SIMILARITY SOLUTIONS TO EVOLUTION EQUATIONS IN ONE-DIMENSIONAL INTERFACES

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ABSTRACT. In this note, we study the evolution equation
$$\partial_t h = -\nu \partial_x^2 h - K \partial_x^4 h + \lambda_1 (\partial_x h)^2 - \lambda_2 \partial_x^2 (\partial_x h)^2.$$
which was introduced by Muñoz-Garcia [9] in the context of erosion by ion beam sputtering. We obtain an analytic solution that has the similarity form, which is used in obtaining the coarsening behavior. This solution has amplitude and wavelength that increase like $\ln(t)$ and $\sqrt{t \ln(t)}$, respectively.

1. RESULTS

Muñoz-Garcia, Cuerno and Castro [9] derived and analyzed numerically a one-dimensional class of unstable surface growth models in the context of erosion by ion beam sputtering. They showed that the surface morphology is described by the "interfacial height" equation

$$\partial_t h = -\nu \partial_x^2 h - K \partial_x^4 h + \lambda_1 (\partial_x h)^2 - \lambda_2 \partial_x^2 (\partial_x h)^2, \tag{1.1}$$

or, after some rescaling,

$$\partial_t h = -\partial_x^2 h - \partial_x^4 h + (\partial_x h)^2 - r \partial_x^2 (\partial_x h)^2, \tag{1.2}$$

where $\nu > 0$, $K > 0$, λ_1 and λ_2 are real parameters and $r = \nu \lambda_2 / (K \lambda_1)$. The parameters λ_1 and λ_2 have the same sign; i.e., $r > 0$, for mathematical well-posedness and are positive.

The above equations are referred to as the mixed Kuramoto-Sivashinsky equation. Equation (1.1) was also proposed for amorphous thin films in the presence of potential density variations [14]. This equation is also referred to as the snow equation. Under some conditions on the parameters, (1.1) models a snow surface growth based on solar radiation [15].

The first linear term on the right-hand side of (1.1) is the instability term. This term is balanced by the classical stabilizing linear term $\partial_x^4 h$. The nonlinear term $\partial_x^2 (\partial_x h)^2$ is responsible for the coarsening dynamics; i.e., the amplitude and wavelength, or the lateral width, increase with time without bound (see below), while the nonlinear term $(\partial_x h)^2$ interrupts the coarsening process [9, 15]. For $\lambda_2 = 0$, or $r = 0$ (1.1) reduces to the famous Kuaramoto-Sivashinsky equation

which is known to produce a spatio-temporal chaos and no coarsening. For $\lambda_1 = 0$ ($r \to \infty$), (1.1) reads

$$\partial_t h = -\nu \partial_x^2 h - K \partial_x^4 h - \lambda_2 \partial_x^2 (\partial_x h)^2. \tag{1.3}$$

Such an equation, referred to as the Conserved Kuaramoto-Sivashinsky (CKS) equation, appears in different physical contexts. Recently, the CKS equation has been derived and used by Frisch and Verga [2] to study the step meandering instability on a vicinal surface. The authors obtained a $l(t) = t^{1/2}$ scaling, for large time (uninterrupted coarsening), and also demonstrated a linear time growth of the characteristic meander amplitude ($A(t) = t$) (here the wavelength $l(t)$ and the amplitude $A(t)$ are the mean lateral distance between two consecutive local minima and mean vertical distance from a local minimum to the next local maximum, respectively). Moreover it is shown that the solution of (1.3) is a periodic juxtaposition of parabolas of the form (see also [13])

$$h(x) = a - \frac{4a}{b^2} x^2, \tag{1.4}$$

where a and b are real parameters. In fact, (1.4) is a stationary solution to the CKS equation for $a/b^2 = \nu/(16\lambda_2)$; $h(x) = a - \left(\nu x^2/(16\lambda_2)\right)$ for any a, irrespective of K [2].

Recently, using (1.4), a class of exact solutions to (1.3) are derived by Guedda et al. [4]. To be more precise the authors obtained a family of exact solutions to the CKS equation having the form

$$h(x,t) = -\frac{\nu}{4\lambda_2} x^2, \quad \text{for } |x| \le y(t), \tag{1.5}$$

and zero elsewhere, where the parabola edge $y = l(t)/2$ satisfies a nonlinear ordinary differential equation. Different scenarios are found: the lateral coarsening $l(t)$ (i) grows with time like $\sqrt{t}$, (ii) disappears, or (iii) does not change for all time, depending on the initial lateral coarsening $l(0)$.

Let us return to (1.2) which may present a transition or interpolation between the chaotic behavior of the KS equation (as $r \to 0$) and the coarsening behavior of the CKS equation (as $r \to \infty$). From a mathematical point of view, since (1.1) can be considered as a nonlinear perturbation of the CKS equation, we are concerned wit the effect of the KPZ term $\lambda_1 (\partial_x h)^2$ on the coarsening property of (1.1), or (1.2). To be more precise, we will present an exact non stationary solution to (1.2) with a logarithmic growth of the amplitude and unbounded wavelength (perpetual coarsening) which indicates, in particular, that equation (1.2) may have different behaviors.

As in [9] we take $r = 50$. The main line of argument used here is similar to the one used in [2] and [4]. In fact we are looking for an exact non stationary solution of (1.2) having the form

$$h(x,t) = A(t)\left[1 - \frac{4}{l^2(t)} x^2\right], \tag{1.6}$$

for $|x| \le l(t)/2$ and zero elsewhere, presenting one individual cell, where A and l are unknown functions to be determined explicitly. The form of the relation (1.6) is instructive. Equation (1.2) may have a cell or a mound centered at the origin with an amplitude or a height scale $A(t)$ and a lateral width or a linear scale $l(t)$.

Ansatz (1.6) is typical. This means that the ratio $h(x,t)/A(t)$ depends only on a single variable $\eta = x/l(t)$;

$$\frac{h(x,t)}{A(t)} = \varphi(\eta), \quad \eta = x/l(t). \tag{1.7}$$

A class of solutions (1.7) are called invariant or similarity solutions and are commonly used for extracting the coarsening behavior from a nonlinear continuum equation (see for example [6, 7, 10, 11]).

For a useful comparison, we first consider equation (1.2) where the KPZ term is absent (i.e. CKS equation);

$$\partial_t h = -\partial_x^2 h - \partial_x^4 h - r\partial_x^2 (\partial_x h)^2. \tag{1.8}$$

If in (1.6) we set

$$A(t) = 1 + \frac{t}{400}, \quad l(t) = 2\sqrt{400 + t},$$

we obtain an exact similarity solution to (1.8) with $r = 50$. Clearly $A(t) \sim t$ and $l(t) \sim \sqrt{t}$, as $t \to \infty$, which agree with the result of [2].

Next, we study equation (1.2). Defining

$$A(t) = 7 - \frac{640q^2 t}{1 + 16qt} + \frac{1}{2}\ln(1 + 16qt), \tag{1.9}$$

and

$$l(t) = \sqrt{1200 + (112 - 640q)t + \frac{1}{2q}(1 + 16qt)\ln(1 + 16qt)}, \tag{1.10}$$

where $q = 7/1200$, it can be verified that (1.6)-(1.10) is an exact solution to (1.2), where $r = 50$, which begins with the cell defined by the truncated parabola

$$h(x,0) = 7[1 - \frac{1}{300}x^2], \tag{1.11}$$

for $|x| \leq 10\sqrt{3}$, and zero elsewhere. The initial amplitude and lateral width are $A(0) = 7$ and $l(0) = 20\sqrt{3}$.

We may deduce from this that equation (1.1), or (1.2), has solutions with amplitude and wavelength that tend to infinity with t, and behave like, as $t \to \infty$,

$$A(t) \sim \frac{1}{2}\ln(t), \quad l(t) \sim 2\sqrt{2t\ln(t)}. \tag{1.12}$$

Interestingly, the above explicit solution reveals that the structure may undergo logarithm-law coarsening. Similar behavior is obtained for the one dimensional convective Cahn-Hilliard equation [3]. The dynamic is controlled not only by the KPZ term, but also by the Conserved KPZ term $\partial_x^2 (\partial_x h)^2$ (recall that the KS equation produces a chaotic behavior without coarsening and the CKS equation exhibits a coarsening behavior with the power-law $l(t) \sim \sqrt{t}$). Note that $\max |\partial_x h|^2$ and $\max |\partial_x^2 (\partial_x h)^2|$ go to zero as t tends to infinity and that

$$\frac{\max |\partial_x h|^2}{\max |\partial_x^2 (\partial_x h)^2|} = l^2(t)/8, \tag{1.13}$$

which approaches infinity with t. This indicates that $\partial_x^2 (\partial_x h)^2$ is small compared to $(\partial_x h)^2$, for large t.

In summary, a solution which begins with the cell defined by (1.11) is found in a closed form. This solution displays unbounded coarsening. An important and interesting task for future investigation is to exhibit, for large r, different behaviors

(if any) and to understand their dependence on the initial lateral coarsening or initial amplitude.

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