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HYERS-ULAM STABILITY FOR SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS

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Abstract. We prove the Hyers-Ulam stability of linear differential equations of second-order with boundary conditions or with initial conditions. That is, if y is an approximate solution of the differential equation $y'' + \beta(x)y = 0$ with $y(a) = y(b) = 0$, then there exists an exact solution of the differential equation, near y.

1. Introduction and preliminaries

In 1940, Ulam [17] posed the following problem concerning the stability of functional equations:

> Give conditions in order for a linear mapping near an approximately linear mapping to exist.

The problem for approximately additive mappings, on Banach spaces, was solved by Hyers [2]. The result by Hyers was generalized by Rassias [13]. Since then, the stability problems of functional equations have been extensively investigated by several mathematicians [3, 12, 13].

Alsina and Ger [1] were the first authors who investigated the Hyers-Ulam stability of a differential equation. In fact, they proved that if a differentiable function $y : I \to \mathbb{R}$ satisfies $|y'(t) - y(t)| \le \varepsilon$ for all $t \in I$, then there exists a differentiable function $g : I \to \mathbb{R}$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|y(t) - g(t)| \le 3\varepsilon$ for every $t \in I$.

The above result by Alsina and Ger was generalized by Miura, Takahasi and Choda [11], by Miura [8], also by Takahasi, Miura and Miyajima [15]. Indeed, they dealt with the Hyers-Ulam stability of the differential equation $y'(t) = \lambda y(t)$, while Alsina and Ger investigated the differential equation $y'(t) = y(t)$.

Miura et al [10] proved the Hyers-Ulam stability of the first-order linear differential equations $y'(t) + g(t)y(t) = 0$, where g(t) is a continuous function, while Jung [4] proved the Hyers-Ulam stability of differential equations of the form $\varphi(t)y'(t) = y(t)$.

Furthermore, the result of Hyers-Ulam stability for first-order linear differential equations has been generalized in [5, 6, 10, 16, 18, 19].

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Let us consider the Hyers-Ulam stability of the $y'' + \beta(x)y = 0$, it may be not stable for unbounded intervals. Indeed, for $\beta(x) = 0$, $\varepsilon = 1/4$ and $y(x) = x^2/16$ condition $-\varepsilon < y'' < -\varepsilon$ is fulfilled and the function $y_0(x) = C_1x + C_2$, for which $|y(x) - y_0(x)| = |\frac{x^2}{16} - C_1x + C_2|$ is bounded, does not exist.

The aim of this paper is to investigate the Hyers-Ulam stability of the second-order linear differential equation

$$y'' + \beta(x)y = 0 \tag{1.1}$$

with boundary conditions

$$y(a) = y(b) = 0 \tag{1.2}$$

or with initial conditions

$$y(a) = y'(a) = 0, \tag{1.3}$$

where $y \in C^2[a, b]$, $\beta(x) \in C[a, b]$, $-\infty < a < b < +\infty$.

First of all, we give the definition of Hyers-Ulam stability with boundary conditions and with initial conditions.

Definition 1.1. We say that (1.1) has the Hyers-Ulam stability with boundary conditions (1.2) if there exists a positive constant K with the following property: For every $\varepsilon > 0$, $y \in C^2[a, b]$, if

$$|y'' + \beta(x)y| \le \varepsilon,$$

and $y(a) = y(b) = 0$, then there exists some $z \in C^2[a, b]$ satisfying

$$z'' + \beta(x)z = 0$$

and $z(a) = z(b) = 0$, such that $|y(x) - z(x)| < K\varepsilon$.

Definition 1.2. We say that (1.1) has the Hyers-Ulam stability with initial conditions (1.3) if there exists a positive constant K with the following property: For every $\varepsilon > 0$, $y \in C^2[a, b]$, if

$$|y'' + \beta(x)y| \le \varepsilon,$$

and $y(a) = y'(a) = 0$, then there exists some $z \in C^2[a, b]$ satisfying

$$z'' + \beta(x)z = 0$$

and $z(a) = z'(a) = 0$, such that $|y(x) - z(x)| < K\varepsilon$.

2. Main Results

In the following theorems, we will prove the Hyers-Ulam stability with boundary conditions and with initial conditions.

Let $\beta(x) = 1$, $a = 0$, $b = 1$, then it is easy to see that for any $\varepsilon > 0$, there exists $y(t) = \frac{\varepsilon x^2}{H} - \frac{\varepsilon x}{H}$, with $H > 4$, such that $|y'' + \beta(x)y| < \varepsilon$ with $y(0) = y(1) = 0$.

Theorem 2.1. *If* $\max|\beta(x)| < 8/(b-a)^2$. *Then* (1.1) *has the Hyers-Ulam stability with boundary conditions* (1.2).

Proof. For every $\varepsilon > 0$, $y \in C^2[a, b]$, if $|y'' + \beta(x)y| \le \varepsilon$ and $y(a) = y(b) = 0$. Let $M = \max\{|y(x)| : x \in [a, b]\}$, since $y(a) = y(b) = 0$, there exists $x_0 \in (a, b)$ such that $|y(x_0)| = M$. By Taylor formula, we have

$$y(a) = y(x_0) + y'(x_0)(x_0 - a) + \frac{y''(\xi)}{2}(x_0 - a)^2,$$
$$y(b) = y(x_0) + y'(x_0)(b - x_0) + \frac{y''(\eta)}{2}(b - x_0)^2;$$

thus

$$|y''(\xi)| = \frac{2M}{(x_0 - a)^2}, \quad |y''(\eta)| = \frac{2M}{(x_0 - b)^2}$$

On the case $x_0 \in (a, \frac{a+b}{2}]$, we have

$$\frac{2M}{(x_0 - a)^2} \geq \frac{2M}{(b-a)^2/4} = \frac{8M}{(b-a)^2}$$

On the case $x_0 \in [\frac{a+b}{2}, b)$, we have

$$\frac{2M}{(x_0 - b)^2} \geq \frac{2M}{(b-a)^2/4} = \frac{8M}{(b-a)^2}.$$

So

$$\max |y''(x)| \geq \frac{8M}{(b-a)^2} = \frac{8}{(b-a)^2} \max |y(x)|.$$

Therefore,

$$\max |y(x)| \leq \frac{(b-a)^2}{8} \max |y''(x)|.$$

Thus

$$\begin{aligned} \max |y(x)| &\leq \frac{(b-a)^2}{8} [\max |y''(x) - \beta(x) y| + \max |\beta(x)| \max |y(x)|], \\ &\leq \frac{(b-a)^2}{8} \varepsilon + \frac{(b-a)^2}{8} \max |\beta(x)| \max |y(x)|]. \end{aligned}$$

Let $\eta = (b-a)^2 \max |\beta(x)|/8$, $K = (b-a)^2/\big(8(1-\eta)\big)$. Obviously, $z_0(x) = 0$ is a solution of $y'' - \beta(x) y = 0$ with the boundary conditions $y(a) = y(b) = 0$.

$$|y - z_0| \leq K\varepsilon.$$

Hence (1.1) has the Hyers-Ulam stability with boundary conditions (1.2). $\square$

Next, we consider the Hyers-Ulam stability of $y'' + \beta(x) y = 0$ in $[a, b]$ with initial conditions (1.3). For example, let $\beta(x) = 1$, $a = 0$, $b = 1$, then for any $\varepsilon > 0$, there exists $y(t) = \frac{\varepsilon x^2}{H}$ with $H > 3$, such that $|y'' + \beta(x) y| < \varepsilon$ with $y(0) = y'(0) = 0$.

Theorem 2.2. *If* $\max |\beta(x)| < 2/(b-a)^2$. *Then* (1.1) *has the Hyers-Ulam stability with initial conditions* (1.3).

Proof. For every $\varepsilon > 0$, $y \in C^2[a, b]$, if $|y'' + \beta(x) y| \leq \varepsilon$ and $y(a) = y'(a) = 0$. By Taylor formula, we have

$$y(x) = y(a) + y'(a)(x-a) + \frac{y''(\xi)}{2}(x-a)^2.$$

Thus

$$|y(x)| = |\frac{y''(\xi)}{2}(x-a)^2| \leq \max |y''(x)| \frac{(b-a)^2}{2};$$

so, we obtain

$$\begin{aligned} \max |y(x)| &\leq \frac{(b-a)^2}{2} [\max |y''(x) - \beta(x) y| + \max |\beta(x)| \max |y(x)|] \\ &\leq \frac{(b-a)^2}{2} \varepsilon + \frac{(b-a)^2}{2} \max |\beta(x)| \max |y(x)|]. \end{aligned}$$

Let $\eta = (b-a)^2 \max|\beta(x)|/2$, $K = (b-a)^2/\big(2(1-\eta)\big)$. It is easy to see that $z_0(x) = 0$ is a solution of $y'' - \beta(x)y = 0$ with the initial conditions $y(a) = y'(a) = 0$.

$$|y - z_0| \le K\varepsilon.$$

Hence (1.1) has the Hyers-Ulam stability with initial conditions (1.3). □

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