Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 90, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

COMPACTNESS RESULTS FOR QUASILINEAR PROBLEMS WITH VARIABLE EXPONENT ON THE WHOLE SPACE

OLFA ALLEGUE, ASMA KAROUI SOUAYAH

Abstract. In this work we give a compactness result which allows us to prove the point-wise convergence of the gradients of a sequence of solutions to a quasilinear inequality and for an arbitrary open set. This result suggests solutions to many problems, notably nonlinear elliptic problems with critical exponent.

1. Introduction and preliminary results

In their recent work El Hamidi and Rakotoson [5] gave a compactness result to prove the point-wise convergence of the gradients of a sequence of solutions to a general quasilinear inequality and for an arbitrary open set. They proved the following result.

Lemma 1.1. *Let $\hat{a}$ be a Carathéodory function from $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}^N$ satisfying the usual Leray-Lions growth and monotonicity conditions. Let (u_n) be a bounded sequence of $W^{1,p}_{\rm loc}(\mathbb{R}^N) = \{v \in L^p_{\rm loc}(\mathbb{R}^N), |\nabla v| \in L^p_{\rm loc}(\mathbb{R}^N)\}$, with $1 < p < +\infty$, (f_n) be a bounded sequence of $L^1_{\rm loc}(\mathbb{R}^N)$ and (g_n) be a sequence of $W^{-1,p'}_{\rm loc}(\mathbb{R}^N)$ tending strongly to zero. Assume that (u_n) satisfies:*

$$\int_{\mathbb{R}^N} \hat{a}(x, u_n(x), \nabla u_n(x)).\nabla\phi\, dx = \int_{\mathbb{R}^N} f_n\phi\, dx + \langle g_n, \phi\rangle,$$

for all $\phi \in W^{1,p}_{\rm comp}(\mathbb{R}^N) = \{v \in W^{1,p}(\mathbb{R}^N)$, with compact support$\}$, ϕ bounded. Then:

1. *there exists a function u such that $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$,*
2. *$u \in W^{1,p}_{\rm loc}(\mathbb{R}^N)$,*
3. *there exists a subsequence, still denoted (u_n), such that $\nabla u_n(x) \to \nabla u(x)$ a.e. in $\mathbb{R}^N$.*

In the present work, we generalize Lemma 1.1 for the $p(x)$-Laplace operator. Our principal result can be applied to a large class of quasilinear elliptic problems where there holds a lack of compactness, especially for the critical exponent equations.

In the sequel, we start with some preliminary basic results on the theory of Lebesgue-Sobolev spaces with variable exponent. We refer to the book by Musielak

---

2000 *Mathematics Subject Classification.* 35J655, 35J60, 35J70, 58E05.
Key words and phrases. $p(x)$-Laplace operator; critical Sobolev exponent; compactness; Palais-Smale condition.
©2011 Texas State University - San Marcos.
Submitted May 2, 2011. Published July 6, 2011.

[19], the papers by Kovacik and Rakosnik [13] and by Fan et al. [6, 7, 8]. Set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u : u \text{ is a Borel real-valued function on } \Omega, \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}.$$

We define on $L^{p(x)}$, the so-called *Luxemburg norm*, by the formula

$$|u|_{p(x)} := \inf\big\{\mu > 0 : \int_\Omega |\frac{u(x)}{\mu}|^{p(x)}\,dx \le 1\big\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many aspects: they are separable and Banach spaces [13, Theorem 2.5; Corollary 2.7] and the Hölder inequality holds [13, Theorem 2.1]. The inclusions between Lebesgue spaces are also naturally generalized [13, Theorem 2.8]: if $0 < |\Omega| < \infty$ and r_1, r_2 are variable exponents so that $r_1(x) \le r_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x)+1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder type inequality

$$\big|\int_\Omega uv\,dx\big| \le \big(\frac{1}{p^-} + \frac{1}{p'^-}\big)|u|_{p(x)}|v|_{p'(x)}, \tag{1.1}$$

is held.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

The space $W^{1,p(x)}(\Omega)$ is equipped by the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

We recall that if $(u_n), u, \in W^{1,p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold:

$$\min(|u|^{p^-}_{p(x)}, |u|^{p^+}_{p(x)}) \le \rho_{p(x)}(u) \le \max(|u|^{p^-}_{p(x)}, |u|^{p^+}_{p(x)}), \tag{1.2}$$

$$\min(|\nabla u|^{p^-}_{p(x)}, |\nabla u|^{p^+}_{p(x)}) \le \rho_{p(x)}(|\nabla u|) \le \max(|\nabla u|^{p^-}_{p(x)}, |\nabla u|^{p^+}_{p(x)}), \tag{1.3}$$

$$\begin{gathered} |u|_{p(x)} \to 0 \iff \rho_{p(x)}(u) \to 0, \\ \lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \iff \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0, \\ |u_n|_{p(x)} \to \infty \iff \rho_{p(x)}(u_n) \to \infty. \end{gathered} \tag{1.4}$$

We define also $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}.$$

The space $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space.

Next, we recall some embedding results regarding variable exponent Lebesgue-Sobolev spaces. We note that if $s(x) \in C_+(\overline{\Omega})$ and $s(x) < p^*(x)$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ is compact and continuous, where $p^*(x) = Np(x)/(N-p(x))$ if $p(x) < N$ or $p^*(x) = +\infty$ if $p(x) \geq N$. We refer to [13] for more properties of Lebesgue and Sobolev spaces with variable exponent. We also refer to the recent papers [1, 3, 4, 10, 11, 16, 17, 18] for the treatment of nonlinear boundary value problems in Lebesgue-Sobolev spaces with variable exponent. For relevant applications and related results we refer to the recent books by Ghergu and Rădulescu [9] and Kristály, Rădulescu and Varga [12].

2. Notation and compactness result

Let Ω be an arbitrary open set of $\mathbb{R}^N$, we shall denote by $\omega \subset\subset \Omega$ any relatively compact open subset ω of Ω (that is $\overline{\omega} \subset \Omega$, where $\overline{\omega}$ is the closure of ω). Let $1 < p(x) < +\infty$, we set

$$W^{1,p(x)}_{\rm loc}(\Omega) = \{v \in L^{p(x)}_{\rm loc}(\Omega); \nabla v \in L^{p(x)}_{\rm loc}(\Omega)\}.$$

For a given $q(x) \in (1,+\infty)$, we denote by $q'(x) := \frac{q(x)}{q(x)-1}$ its conjugate exponent. We shall use the following globally real Lipschitz functions: For $\epsilon > 0$, $\sigma \in \mathbb{R}$, let

$$S_\epsilon(\sigma) = \begin{cases} \sigma & \text{if } |\sigma| \leq \epsilon \\ \epsilon \operatorname{sign}(\sigma) & \text{otherwise,} \end{cases}$$

and $\sigma^k := S_k(\sigma)$ for $k \geq 1$.

We shall consider a nonlinear map $\hat{a} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying the following conditions:

- (L1) $\hat{a}(x,.,.)$ is a continuous map for almost every x and for all $(\sigma,\xi) \in \mathbb{R} \times \mathbb{R}^N$, $\hat{a}(.,\sigma,\xi)$ is measurable (such a property is called Carathéodory property),
- (L2) $\hat{a}$ maps bounded sets of $W^{1,p(x)}_{\rm loc}(\Omega)$ into bounded sets of $L^{p'(x)}_{\rm loc}(\Omega)$, and for almost all $x \in \Omega$, for all (σ,ξ) in $\mathbb{R} \times \mathbb{R}^N$, $\hat{a}(x,\sigma,\xi) \cdot \xi \geq 0$, for almost every $x \in \Omega$ and for all $v \in W^{1,p(x)}_{\rm loc}(\Omega)$, the mapping $u \mapsto \hat{a}(x,u,\nabla v)$ is continuous from $W^{1,p(x)}(\omega)$-weak into $L^{p'(x)}(\omega)$-strong, for all $\omega \subset\subset \Omega$,
- (L3) for almost every $x \in \Omega$, for all $(\sigma,\xi_i) \in \mathbb{R} \times \mathbb{R}^N$, $i = 1, 2$,
$$[\hat{a}(x,\sigma,\xi_1) - \hat{a}(x,\sigma,\xi_2)][\xi_1 - \xi_2] > 0, \quad \text{for } \xi_1 \neq \xi_2.$$
- (L4) if for some $x \in \Omega$, there is a sequence $(\sigma_n,\xi_{1n}) \in \mathbb{R} \times \mathbb{R}^N$, $\xi_2 \in \mathbb{R}^N$ such that $[\hat{a}(x,\sigma_n,\xi_{1n}) - \hat{a}(x,\sigma_n,\xi_2)][\xi_{1n} - \xi_2]$ and σ_n are bounded as $n \to +\infty$ then $|\xi_{1n}|$ remains in a bounded set of $\mathbb{R}$ as $n \to +\infty$.

As a corollary of our main result, we state the following result.

Lemma 2.1. *Let $\hat{a}$ be a Carathéodory function from $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}^N$ satisfying the usual Leray-Lions growth and monotonicity conditions. Let (u_n) be a bounded sequence of*

$$W^{1,p(x)}_{\rm loc}(\mathbb{R}^N) = \{v \in L^{p(x)}_{\rm loc}(\mathbb{R}^N), |\nabla v| \in L^{p(x)}_{\rm loc}(\mathbb{R}^N)\},$$

with $1 < p(x) < +\infty$, (f_n) be a bounded sequence of $L^1_{\rm loc}(\mathbb{R}^N)$ and (g_n) be a sequence of $W^{-1,p'(x)}_{\rm loc}(\mathbb{R}^N)$ tending strongly to zero. Assume that (u_n) satisfies

$$\int_{\mathbb{R}^N} a(x,u_n(x),\nabla u_n(x)).\nabla\phi \, dx = \int_{\mathbb{R}^N} f_n \phi \, dx + \langle g_n, \phi \rangle,$$

for all $\phi \in W^{1,p(x)}_{\rm comp}(\mathbb{R}^N) = \{W^{1,p(x)}(\mathbb{R}^N)$ *with compact support*$\}$, ϕ *bounded. Then:*

(1) *there exists a function* u *such that* $u_n(x) \to u(x)$ *a.e in* $\mathbb{R}^N$,
(2) $u \in W^{1,p(x)}_{\rm loc}(\mathbb{R}^N)$,
(3) *there exists a subsequence, still denoted* (u_n), *such that* $\nabla u_n(x) \to \nabla u(x)$ *a.e. in* $\mathbb{R}^N$.

This Lemma generalizes the result in [5, Lemma 1], and it will be used for the critical exponent equation, to show that suitable Palais-Smale sequences are relatively compact. Our main result is concerned with the convergence almost everywhere of the gradients.

Theorem 2.2. *Let* (u_n) *be a bounded sequence of* $W^{1,p(x)}_{\rm loc}(\Omega)$. *Then*

(i) *There is a subsequence still denoted* (u_n) *and a function* $u \in W^{1,p(x)}_{\rm loc}(\Omega)$ *such that*
$$u_n(x) \to u(x) \quad \text{a.e. in } \Omega \text{ as } n \to +\infty$$
(ii) *If furthermore, we assume* (L1)–(L4) *and that for all* $\phi \in C^\infty_c(\Omega)$, *and all* $k \geq k_0 > 0$:
$$\limsup_{n\to+\infty} \int_\Omega \hat{a}(x, u_n(x), \nabla u_n(x)) \cdot \nabla(\phi S_\epsilon(u_n - u^k)) \leq o(1)$$
as $\epsilon \to 0$ *then there exists a subsequence still denoted* (u_n) *such that*
$$\nabla u_n(x) \to \nabla u(x) \quad \text{a.e. in } \Omega \text{ as } n \to +\infty.$$

Remark 2.3. (1) The term $o(1)$ in (ii) might depend on k and ϕ.

(2) (L2) is satisfied if for all $\omega \subset\subset \Omega$, there is a constant $c_\omega > 0$ and a function $a_0 \in L^{p'(x)}(\omega)$ such that for almost every $x \in \omega$, for all $(\sigma, \xi) \in \mathbb{R} \times \mathbb{R}^N$:
$$|\hat{a}(x, \sigma, \xi)| \leq c_\omega[|\sigma|^{p(x)-1} + |\xi|^{p(x)-1} + a_0(x)].$$
and (L4) is true if $\hat{a}(x, \sigma, \xi) \cdot \xi \geq c^1_\omega |\xi|^{p(x)} - c^2_\omega$, $c^1_\omega > 0$.

(3) Bounded sets in $W^{1,p(x)}_{\rm loc}(\Omega)$ will be bounded in
$$W^{1,p(x)}(\omega) = \{v \in L^{p(x)}(\omega), \nabla v \in L^{p(x)}(\omega)\}, \quad \text{for every } \omega \subset\subset \Omega.$$

Proof of theorem 2.2. (i) Let $(w_j)_{j\geq0}$ be a sequence of bounded relatively compact subsets of Ω such that $\overline{\omega}_j \subset \omega_{j+1}$ and $\cup_{j=0}^{+\infty} \omega_j = \Omega$. Since $(u_n)_n$ is bounded in $W^{1,p(x)}(\omega_j)$, by the usual embeddings, we deduce that there is a subsequence u_{n_j} and a function u in $W^{1,p(x)}(\omega_j)$ such that $u_{n_j}(x) \to u(x)$ as $n \to \infty$. We conclude with the usual diagonal Cantor process.

(ii) Let $\phi \in C^\infty_c(\Omega), 0 \leq \phi \leq 1, \phi = 1$ on ω_j and $\text{supp}(\phi) \subset \omega_{j+1}$, and set
$$\Delta(u_n, u)(x) = [\hat{a}(x, u_n(x), \nabla u_n(x)) - \hat{a}(x, u_n(x), \nabla u(x))]\nabla(u_n - u)(x).$$
Then one has:

(ii.1) $\Delta(u_n, u)(x) \geq 0$ a.e. on Ω (due to (L3)).
(ii.2) $\sup_n \int_{\omega_{j+1}} \Delta(u_n, u)dx$ is finite (since (u_n) is in a bounded set of $W^{1,p(x)}_{\rm loc}(\Omega)$ and the growth condition (L2)).

Let us show that $\lim_n \int_\Omega \phi \Delta(u_n, u)^{\frac{1}{p(x)}} dx = 0$. On one hand,
$$\int_\Omega \phi\Delta(u_n, u)^{\frac{1}{p(x)}} dx = \int_{\{|u|>k\}} \phi\Delta(u_n, u)^{\frac{1}{p(x)}} dx + \int_{\{|u|\leq k\}} \phi\Delta(u_n, u)^{\frac{1}{p(x)}} dx. \tag{2.1}$$

By the Hölder inequality

$$\int_{\{|u|>k\}} \Delta(u_n,u)^{\frac{1}{p(x)}}\phi\,dx \le |\Delta(u_n,u)^{\frac{1}{p(x)}}|_{p(x)}|\phi|_{\frac{p(x)}{p(x)-1}} \le a_1(j)|\phi|_{\frac{p(x)}{p(x)-1}},$$

where $a_s(j)$ are different constants depending on j but independent of n, ϵ and k. Noticing that

$$\text{meas}\{x\in w_{j+1}:|u|>k\}\le \frac{c_1(j)}{k^{p^-}},$$

one deduces that

$$\rho_{\frac{p(x)}{p(x)-1}}(\phi)=\int_{\{|u|>k\}}\phi^{\frac{p(x)}{p(x)-1}}\,dx\le\frac{c_1(j)}{k^{p^-}} \tag{2.2}$$

where $c_m(j)$ are different constants depending on j and ϕ but independent of n, ϵ and k. We conclude that

$$\limsup_{n\to\infty}\int_{\{|u|>k\}}\Delta(u_n,u)^{\frac{1}{p(x)}}\phi\,dx\le o(1)\quad\text{as } k\to\infty \tag{2.3}$$

While for the second integral, we have

$$\begin{aligned}\int_{\{|u|\le k\}}\Delta(u_n,u)^{\frac{1}{p(x)}}\phi\,dx &= \int_{\{|u|\le k\}\cap\{|u_n-u|\le\epsilon|\}}\Delta(u_n,u)^{\frac{1}{p(x)}}\phi\,dx\\ &\quad+\int_{\{|u|\le k\}\cap\{|u_n-u|>\epsilon\}}\Delta(u_n,u)^{\frac{1}{p(x)}}\phi\,dx.\end{aligned} \tag{2.4}$$

Moreover, the second term in the right hand side in the last inequality satisfies

$$\int_{\{|u|\le k\}\cap\{|u_n-u|>\epsilon\}}\Delta(u_n,u)^{\frac{1}{p(x)}}\phi\,dx\le|\Delta(u_n,u)^{\frac{1}{p(x)}}|_{p(x)}|\phi|_{\frac{p(x)}{p(x)-1}}\le a_2(j)|\phi|_{\frac{p(x)}{p(x)-1}}$$

and

$$\rho_{\frac{p(x)}{p(x)-1}}(\phi)\le a_2(\phi)\,\text{meas}\{x\in w_{j+1}:|u_n-u|>\epsilon\}.$$

Since (u_n) converges to u in measure, we deduce that, for n sufficiently large, $\text{meas}\{x\in w_{j+1}:|u_n-u|>\epsilon\}\le\epsilon$. It follows that

$$\limsup_{n\to+\infty}\int_{\{|u|\le k\}\cap\{|u_n-u|>\epsilon\}}\Delta(u_n,u)^{\frac{1}{p(x)}}\phi\,dx\le o(1)\quad\text{as }\epsilon\to0. \tag{2.5}$$

Setting $A^\epsilon_{n,k}=w_{j+1}\cap\{|u|\le k\}\cap\{|u_n-u|\le\epsilon\}$, we obtain from the Hölder inequality

$$\int_{A^\epsilon_{n,k}}\Delta(u_n,u)^{\frac{1}{p(x)}}\phi\,dx\le c_2(j)|\Delta(u_n,u)^{\frac{1}{p(x)}}\phi^{\frac{1}{p(x)}}|_{p(x)}, \tag{2.6}$$

and

$$\rho_{p(x)}(\Delta(u_n,u)^{\frac{1}{p(x)}}\phi^{\frac{1}{p(x)}})=I^1_{n,k}(\epsilon)-I^2_{n,k}(\epsilon),$$

with

$$I^1_{n,k}(\epsilon)=\int_{A^\epsilon_{n,k}}\hat a(x,u_n,\nabla u_n)\cdot\nabla(u_n-u)\phi\,dx,$$

$$I^2_{n,k}(\epsilon)=\int_{\{|u|\le k\}}\hat a(x,u_n,\nabla u)\cdot\nabla S_\epsilon(u_n-u)\phi\,dx.$$

Since $\hat a(x,u_n,\nabla u)\to\hat a(x,u,\nabla u)$ strongly in $L^{p'(x)}(w_{j+1})$ (by the last statement of (L2)) and $\nabla S_\epsilon(u_n-u)\rightharpoonup0$ in $L^{p(x)}(w_{j+1})$-weak, we deduce that

$$\lim_{n\to+\infty}I^2_{n,k}(\epsilon)=0, \tag{2.7}$$

while for the term $I^1_{n,k}(\epsilon)$, we obtain

$$I^1_{n,k}(\epsilon) \le \int_\Omega \hat{a}(x,u_n,\nabla u_n)\cdot\nabla(\phi S_\epsilon(u_n-u^k)) - \int_\Omega \hat{a}(x,u_n,\nabla u_n)\cdot\nabla\phi S_\epsilon(u_n-u^k)dx. \tag{2.8}$$

Since

$$\Big|\int_\Omega \hat{a}(x,u_n,\nabla u_n)\cdot\nabla\phi S_\epsilon(u_n-u^k)dx\Big| \le c_3(j)\epsilon; \tag{2.9}$$

then assumption (ii) implies

$$\limsup_{n\to+\infty} I^1_{n,k}(\epsilon) \le c_3(j)\epsilon + \circ(1) \quad \text{as } \epsilon\to 0. \tag{2.10}$$

Combining relations (2.6), (2.7) and (2.10), it follows that

$$\limsup_{n\to+\infty}\int_{A^\epsilon_{n,k}} \Delta(u_n,u)^{\frac{1}{p(x)}}\phi\,dx \le o(1) \quad \text{as } \epsilon\to 0. \tag{2.11}$$

Letting first $\epsilon\to 0$ and then k to infinity, by relations (2.1), (2.3), (2.4), (2.5) and (2.11), we deduce

$$\lim_{n\to+\infty}\int_\Omega \Delta(u_n,u)^{\frac{1}{p(x)}}\phi\,dx = 0.$$

We then obtain that for a subsequence (u_{j_n}),

$$\Delta(u_{j_n},u)(x)\to 0 \quad \text{a.e. on } w_j.$$

Arguing as Leray-Lions [14, 15], we deduce from (L4) that $\nabla u_{j_n}(x)\to\nabla u(x)$ a.e. on w_j. The proof is achieved by the diagonal process of Cantor. □

Proof of lemma 2.1. Since (u_n) belongs to a bounded set of $W^{1,p(x)}_{\rm loc}(\mathbb{R}^N)$, statement (i) of Theorem 2.2 implies that there is a function u and a subsequence still denoted by (u_n) such that

$$u_n(x)\to u(x) \quad \text{a.e. in } \mathbb{R}^N, \text{ as } n\to\infty$$

and

$$u\in W^{1,p(x)}_{\rm loc}(\mathbb{R}^N).$$

Then for all $\phi\in C^\infty_c(\mathbb{R}^N)$, $\phi S_\epsilon(u_n-u^k)$ is an element of $W^{1,p(x)}_{\rm comp}(\mathbb{R}^N)$ and

$$\Big|\int_{\mathbb{R}^N} f_n\phi S_\epsilon(u_n-u^k)dx\Big| \le \epsilon|\phi|_\infty|f_n|_{L^1(\omega)} \le c(\phi)\epsilon, \tag{2.12}$$

(for every ϕ such that $\mathrm{supp}(\phi)\subset\omega$, $\overline{\omega}$ is a compact of $\mathbb{R}^N$), and

$$|\langle g_n,\phi S_\epsilon(u_n-u^k)\rangle| \le |g_n|_{W^{-1,p'(x)}(\omega)}|\phi S_\epsilon(u_n-u^k)|_{W^{1,p(x)}(\mathbb{R}^N)}.$$

Using the fact that $|\phi S_\epsilon(u_n-u^k)|_{W^{1,p(x)}(\mathbb{R}^N)}$ is bounded independently of ϵ, n, k and that $|g_n|_{W^{-1,p'(x)}(\omega)}\to 0$, it holds:

$$\limsup_n \int_{\mathbb{R}^N}\hat{a}(x,u_n,\nabla u_n)\cdot\nabla(\phi S_\epsilon(u_n-u^k))dx \le O(\epsilon).$$

Finally, by Theorem 2.2 we complete the proof. □

3. Examples of applications

In this section, we are interested in the existence of solutions to the problem

$$\begin{gathered} -\operatorname{div}\Big(\Big(|\nabla u(x)|^{p(x)-2}\Big)\nabla u(x)\Big)=\lambda f(u)+g(u)\quad \text{for } x\in\Omega,\\ u\geq 0\quad \text{for } x\in\Omega,\\ u=0\quad \text{for } x\in\partial\Omega, \end{gathered}\tag{3.1}$$

where $\Omega\subset\mathbb{R}^N$, $(N\geq 3)$ is a bounded domain with smooth boundary, λ is a positive real number and p is a continuous function on $\overline{\Omega}$ with $p^+<N$.

In the first result, we assume that f and g are continuous and satisfy the following hypotheses (see [2]):

(F1) There exist positive constants $C_1, C_2>0$ and $q:\bar{\Omega}\to\mathbb{R}$ a continuous function such that

$$C_1 t^{q(x)-1}\leq f(t)\leq C_2 t^{q(x)-1},\quad \forall t\geq 0.$$

(G1) There exists a positive constant $C_3>0$ such that

$$|g(t)|\leq C_3|t|^{p^*(x)-1},\quad \forall t\in\mathbb{R}.$$

(G2) There exists $\gamma\in(p^+,p^{*-}]$ such that

$$0<\gamma G(t)\leq tg(t),\quad \forall t\in\mathbb{R},$$

where $G(t)=\int_0^t g(s)ds$.

We prove the following result.

Theorem 3.1. *If $1<q^+<p^{*-}$, $q^-<p^-$, and (F1), (G1).(G2) hold, then there exists λ^* such that for all $\lambda\in(0,\lambda^*)$, problem (3.1) has a non trivial solution.*

In the second result, we are concerned with the special case $f(u)=-|u|^{q(x)-2}u$ and $g(u)=|u|^{p^*(x)-2}u$. We prove the following result.

Theorem 3.2. *For any $\lambda>0$ problem (3.1) has infinitely many weak solutions provided that $p^{*-}>max(p^+,q^+)$.*

Proof of Theorem 3.1. Let E denote the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$. The energy functional corresponding to (3.1) is $J_\lambda:E\to\mathbb{R}$, defined as

$$J_\lambda(u):=\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}dx-\lambda\int_\Omega F(u_+)dx-\int_\Omega G(u_+)dx,$$

where $u_+(x)=\max\{u(x),0\}$ and F is defined by $F(t)=\int_0^t f(s)ds$.

Remark 3.3. Assume that condition (G1) is fulfilled, it is clear that for every $t\geq 0$, we obtain

$$-\frac{C_3}{p^{*-}}t^{p^*(x)}\leq G(t)\leq\frac{C_3}{p^{*-}}t^{p^*(x)}$$

Proposition 3.4. *The functional J_λ is well-defined on E and $J_\lambda\in C^1(E,\mathbb{R})$.*

Proof. We have the following continuous embedding (see [13, Theorem 2.8])

$$W_0^{1,p(x)}(\Omega)\hookrightarrow L^{p^*(x)}(\Omega)$$

using the fact that Ω is bounded, we obtain the continuous embedding

$$W_0^{1,p(x)}(\Omega)\hookrightarrow L^{s(x)}(\Omega),\quad s\in[1,p^*],$$

which implies that J_λ is well-defined on E and $J_\lambda \in C^1(E,\mathbb{R})$, with the derivative given by

$$\langle dJ_\lambda(u), v\rangle = \int_\Omega \big(|\nabla u|^{p(x)-2}\nabla u \nabla v - \lambda f(u)v - g(u)v\big)dx, \quad \forall v \in E.$$

□

The proof of Theorem 3.1 is related to Ekeland's variational principle. In order to apply it we need the following lemmas:

Lemma 3.5. *Under hypotheses of theorem 3.1, there exists $M_2 > 0$ such that for all $\rho \in (0,1)$ for all $C_3 < \frac{q^-}{p^+ M_2^{p^{*-}}}\rho^{p^+-q^-}$, there exists $\lambda^* > 0$ and $r > 0$ such that, for all $u \in E$ with $\|u\| = \rho$, $J_\lambda(u) \geq r > 0$ for all $\lambda \in (0,\lambda^*)$.*

Proof. Since $E \hookrightarrow L^{q(x)}(\Omega)$ and $E \hookrightarrow L^{p^*(x)}(\Omega)$ are continuous, there exists $M_1 > 0$ and $M_2 > 0$ such that

$$|u|_{q(x)} \leq M_1\|u\| \quad \text{and} \quad |u|_{p^*(x)} \leq M_2\|u\|, \quad \forall u \in E. \tag{3.2}$$

We fix $\rho \in (0,1)$ such that $\rho < \min(1, 1/M_1, 1/M_2)$. Then for all $u \in E$, with $\|u\| = \rho$, we deduce that

$$|u|_{q(x)} < 1 \quad \text{and} \quad |u|_{p^*(x)} < 1.$$

Furthermore, by (1.2) for all $u \in E$ with $\|u\| = \rho$, we have

$$\int_\Omega |u|^{q(x)}\,dx \leq |u|_{q(x)}^{q^-}, \quad \text{and} \quad \int_\Omega |u|^{p^*(x)}\,dx \leq |u|_{p(x)}^{p^{*-}}.$$

The above inequality and relation (3.2) imply that for all $u \in E$ with $\|u\| = \rho$,

$$\int_\Omega |u|^{q(x)}\,dx \leq M_1^{q^-}\|u\|^{q^-}, \quad \textit{and} \quad \int_\Omega |u|^{p^*(x)}\,dx \leq M_2^{p^{*-}}\|u\|^{p^{*-}}. \tag{3.3}$$

Using relation (3.3) we deduce that, for any $u \in E$ with $\|u\| = \rho$, the following inequalities hold:

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p^+}\|u\|^{p^+} - \frac{\lambda}{q^-}C_2 M_1^{q^-}\|u\|^{q^-} - \frac{C_3}{p^{*-}}M_2^{p^{*-}}\|u\|^{p^{*-}}, \\ &\geq \frac{1}{p^+}\rho^{p^+} - \frac{\lambda}{q^-}C_2 M_1^{q^-}\rho^{q^-} - \frac{C_3}{p^{*-}}M_2^{p^{*-}}\rho^{p^{*-}}. \end{aligned}$$

By the above inequality we remark that if we define for all $C_3 < \frac{q^-}{p^+ M_2^{p^{*-}}}\rho^{p^+-q^-}$

$$\lambda^* = \frac{q^-}{2C_2 M_1^{q^-}}\big[\frac{1}{p^+}\rho^{p^+-q^-} - \frac{C_3}{q^-}M_2^{p^{*-}}\big], \tag{3.4}$$

then for any $\lambda \in (0,\lambda^*)$, there exists $r > 0$ such that $J_\lambda(u) \geq r > 0$ for all $u \in E$ with $\|u\| = \rho$. The proof is complete. □

Lemma 3.6. *There exists $\phi \in E$ such that $\phi \geq 0$, $\phi \neq 0$ and $J_\lambda(t\phi) < 0$, for $t > 0$ small enough.*

Proof. Since $q^- < p^-$, then let $\epsilon_0 > 0$ be such that $q^- + \epsilon_0 < p^-$. On the other hand, since $q \in C(\overline{\Omega})$ it follows that there exists an open set $\Omega_0 \subset\subset \Omega$ such that $|q(x) - q^-| < \epsilon_0$ for all $x \in \Omega_0$. Thus, we conclude that $q(x) \leq q^- + \epsilon_0 < p^-$ for all

$x \in \overline{\Omega}_0$. Let $\phi \in C_0^\infty(\Omega)$ be such that $\operatorname{supp}(\phi) \supset \overline{\Omega}_0$, $\phi(x) = 1$ for all $x \in \overline{\Omega}_0$ and $0 \le \phi \le 1$ in Ω. Then using the above information for any $t \in (0,1)$ we have

$$\begin{aligned}
J_\lambda(t\phi) &= \int_\Omega \frac{t^{p(x)}}{p(x)} |\nabla \phi|^{p(x)}\,dx - \lambda \int_\Omega F(t\phi)dx - \int_\Omega G(t\phi)dx,\\
&\le \int_\Omega \frac{t^{p(x)}}{p(x)} |\nabla \phi|^{p(x)}\,dx - C_1\lambda \int_\Omega \frac{t^{q(x)}}{q(x)} |\phi|^{q(x)}dx + C_3 \int_\Omega \frac{t^{p^*(x)}}{p^*(x)} |\phi|^{p^*(x)}dx,\\
&\le \frac{t^{p^-}}{p^-} \int_\Omega |\nabla \phi|^{p(x)}dx - \frac{C_1\lambda}{q^+} \int_\Omega t^{q(x)}|\phi|^{q(x)} + C_3 \frac{t^{p^{*-}}}{p^{*-}} \int_\Omega |\phi|^{p^*(x)}\,dx,\\
&\le \frac{t^{p^-}}{p^-} \Big[\int_\Omega |\nabla \phi|^{p(x)}dx + C_3 \int_\Omega |\phi|^{p^*(x)}dx \Big] - \frac{\lambda t^{q^- + \epsilon_0}}{q^+} \int_{\Omega_0} |\phi|^{q(x)}\,dx,\\
&= \frac{t^{p^-}}{p^-} \Big[\int_\Omega |\nabla \phi|^{p(x)}dx + C_3 \int_\Omega |\phi|^{p^*(x)}dx \Big] - \frac{\lambda t^{q^- + \epsilon_0}}{q^+} |\Omega_0|.
\end{aligned}$$

Therefore, $J_\lambda(t\phi) < 0$, for $t < \delta^{1/(p^- - q^- - \epsilon_0)}$ with

$$0 < \delta < \min\{1, \frac{p^-\lambda|\Omega_0|}{q^+\big[\int_\Omega |\nabla\phi|^{p(x)}dx + C_3 \int_\Omega |\phi|^{p^*(x)}dx\big]}\}.$$

Finally, we point out that $\int_\Omega |\nabla\phi|^{p(x)}dx + C_3 \int_\Omega |\phi|^{p^*(x)}dx > 0$. In fact if

$$\int_\Omega |\nabla\phi|^{p(x)}dx + C_3 \int_\Omega |\phi|^{p^*(x)}dx = 0,$$

then $\int_\Omega |\phi|^{p^*(x)}dx = 0$. Using relation (1.2), we deduce that $|\phi|_{p^*(x)} = 0$ and consequently $\phi = 0$ in Ω which is a contradiction. The proof is complete. $\square$

Proof of theorem 3.1. Let λ^* be defined as in (3.4) and $\lambda \in (0, \lambda^*)$. By Lemma 3.5 it follows that on the boundary of the ball centered at the origin and of radius ρ in E, denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)} J_\lambda > 0. \tag{3.5}$$

On the other hand, by Lemma 3.6, there exists $\phi \in E$ such that $J_\lambda(t\phi) < 0$, for all $t > 0$ small enough. Moreover, relations (1.2) and (3.2) imply, that for any $u \in B_\rho(0)$, we have

$$J_\lambda(u) \ge \frac{1}{p^+}\|u\|^{p^+} - \frac{\lambda}{q^-} C_2^{q^-} M_1^{q^-} \|u\|^{q^-} - \frac{C_3^{p^{*-}}}{q^-} M_2^{q^-} \|u\|^{p^{*-}}.$$

It follows that

$$-\infty < J_\infty := \inf_{\overline{B_\rho(0)}} J_\lambda < 0.$$

We let now $0 < \epsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda$. Using the above information, the functional $J_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, is lower bounded on $\overline{B_\rho(0)}$ and $J_\lambda \in C^1(\overline{B_\rho(0)}, \mathbb{R})$. Then by Ekeland's variational principle there exists $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$J_\infty \le J_\lambda(u_\epsilon) \le J_\infty + \epsilon,$$
$$0 < J_\lambda(u) - J_\lambda(u_\epsilon) + \epsilon \cdot \|u - u_\epsilon\|, \quad u \ne u_\epsilon.$$

Since

$$J_\lambda(u_\epsilon) \le \inf_{\overline{B_\rho(0)}} J_\lambda + \epsilon \le \inf_{B_\rho(0)} J_\lambda + \epsilon < \inf_{\partial B_\rho(0)} J_\lambda,$$

we deduce that $u_\epsilon \in B_\rho(0)$.

Now, we define $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $I_\lambda(u) = J_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\epsilon + t \cdot v) - I_\lambda(u_\epsilon)}{t} \geq 0,$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{J_\lambda(u_\epsilon + t \cdot v) - J_\lambda(u_\epsilon)}{t} + \epsilon \cdot \|v\| \geq 0.$$

Letting $t \to 0$ it follows that $\langle dJ_\lambda(u_\epsilon), v\rangle + \epsilon \cdot \|v\| \geq 0$ we have $\|dJ_\lambda(u_\epsilon)\| \leq \epsilon$. We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$J_\lambda(u_n) \to J_\infty \quad \text{and} \quad dJ_\lambda(u_n) \to 0_{E^*}. \tag{3.6}$$

From where we can conclude that $\{u_n\}$ is a bounded $(PS)_{J_\infty}$ sequence to J_λ. By a subsequence still denoted by $\{u_n\}$, we may assume that $\{u_n\}$ has a weak limit $u_\lambda \in W_0^{1,p(x)}(\Omega)$. Moreover, from the definition of the functional J_λ, we can assume that $\{u_n\}$ is a sequence of non negative functions. Now, we need the following lemma.

Lemma 3.7. *The weak limit u_λ of $\{u_n\}$ is a non negative solution to* (3.1) *for $\lambda \in (0, \lambda^*)$.*

Proof. In what follows, we will show $dJ_\lambda(u_\lambda) = 0$ and $u_\lambda \neq 0, \forall \lambda \in (0, \lambda^*)$ which imply that lemma 3.7 holds true. Firstly note that

$$\begin{gathered}
J_\lambda(u_n) = \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx - \lambda \int_\Omega F(u_{n+}) dx - \int_\Omega G(u_{n+}) dx, \\
\langle dJ_\lambda(u_n), u_n\rangle = \int_\Omega |\nabla u_n|^{p(x)} dx - \lambda \int_\Omega f(u_n) u_n \, dx - \int_\Omega g(u_n) u_n \, dx.
\end{gathered}$$

Then $J_\lambda(u_n) - \frac{1}{\gamma}\langle dJ_\lambda(u_n), u_n\rangle = J_\infty + o_n(1)$. Thus, since $uf(u) \geq 0$ for every $u \geq 0$, we obtain

$$\begin{aligned}
&\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx - \frac{1}{\gamma} \int_\Omega |\nabla u_n|^{p(x)} dx - \lambda \int_\Omega F(u_{n+}) dx \\
&+ \frac{\lambda}{\gamma} \int_\Omega f(u_n) u_n \, dx - \int_\Omega G(u_{n+}) dx + \frac{1}{\gamma} \int_\Omega g(u_n) u_n \, dx \\
&\geq \big(\frac{1}{p^+} - \frac{1}{\gamma}\big) \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx - \lambda \int_\Omega F(u_{n+}) dx \\
&\quad + \frac{1}{\gamma}\Big(\int_\Omega \big(g(u_n) u_n - \gamma G(u_{n+})\big) dx\Big).
\end{aligned}$$

Since $\gamma > p^+$ and applying (G2) we have

$$\begin{aligned}
J_\lambda(u_n) - \frac{1}{\gamma}\langle dJ_\lambda(u_n), u_n\rangle &\geq \big(\frac{1}{p^+} - \frac{1}{\gamma}\big) \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx - \lambda \int_\Omega F(u_{n+}) dx, \\
&\geq -\lambda \int_\Omega F(u_{n+}) dx, \\
&\geq -\lambda C_2 \int_\Omega \frac{1}{q(x)} u_n^{q(x)} dx.
\end{aligned}$$

Using (1.2) we deduce that $-\frac{\lambda C_2}{q^+}|u_n|_{q(x)}^{q^+} \le J_\infty + o_n(1)$. Moreover $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and passing to the limit as $n \to \infty$, we obtain

$$-\frac{\lambda C_2}{q^+}|u_\lambda|_{q(x)}^{q^+} \le J_\infty < 0.$$

We deduce that $u_\lambda \neq 0$. To conclude that u_λ is a solution to (3.1), we use Theorem 2.2, which implies $\nabla u_n(x) \to \nabla u_\lambda(x)$ a.e. in Ω as $n \to \infty$. $\square$

Proof of Theorem 3.2. Now, we are concerned with the special case of problem (3.1),

$$\begin{gathered} -\operatorname{div}\left(\left(|\nabla u(x)|^{p(x)-2}\right)\nabla u(x)\right) = -\lambda|u|^{q(x)-2}u + |u|^{p^*(x)-2}u \quad \text{for } x \in \Omega, \\ u \ge 0 \quad \text{for } x \in \Omega, \\ u = 0 \quad \text{for } x \in \partial\Omega, \end{gathered} \tag{3.7}$$

where $\Omega \subset \mathbb{R}^N$, $(N \ge 3)$ is a bounded domain with smooth boundary, λ is a positive real number and p is a continuous function on $\overline{\Omega}$. The proof of Theorem 3.2 is based on a $\mathbb{Z}_2$-symmetric version for even functionals of the Mountain pass Theorem (see [20, Theorem 9.12]).

The energy functional corresponding to the problem (3.7) is $J_\lambda : E \to \mathbb{R}$, defined as

$$J_\lambda(u) := \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \lambda\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx - \int_\Omega \frac{1}{p^*(x)}|u|^{p^*(x)}dx,$$

It is clear that the functional J_λ is well-defined on E and $J_\lambda \in C^1(E,\mathbb{R})$, with the derivative given by

$$\langle dJ_\lambda(u), v\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla v + \lambda|u|^{q(x)-2}uv - |u|^{p^*(x)-2}uv\right)dx, \quad \forall v \in E.$$

To use the mountain pass theorem, we need the following lemmas:

Lemma 3.8. *For any $\lambda > 0$ there exists $r, a > 0$ such that $J_\lambda(u) \ge a > 0$ for any $u \in E$ with $\|u\| = r$.*

Proof. Recall that E is continuously embedded in $L^{p^*(x)}(\Omega)$. So there exists a positive constant C_4 such that, for all $u \in E$,

$$|u|_{p^*(x)} \le C_4\|u\|. \tag{3.8}$$

Suppose that $\|u\| < \min(1, \frac{1}{C_4})$, then for all $u \in E$ with $\|u\| = \rho$ we have $|u|_{p^*(x)} < 1$. Furthermore, relation (1.2) yields for all $u \in E$ with $\|u\| = \rho$ we have

$$\int_\Omega |u|^{p^*(x)}\,dx \le |u|_{p^*(x)}^{p^{*-}}.$$

The above inequality and relation (3.8) imply that for all $u \in E$ with $\|u\| = \rho$, we have

$$\int_\Omega |u|^{p^*(x)}\,dx \le C_4^{p^{*-}}\|u\|^{p^{*-}}. \tag{3.9}$$

Then using relation (3.9), we deduce that, for any $u \in E$ with $\|u\| = \rho$, the following inequalities hold

$$J_\lambda(u) \ge \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - \frac{1}{p^{*-}}\int_\Omega |u|^{p^*(x)}\,dx,$$

$$\geq \frac{1}{p^+}\|u\|^{p^+} - \frac{1}{p^{*-}} C_4^{p^{*-}} \|u\|^{p^{*-}}.$$

Let $h(t) = \frac{1}{p^+}t^{p^+} - \frac{1}{p^{*-}}C_4^{p^{*-}} t^{p^{*-}}$, $t > 0$. It is easy to see that $h(t) > 0$ for all $t \in (0, t_1)$, where $t_1 < \big(\frac{p^{*-}}{p^+ C_4^{p^{*-}}}\big)^{\frac{1}{p^{*-}-p^+}}$.

So for any $\lambda > 0$, we can choose $r, a > 0$ such that $J_\lambda(u) \geq a > 0$ for all $u \in E$ with $\|u\| = r$. The proof is complete. $\square$

Lemma 3.9. *If $E_1 \subset E$ is a finite dimensional subspace, the set $S = \{u \in E_1; J_\lambda(u) \geq 0\}$ is bounded in E.*

Proof. We have

$$\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx \leq K_1(\|u\|^{p^-} + \|u\|^{p^+}) \quad \forall u \in E, \tag{3.10}$$

where K_1 is a positive constant. Also we have

$$\int_\Omega |u|^{q(x)}\, dx \leq |u|_{q(x)}^{q^-} + |u|_{q(x)}^{q^+} \quad \forall u \in E. \tag{3.11}$$

The fact that E is continuously embedded in $L^{q(x)}(\Omega)$, assures the existence of a positive constant C_5 such that

$$|u|_{q(x)} \leq C_5\|u\| \quad \forall u \in E. \tag{3.12}$$

The last two inequalities show that there exists a positive constant $K_2(\lambda)$ such that

$$\lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)}\, dx \leq K_2(\lambda)\left(\|u\|^{q^-} + \|u\|^{q^+}\right) \quad \forall u \in E. \tag{3.13}$$

By inequalities (3.10) and (3.13), we obtain

$$J_\lambda(u) \leq K_1(\|u\|^{p^-} + \|u\|^{p^+}) + K_2(\lambda)\left(\|u\|^{q^-} + \|u\|^{q^+}\right) - \frac{1}{p^{*+}} \int_\Omega |u|^{p^*(x)}\, dx, \tag{3.14}$$

for all $u \in E$.

Let $u \in E$ be arbitrary but fixed. We define

$$\Omega_< = \{x \in \Omega; |u(x)| < 1\}. \quad \Omega_\geq = \Omega \backslash \Omega_<.$$

Then we have

$$\begin{aligned}
J_\lambda(u) &\leq K_1(\|u\|^{p^-} + \|u\|^{p^+}) + K_2(\lambda)\big(\|u\|^{q^-} + \|u\|^{q^+}\big) - \frac{1}{p^{*+}} \int_\Omega |u|^{p^*(x)}\, dx, \\
&\leq K_1(\|u\|^{p^-} + \|u\|^{p^+}) + K_2(\lambda)\big(\|u\|^{q^-} + \|u\|^{q^+}\big) - \frac{1}{p^{*+}} \int_{\Omega_\geq} |u|^{p^*(x)}\, dx, \\
&\leq K_1(\|u\|^{p^-} + \|u\|^{p^+}) + K_2(\lambda)\big(\|u\|^{q^-} + \|u\|^{q^+}\big) - \frac{1}{p^{*+}} \int_{\Omega_\geq} |u|^{p^{*-}}\, dx, \\
&\leq K_1(\|u\|^{p^-} + \|u\|^{p^+}) + K_2(\lambda)\big(\|u\|^{q^-} + \|u\|^{q^+}\big) - \frac{1}{p^{*+}} \int_\Omega |u|^{p^{*-}}\, dx \\
&\quad + \frac{1}{p^{*+}} \int_{\Omega_<} |u|^{p^{*-}}\, dx.
\end{aligned}$$

But there exists positive constant K_3 such that

$$\frac{1}{p^{*+}} \int_{\Omega_<} |u|^{p^{*-}}\, dx \leq K_3 \quad \forall u \in E.$$

The functional $|.|_{p^{*-}} : E \to \mathbb{R}$ defined by

$$|u|_{p^{*-}} = \Big(\int_\Omega |u|^{p^{*-}}\,dx \Big)^{1/p^{*-}}$$

is a norm in E. In the finite dimensional subspace E_1 the norm $|u|_{p^{*-}}$ and $\|u\|$ are equivalent, so there exists a positive constant $K = K(E_1)$ such that

$$\|u\| \le K|u|_{p^{*-}} \quad \forall u \in E_1.$$

So that there exists a positive constant K_4 such that

$$J_\lambda(u) \le K_1(\|u\|^{p^-} + \|u\|^{p^+}) + K_2(\lambda)\big(\|u\|^{q^-} + \|u\|^{q^+}\big) + K_3 - K_4\|u\|^{p^{*-}}, \quad \forall u \in E_1.$$

Hence

$$K_1(\|u\|^{p^-} + \|u\|^{p^+}) + K_2(\lambda)\big(\|u\|^{q^-} + \|u\|^{q^+}\big) + K_3 - K_4\|u\|^{p^{*-}} \ge 0, \quad \forall u \in S.$$

And since $p^{*-} > \max(p^+, q^+)$, we conclude that S is bounded in E. □

Lemma 3.10. *If $\{u_n\} \subset E$ is a sequence which satisfies the properties*

$$|J_\lambda(u_n)| < C_6, \tag{3.15}$$

$$dJ_\lambda(u_n) \to 0 \quad as \quad n \to \infty, \tag{3.16}$$

where C_6 is a positive constant, then $\{u_n\}$ possesses a convergent subsequence.

Proof. First we show that $\{u_n\}$ is bounded in E. If not,we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. Thus we may consider that $\|u_n\| > 1$ for any integer n. Using (3.16) it follows that there exists $N_1 > 0$ such that for any $n > N_1$ we have

$$\|dJ_\lambda(u_n)\| \le 1.$$

On the other hand, for all $n > N_1$ fixed, the application $E \ni v \to \langle dJ_\lambda(u_n), v\rangle$ is linear and continuous. The above information yield that

$$|\langle dJ_\lambda(u_n), v\rangle| \le \|dJ_\lambda(u_n)\|\|v\| \le \|v\| \quad \forall v \in E, \quad n > N_1.$$

Setting $v = u_n$, we have

$$-\|u_n\| \le \int_\Omega |\nabla u_n|^{p(x)}\,dx - \int_\Omega |u_n|^{p^*(x)}\,dx + \lambda \int_\Omega |u_n|^{q(x)}\,dx \le \|u_n\|,$$

for all $n > N_1$. We obtain

$$-\|u_n\| - \int_\Omega |\nabla u_n|^{p(x)}\,dx - \lambda \int_\Omega |u_n|^{q(x)}\,dx \le -\int_\Omega |u_n|^{p^*(x)}\,dx, \tag{3.17}$$

for all $n > N_1$. Provided that $\|u_n\| > 1$ relation (3.15) and (3.17) imply

$$\begin{aligned}
C_6 &> J_\lambda(u_n)\\
&\ge (\frac{1}{p^+} - \frac{1}{p^{*-}}) \int_\Omega |\nabla u_n|^{p(x)}\,dx + \lambda(\frac{1}{q^+} - \frac{1}{p^{*-}}) \int_\Omega |u_n|^{q(x)}\,dx - \frac{1}{p^{*-}}\|u_n\|,\\
&\ge (\frac{1}{p^+} - \frac{1}{p^{*-}})\|u_n\||^{p^-}\,dx - \frac{1}{p^{*-}}\|u_n\|.
\end{aligned}$$

Letting $n \to \infty$ we obtain a contradiction. It follows that $\{u_n\}$ is bounded in E. And we deduce that there exists a subsequence, again denoted by $\{u_n\}$, and $u \in E$ such that $\{u_n\}$ converges weakly to u in E. Now by Theorem 2.2 we have $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. The proof is complete. □

Proof of Theorem 3.2. It is clear that the functional J_λ is even and verifies $J_\lambda(0) = 0$. Lemma 3.8, lemma 3.9 and Lemma 3.10 implies that J_λ satisfies the the Mountain Pass Theorem condition. Thus we conclude that problem (3.7) has infinitely many weak solutions in E. The proof is complete. $\square$

References

[1] O. Allegue and M. Bezzarga; A multiplicity of solutions for a nonlinear degenerate problem involving a $p(x)$-Laplace type operator, *Complex Variables and Elliptic Equations*, in press.

[2] C. O. Alves and A. El Hamidi; Existence of solutions for anisotropic equations with critical exponents, *Differential and Integral Equations*, **21** (2008), 25-40.

[3] K. Ben Ali and M. Bezzarga; On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Potential Theory and Stochastics in Albac Aurel Cornea Memorial Volume*, Conference Proceedings Albac, September 4-8, 2007, Theta 2008, pp. 21-34.

[4] K. Ben Ali and K. Kefi; Mountain pass and Ekeland's principle for eigenvalue problem with variable exponent , *Complex variables and elliptic equations*, **54**, (2009), No. 8, 795-809.

[5] A. El Hamidi and J. M. Rakotoson; Compactness and quasilinear problems with critical exponents , *Diff. Int. Equ.* **18** (2005), 1201-1220.

[6] X. Fan, J. Shen and D. Zhao; Sobolev Embedding Theorems for Spaces $W^{k,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **262** (2001), 749-760.

[7] X. L. Fan and D. Zhao; On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263** (2001), 424-446.

[8] X. Fan, Q. Zhang and D. Zhao; Eigenvalues of p(x)-Laplacian Dirichlet problem, *J. Math. Anal. Appl.*, **302** (2005), 306-317.

[9] M. Ghergu and V. Rădulescu; *Singular Elliptic Problems: Bifurcation and Asymptotic Analysis*, Oxford Lecture Series in Mathematics and its Applications, vol. 37, Oxford University Press, New York, 2008.

[10] A. Karoui Souayah and K. Kefi; On a class of nonhomogenous quasilinear problem involving Sobolev spaces with variable exponent, *An.Şt. Univ. Ovidius Constanţa*, Vol. 18(1), 2010, 309-328.

[11] K. Kefi; Nonhomogeneous boundary value problems in Sobolev spaces with variable exponent, *International Journal of Applied Mathematical Sciences*, Vol.3, No.2 (2006), pp. 103-115.

[12] A. Kristály, V. Rădulescu and C. Varga; *Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Cambridge University Press, Cambridge, 2010.

[13] O. Kováčik and J. Rákosník; On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.* **41** (1991), 592-618.

[14] J.-L. Lions; Quelques Mthodes de Rsolution de Problèmes aux Limites Non Linaires, *Dunod Paris* (1969).

[15] J. Leray and J.-L. Lions; Quelques rsultats de Visik sur les problmes elliptiques non linaires par les mthodes de Minty-Browder, *Bull. Soc. Math. France*, Vol**93** (1965), 97-107.

[16] M. Mihăilescu and V. Rădulescu; A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proc. Roy. Soc. London Ser. A* **462** (2006), 2625-2641.

[17] M. Mihăilescu and V. Rădulescu; On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proceedings of the American Mathematical Society*, **135** (2007).

[18] M. Mihăilescu and V. Rădulescu; Existence and multiplicity of solutions for quasilinear nonhomogeneous problems: an Orlicz-Sobolev space setting, *J. Math. Anal. Appl.*, **330** (2007), 416-432.

[19] J. Musielak; *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, Vol. 1034, Springer, Berlin, 1983.

[20] P. Rabinowitz; Minimaux methods in critical point theory with applications to differential equation, Expository Lectures from the CBMS Regional Conference held at the University of Miami, *American Mathematical Society*, Providence, RI, (1984).

Olfa Allegue
Institut Préparatoire aux Etudes d'ingénieurs de Tunis, 2 Rue Jawaher Lel Nehru, 1008, Montfleury-Tunis, Tunisia
E-mail address: allegue_olfa@yahoo.fr

Asma Karoui Souayah
Institut Préparatoire aux Etudes d'ingénieurs de Tunis, 2 Rue Jawaher Lel Nehru, 1008, Montfleury-Tunis, Tunisia
E-mail address: asma.souayah@yahoo.fr