Banach space with respect to this norm. If $0 < \alpha \leq 1$, the embedding $X_1 \hookrightarrow X_\alpha \hookrightarrow X$ are dense and continuous. We define, for each $\alpha > 0$, $X_{-\alpha} = (X_\alpha)^*$, the dual space of $X_\alpha$, endowed with the natural norm

$$\|x\|_{-\alpha} = \|A_0^{-\alpha} x\|.$$

Let $R, R' > 0$ and $B_\alpha = \{x \in X_\alpha : \|x\|_\alpha < R\}$, $B_{\alpha-1} = \{y \in X_{\alpha-1} : \|y\|_{\alpha-1} < R'\}$. We shall also use the following assumptions:

(F2) For some $\alpha \in [0, 1)$ and for any $v \in B_\alpha$, the operator $A(t, v)$ is well defined on $D_0$ for all $t \in [0, T]$. Furthermore, for any $t, s \in [0, T]$ and $v, w \in B_\alpha$, the following condition holds

$$\|[A(t, v) - A(s, w)]A^{-1}(s, w)\| \leq C(R)[|t - s|^{\theta_1} + \|v - w\|_\alpha] \tag{2.7}$$

for some $0 < \theta_1 \leq 1$.

(F3) (a) For every $t, s \in [0, T]$; $x, y \in B_\alpha$ and $x', y' \in B_{\alpha-1}$, there exist constants $L_f = L_f(t, R, R') > 0$ and $0 < \theta_1 \leq 1$, such that the nonlinear map $f : [0, T] \times B_\alpha \times B_{\alpha-1} \to X$ satisfies the condition

$$\|f(t, x, x') - f(s, y, y')\| \leq L_f(|t - s|^{\theta_1} + \|x - y\|_\alpha + \|x' - y'\|_{\alpha-1}), \tag{2.8}$$

(b) There exist constants $L_h = L_h(t, R) > 0$ and $0 < \theta_2 \leq 1$, such that $h(\cdot, 0) = 0$, $h : B_\alpha \times [0, T] \to [0, T]$ satisfies the following condition,

$$|h(x, t) - h(y, s)| \leq L_h(\|x - y\|_\alpha + |t - s|^{\theta_2}), \tag{2.9}$$

for all $x, y \in B_\alpha$ and for all $s, t \in [0, T]$.

(F4) Let $u_0 \in X_\beta$ for some $\beta > \alpha$ and

$$\|u_0\|_\alpha < R. \tag{2.10}$$

Let us state the following Lemmas that will be used in the subsequent sections.

**Lemma 2.3** ([5, Lemma 1.1]). *Let $h \in C^\beta([t_0, T]; X)$. Define the function $H : C^\beta([t_0, T]; X) \to C([t_0, T]; X_1)$ by*

$$Hh(t) = \int_{t_0}^t U(t, s)h(s)ds, \quad t_0 \leq t \leq T.$$

*Then $H$ is a bounded mapping, and $\|Hh\|_{C([t_0,T];X_1)} \le C\|h\|_{C^\beta([t_0,T];X)}$, for some constant $C > 0$.*

We have the following corollary.

**Corollary 2.4.** *For $y \in X_1$, define*

$$P(y;h) = U(t,0)y + \int_0^t U(t,s)h(s)ds,\ 0 \le t \le T.$$

*Then $P$ is a bounded linear mapping from $X_1 \times C^\beta([t_0,T];X)$ into $C([t_0,T];X_1)$.*

## 3. Existence of a Solution

In this section, we will establish the existence and uniqueness of a local solution to (1.1). Let $I = [0,\delta]$ for some positive number $\delta$ to be specified later. Let $C_\alpha$, $0 \le \alpha \le 1$, denote the space of all $X_\alpha$-valued continuous functions on $I$, endowed with the sup-norm, $\sup_{t\in I} \|\psi(t)\|_\alpha$, $\psi \in C(I;X_\alpha)$. Let

$$Y_\alpha = C_{L_\alpha}(I;X_\alpha) = \{\psi \in C_\alpha : \|\psi(t) - \psi(s)\|_{\alpha-1} \le L_\alpha|t-s|,\ \text{for all } t,s \in I\},$$

where $L_\alpha$ is a positive constant to be specified later. It is clear that $Y_\alpha$ is a Banach space with the sup-norm of $C_\alpha$.

**Definition 3.1.** Given $u_0 \in X_\alpha$, by a solution of problem (1.1), we mean a function $u : I \to X$ that satisfies the followings:

- (i) $u(.) \in C_{L_\alpha}(I;X_\alpha) \cap C^1((0,\delta);X) \cap C(I;X)$;
- (ii) $u(t) \in X_\alpha$, for all $t \in (0,\delta)$;
- (iii) $\frac{du}{dt} + A(t,u(t))u(t) = f(t,u(t),u([h(u(t),t)]))$, for all $t \in (0,\delta)$;
- (iv) $u(0) = u_0$.

Let $K > 0$ and $0 < \eta < \beta - \alpha$ be fixed constants. Let

$$\mathcal{S}_\alpha = \{y \in C_\alpha \cap Y_\alpha : y(0) = u_0, \|y(t) - y(s)\|_\alpha \le K|t-s|^\eta\}.$$

Then $\mathcal{S}_\alpha$ is a non-empty closed and bounded subset of $C_\alpha$.

Now we prove the following theorem concerning the existence and uniqueness of a local solution to (1.1). The proof is based on ideas from Gal [6] and Sobolevskiĭ [23]

**Theorem 3.2.** *Let $u_0 \in X_\beta$, where $0 < \alpha < \beta \le 1$. Let the assumptions* (F1)–(F4) *hold. Then there exists a positive number $\delta = \delta(\alpha,u_0), 0 < \delta \le T$ and a unique solution $u(t)$ to* (1.1) *in $[0,\delta]$ such that $u \in \mathcal{S}_\alpha \cap C^1((0,\delta);X)$.*

*Proof.* Let $v \in \mathcal{S}_\alpha$. Then from the assumption (F4), it follows that if $\delta > 0$ is sufficiently small, then

$$\|v(t)\|_\alpha < R, \quad \text{for } t \in I. \tag{3.1}$$

Hence, the operator

$$A_v(t) = A(t,v(t)) \tag{3.2}$$

is well defined for each $t \in I$. Again from the assumption (F2) and inequality (2.3), it is clear that

$$\|[A_v(t) - A_v(s)]A_0^{-1}\| \le C|t-s|^\mu, \quad \text{for } \mu = \min\{\theta_1,\eta\}, \tag{3.3}$$

where $C > 0$ is a constant independent of $\delta$ and of the particular $v \in \mathcal{S}_\alpha$. It is also to be noted that

$$A_v(0) = A_0. \tag{3.4}$$

If $\delta > 0$ is sufficiently small, then from assumption (F1) and inequality (3.3), we have

$$\|(\lambda I - A_v(t))^{-1}\| \le \frac{C}{1+|\lambda|}; \quad \text{for } \lambda \text{ with } \operatorname{Re} \lambda \le 0,\ t \in I. \tag{3.5}$$

Also from assumption (F2), it follows that

$$\|[A_v(t) - A_v(s)]A_v^{-1}(\tau)\| \le C|t-s|^{\mu}, \quad \text{if } t, \tau, s \in I. \tag{3.6}$$

Thus the operator $A_v(t)$ satisfies conditions (A1), (A2) and (A3). Hence, there exists a fundamental solution $U_v(t,s)$ corresponding to $A_v(t)$ and satisfies all estimates derived in Theorem 2.1 uniformly with respect to $v \in \mathcal{S}_\alpha$.

Put $f_v(t) = f(t, v(t), v([h(v(t),t)]))$. Then the assumption (F3) implies that $f_v$ is Hölder continuous on $I$ of exponent $\gamma = \min\{\theta_1, \theta_2, \eta\}$. Now consider the equation

$$\begin{gathered} \frac{dw}{dt} + A(t, v(t))w(t) = f_v(t), \quad t \in I; \\ w(0) = u_0. \end{gathered} \tag{3.7}$$

By Theorem 2.2, there exists a unique solution $w_v$ to (3.7) that is given by

$$w_v(t) = U_v(t,0)u_0 + \int_0^t U_v(t,s)f_v(s)ds, \quad t \in I. \tag{3.8}$$

For each $v \in \mathcal{S}_\alpha$, define a map $F$ by

$$Fv(t) = U_v(t,0)u_0 + \int_0^t U_v(t,s)f_v(s)ds, \quad \text{for each } t \in I. \tag{3.9}$$

By Lemma 2.3, the map $F$ is well defined. We will claim that $F$ maps from $\mathcal{S}_\alpha$ into itself, for sufficiently small $\delta > 0$. Indeed, if $t_1, t_2 \in I$ with $t_2 > t_1$, then we have

$$\begin{aligned} \|Fv(t_2) - Fv(t_1)\|_{\alpha-1} &\le \|[U_v(t_2,0) - U_v(t_1,0)]u_0\|_{\alpha-1} \\ &\quad + \|\int_0^{t_2} U_v(t_2,s)f_v(s)ds - \int_0^{t_1} U_v(t_1,s)f_v(s)ds\|_{\alpha-1}. \end{aligned} \tag{3.10}$$

We will use the bounded inclusion $X \subset X_{\alpha-1}$ to estimate each of the term on the right-hand side of (3.10). The first term on the right-hand side of (3.10) is estimated as follows [4, see Lemma II. 14.1],

$$\|(U_v(t_2,0) - U_v(t_1,0))u_0\|_{\alpha-1} \le C_1\|u_0\|_\alpha (t_2 - t_1), \tag{3.11}$$

where $C_1$ is some positive constant. We have the following estimate for the second term on the right hand side of (3.10) [4, Lemma II. 14.4],

$$\begin{aligned} &\|\int_0^{t_2} U_v(t_2,s)f_v(s)ds - \int_0^{t_1} U_v(t_1,s)f_v(s)ds\|_{\alpha-1} \\ &\le C_2 N_1 (t_2 - t_1)(|\log(t_2 - t_1)| + 1), \end{aligned} \tag{3.12}$$

where $N_1 = \sup_{s\in[0,T]} \|f_v(s)\|$ and $C_2$ is some positive constant.

Using estimates (3.11) and (3.12), from (3.10), we obtain

$$\|Fv(t_2) - Fv(t_1)\|_{\alpha-1} \le L_\alpha |t_2 - t_1|, \tag{3.13}$$

where $L_\alpha = \max\{C_1(t_2 - t_1)^{\alpha-1}\|u_0\|_\alpha, C_2N_1(|\log(t_2 - t_1)| + 1)\}$ that depends on $C_1, C_2, N_1, \delta$.

Our next aim is to show that $\|Fv(t+h) - Fv(t)\|_\alpha \le Kh^\eta$, for some constant $K > 0$ and $0 < \eta < 1$. If $0 \le \alpha < \beta \le 1$ and $0 \le t \le t+h \le \delta$, then

$$\begin{aligned}\|Fv(t+h) - Fv(t)\|_\alpha &\le \|[U_v(t+h,0) - U_v(t,0)]u_0\|_\alpha \\ &\quad + \|\int_0^{t+h} U_v(t+h,s)f_v(s)ds - \int_0^t U_v(t,s)f_v(s)ds\|_\alpha.\end{aligned}$$

Using [4, Lemmas II.14.1, II.14.4], we obtain the following two estimates

$$\|[U_v(t+h,0) - U_v(t,0)]u_0\|_\alpha \le C(\alpha,u_0)h^{\beta-\alpha}; \tag{3.14}$$

$$\|\int_0^{t+h} U_v(t+h,s)f_v(s)ds - \int_0^t U_v(t,s)f_v(s)ds\|_\alpha \le C(\alpha)N_1h^{1-\alpha}(1+|\log h|). \tag{3.15}$$

From (3.14) and (3.15), it is clear that

$$\|Fv(t+h) - Fv(t)\|_\alpha \le h^\eta[C(\alpha,u_0)\delta^{\beta-\alpha-\eta} + C(\alpha)N_1\delta^\nu h^{1-\alpha-\eta-\nu}(|\log h|+1)]$$

for any $\nu > 0, \nu < 1-\alpha-\eta$. Hence, for sufficiently small $\delta > 0$ , we have

$$\|Fv(t+h) - Fv(t)\|_\alpha \le Kh^\eta,$$

for some constant $K > 0$. Thus, we have shown that $F$ maps $\mathcal{S}_\alpha$ into itself.

Finally, we will show that $F$ is a contraction map. For $v_1, v_2 \in S_\alpha$, put $z_1(t) = w_{v_1}(t)$ and $z_2(t) = w_{v_2}(t)$. Thus, for $j = 1, 2$, we have

$$\begin{gathered}\frac{dz_j}{dt} + A_{v_j}(t)z_j(t) = f_{v_j}(t), \quad t \in I; \\ z_j(0) = u_0.\end{gathered} \tag{3.16}$$

It follows from (3.16) that

$$\frac{d}{dt}(z_1 - z_2) + A_{v_1}(t)(z_1 - z_2) = [A_{v_2}(t) - A_{v_1}(t)]z_2 + [f_{v_1}(t) - f_{v_2}(t)]. \tag{3.17}$$

Using [4, Lemmas II.14.3, II.14.5], we obtain that $A_0(t)z_2(t)$ is uniformly Hölder continuous for $\tau \le t \le \delta$, $\tau > 0$. Also from Lemma 2.3, $A_0\int_0^t U_{v_2}(t,s)f_{v_2}(s)ds$ is a bounded function, and hence we have the bound

$$\|A_0z_2(t)\| \le Ct^{\beta-1}. \tag{3.18}$$

Further, in view of (2.3) and (3.6), the operator $[A_{v_2}(t) - A_{v_1}(t)]A_0^{-1}$ is uniformly Hölder continuous for $\tau \le t \le \delta$, $\tau > 0$. Hence, $[A_{v_2}(t) - A_{v_1}(t)]z_2(t)$ is uniformly Hölder continuous for $\tau \le t \le \delta$, $\tau > 0$. Applying Theorem 2.1, we get that for any $\tau \le t \le \delta$, $\tau > 0$,

$$\begin{aligned}z_1(t) - z_2(t) &= U_{v_1}(t,\tau)[z_1(\tau) - z_2(\tau)] \\ &\quad + \int_\tau^t U_{v_1}(t,s)\{[A_{v_2}(s) - A_{v_1}(s)]z_2(t) + [f_{v_1}(s) - f_{v_2}(s)]\}ds.\end{aligned} \tag{3.19}$$

The bound in (3.18) allows us to take $\tau \to 0$ in (3.19), and passing to the limit, we obtain

$$z_1(t) - z_2(t) = \int_0^t U_{v_1}(t,s)\{[A_{v_2}(s) - A_{v_1}(s)]z_2(t) + [f_{v_1}(s) - f_{v_2}(s)]\}ds.$$

Now using (2.7), (2.8), (2.9) and [23, inequality (1.65), page 23], we obtain

$$\begin{aligned} \|Fv_1(t) - Fv_2(t)\|_\alpha &\le C_3C(R)\int_0^t (t-s)^{-\alpha}(\|v_1(s)-v_2(s)\|_\alpha s^{\beta-1}ds \\ &\quad + C_4L_f\int_0^t (t-s)^{-\alpha}\{\|v_1(s)-v_2(s)\|_\alpha \\ &\quad + \|v_1([h(v_1(s),s)]) - v_2([h(v_2(s),s)])\|_{\alpha-1}\}ds \\ &\le C_3C(R)\int_0^t (t-s)^{-\alpha}(\|v_1(s)-v_2(s)\|_\alpha s^{\beta-1}ds \\ &\quad + \frac{C_4}{1-\alpha}L_f(2+L_\alpha L_h)\delta^{1-\alpha}\sup_{t\in I}\|v_1(t)-v_2(t)\|_\alpha \\ &\le \widetilde{K}\delta^{\beta-\alpha}\sup_{t\in I}\|v_1(t)-v_2(t)\|_\alpha, \end{aligned} \tag{3.20}$$

where $\widetilde{K} = \max\{\frac{C_3C(R)}{1-\alpha}, \frac{C_4}{1-\alpha}L_f(2+L_\alpha L_h)\}$. Choose $\delta > 0$ such that

$$\widetilde{K}\delta^{\beta-\alpha} < \frac{1}{2}.$$

Then, from (3.20), it is clear that $F$ is a contraction map. Since $\mathcal{S}_\alpha$ is a complete metric space, by the Banach fixed-point theorem, there exists $u \in \mathcal{S}_\alpha$ such that $Fu = u$. It follows from Sobolevskiĭ [23, Theorem 5] that $u \in C^1((0,\delta);X)$. Thus $u$ is a solution to (1.1) on $[0,\delta]$. □

## 4. Example

Consider the quasi-linear parabolic differential equation with a deviated argument

$$\begin{gathered} \frac{\partial u}{\partial t} + a(x,t,u,\frac{\partial u}{\partial x})\frac{\partial^2 u}{\partial x^2} = \widetilde{H}(x,u(t,x)) + \widetilde{G}(t,x,u(t,x)); \\ u(t,0) = u(t,1), \quad t > 0; \\ u(0,x) = u_0(x), \quad x \in (0,1), \end{gathered} \tag{4.1}$$

where $a(\cdot,\cdot,\dots,\cdot)$ is a continuously differentiable real valued function in all variables. Here, $\widetilde{H}(x,u(t,x)) = \int_0^x K(x,y)u(\widetilde{g}(t)|u(t,y)|,y)dy$ for all $(t,x) \in (0,\infty)\times(0,1)$. Assume that $\widetilde{g} : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Hölder continuous in $t$ with $\widetilde{g}(0) = 0$ and $K \in C^1([0,1]\times[0,1];\mathbb{R})$. The function $\widetilde{G} : \mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is measurable in $x$, locally Hölder continuous in $t$, locally Lipschitz continuous in $u$, uniformly in $x$.

Here, the parabolically means that for any real vector $\xi \ne 0$ and for arbitrary values of $u, \frac{\partial u}{\partial x}$, it holds

$$-a(x,t,u,\frac{\partial u}{\partial x})\xi^2 > 0.$$

Let $A(t,u)u(t) = a(x,t,u,\frac{\partial u}{\partial x})\frac{\partial^2 u}{\partial x^2}$. If $u_0 \in C^1(0,1)$, then

$$A_0 u \equiv a\big(x,0,u_0,\frac{\partial u_0}{\partial x}\big)\frac{\partial^2 u}{\partial x^2}$$

is strongly elliptic operator with continuous coefficient. Let $X = L^2((0,1);\mathbb{R})$. Then $X_1 = D(A_0) = H^2(0,1)\cap H_0^1(0,1)$, $X_{1/2} = D((A_0)^{1/2}) = H_0^1(0,1)$ and $X_{-1/2} = H^{-1}(0,1)$. It is well known that the assumption (F1) is satisfied. The assumption on $a$ implies that $A(t,x)$ satisfies (2.7).

For $x \in (0,1)$, we define $f : \mathbb{R}_+ \times H_0^1(0,1) \times H^{-1}(0,1) \to L^2(0,1)$ by

$$f(t,\phi,\psi) = \widetilde{H}(x,\psi) + \widetilde{G}(t,x,\phi),$$

where $\widetilde{H}(x,\psi(x,t)) = \int_0^x K(x,y)\psi(y,t)dy$. We also assume that $\widetilde{G} : \mathbb{R}_+ \times [0,1] \times H^{-1}(0,1) \to L^2(0,1)$ satisfies

$$\|\widetilde{G}(t,x,u) - \widetilde{G}(s,x,v)\|_{L^2(0,1)} \le C(|t-s|^{\theta_1} + \|u-v\|_{H^{-1}(0,1)}),$$

for some constant $C > 0$. Then it can be seen that $f$ satisfies the condition (2.8) (see Gal [6]) and $h : H_0^1(0,1) \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by $h(\phi(x,t),t) = \widetilde{g}(t)|\phi(x,t)|$ satisfies (2.9) (see Gal [6]). Thus, we can apply the results of previous section to obtain the existence and uniqueness of a local solution to (4.1).

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