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NONHOMOGENEOUS ELLIPTIC EQUATIONS WITH DECAYING CYLINDRICAL POTENTIAL AND CRITICAL EXPONENT

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Abstract. We prove the existence and multiplicity of solutions for a nonhomogeneous elliptic equation involving decaying cylindrical potential and critical exponent.

1. Introduction

In this article, we consider the problem

$$\begin{gathered} -\operatorname{div}(|y|^{-2a}\nabla u) - \mu |y|^{-2(a+1)}u = h|y|^{-2_* b}|u|^{2_*-2}u + \lambda g \quad \text{in } \mathbb{R}^N, \quad y \neq 0 \\ u \in \mathcal{D}_0^{1,2}, \end{gathered} \tag{1.1}$$

where each point in $\mathbb{R}^N$ is written as a pair $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, k and N are integers such that $N \geq 3$ and k belongs to $\{1,\dots,N\}$; $-\infty < a < (k-2)/2$; $a \leq b < a+1$; $2_* = 2N/(N-2+2(b-a))$; $-\infty < \mu < \bar{\mu}_{a,k} := ((k-2(a+1))/2)^2$; $g \in \mathcal{H}'_\mu \cap C(\mathbb{R}^N)$; h is a bounded positive function on $\mathbb{R}^k$ and λ is real parameter. Here $\mathcal{H}'_\mu$ is the dual of $\mathcal{H}_\mu$, where $\mathcal{H}_\mu$ and $\mathcal{D}_0^{1,2}$ will be defined later.

Some results are already available for (1.1) in the case $k = N$; see for example [10, 11] and the references therein. Wang and Zhou [10] proved that there exist at least two solutions for (1.1) with $a = 0$, $0 < \mu \leq \bar{\mu}_{0,N} = ((N-2)/2)^2$ and $h \equiv 1$, under certain conditions on g. Bouchekif and Matallah [2] showed the existence of two solutions of (1.1) under certain conditions on functions g and h, when $0 < \mu \leq \bar{\mu}_{0,N}$, $\lambda \in (0, \Lambda_*)$, $-\infty < a < (N-2)/2$ and $a \leq b < a+1$, with Λ_* a positive constant.

Concerning existence results in the case $k < N$, we cite [6, 7] and the references therein. Musina [7] considered (1.1) with $-a/2$ instead of a and $\lambda = 0$, also (1.1) with $a = 0$, $b = 0$, $\lambda = 0$, with $h \equiv 1$ and $a \neq 2-k$. She established the existence of a ground state solution when $2 < k \leq N$ and $0 < \mu < \bar{\mu}_{a,k} = ((k-2+a)/2)^2$ for (1.1) with $-a/2$ instead of a and $\lambda = 0$. She also showed that (1.1) with $a = 0$, $b = 0$, $\lambda = 0$ does not admit ground state solutions. Badiale et al [1] studied (1.1) with $a = 0$, $b = 0$, $\lambda = 0$ and $h \equiv 1$. They proved the existence of at least a nonzero nonnegative weak solution u, satisfying $u(y,z) = u(|y|,z)$ when $2 \leq k < N$ and

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$\mu < 0$. Bouchekif and El Mokhtar [3] proved that (1.1) with $a = 0$, $b = 0$ admits two distinct solutions when $2 < k \le N$, $b = N - p(N-2)/2$ with $p \in (2, 2^*]$, $\mu < \bar{\mu}_{0,k}$, and $\lambda \in (0, \Lambda_*)$ where Λ_* is a positive constant. Terracini [9] proved that there are no positive solutions of (1.1) with $b = 0$, $\lambda = 0$ when $a \neq 0$, $h \equiv 1$ and $\mu < 0$. The regular problem corresponding to $a = b = \mu = 0$ and $h \equiv 1$ has been considered on a regular bounded domain Ω by Tarantello [8]. She proved that for g in $H^{-1}(\Omega)$, the dual of $H_0^1(\Omega)$, not identically zero and satisfying a suitable condition, the problem considered admits two distinct solutions.

Before formulating our results, we give some definitions and notation. We denote by $\mathcal{D}_0^{1,2} = \mathcal{D}_0^{1,2}(\mathbb{R}^k \backslash \{0\} \times \mathbb{R}^{N-k})$ and $\mathcal{H}_\mu = \mathcal{H}_\mu(\mathbb{R}^k \backslash \{0\} \times \mathbb{R}^{N-k})$, the closure of $C_0^\infty(\mathbb{R}^k \backslash \{0\} \times \mathbb{R}^{N-k})$ with respect to the norms

$$\|u\|_{a,0} = \Big(\int_{\mathbb{R}^N} |y|^{-2a} |\nabla u|^2 \, dx \Big)^{1/2}$$

and

$$\|u\|_{a,\mu} = \Big(\int_{\mathbb{R}^N} (|y|^{-2a} |\nabla u|^2 - \mu |y|^{-2(a+1)} |u|^2) \, dx \Big)^{1/2},$$

respectively, with $\mu < \bar{\mu}_{a,k} = ((k - 2(a+1))/2)^2$ for $k \neq 2(a+1)$.

From the Hardy-Sobolev-Maz'ya inequality, it is easy to see that the norm $\|u\|_{a,\mu}$ is equivalent to $\|u\|_{a,0}$.

Since our approach is variational, we define the functional $I_{a,b,\lambda,\mu}$ on $\mathcal{H}_\mu$ by

$$I(u) := I_{a,b,\lambda,\mu}(u) := (1/2)\|u\|_{a,\mu}^2 - (1/2_*) \int_{\mathbb{R}^N} h |y|^{-2_* b} |u|^{2_*} \, dx - \lambda \int_{\mathbb{R}^N} g u \, dx.$$

We say that $u \in \mathcal{H}_\mu$ is a weak solution of (1.1) if it satisfies

$$\begin{aligned} \langle I'(u), v \rangle &= \int_{\mathbb{R}^N} (|y|^{-2a} \nabla u \nabla v - \mu |y|^{-2(a+1)} uv - h|y|^{-2_* b} |u|^{2_* - 2} uv - \lambda g v) \, dx \\ &= 0, \quad \text{for } v \in \mathcal{H}_\mu. \end{aligned}$$

Here $\langle \cdot, \cdot \rangle$ denotes the product in the duality $\mathcal{H}'_\mu, \mathcal{H}_\mu$.

Throughout this work, we consider the following assumptions:

- (G) There exist $\nu_0 > 0$ and $\delta_0 > 0$ such that $g(x) \ge \nu_0$, for all x in $B(0, 2\delta_0)$;
- (H) $\lim_{|y| \to 0} h(y) = \lim_{|y| \to \infty} h(y) = h_0 > 0$, $h(y) \ge h_0$, $y \in \mathbb{R}^k$.

Here, $B(a, r)$ denotes the ball centered at a with radius r.

Under some conditions on the coefficients of (1.1), we split $\mathcal{N}$ in two disjoint subsets $\mathcal{N}^+$ and $\mathcal{N}^-$, thus we consider the minimization problems on $\mathcal{N}^+$ and $\mathcal{N}^-$.

Remark 1.1. Note that all solutions of (1.1) are nontrivial.

We shall state our main results.

Theorem 1.2. *Assume that* $3 \le k \le N$, $-1 < a < (k-2)/2$, $0 \le \mu < \bar{\mu}_{a,k}$, *and* (G) *holds, then there exists* $\Lambda_1 > 0$ *such that the* (1.1) *has at least one nontrivial solution on* $\mathcal{H}_\mu$ *for all* $\lambda \in (0, \Lambda_1)$.

Theorem 1.3. *In addition to the assumptions of the Theorem 1.2, if* (H) *holds, then there exists* $\Lambda_2 > 0$ *such that* (1.1) *has at least two nontrivial solutions on* $\mathcal{H}_\mu$ *for all* $\lambda \in (0, \Lambda_2)$.

This article is organized as follows. In Section 2, we give some preliminaries. Section 3 and 4 are devoted to the proofs of Theorems 1.2 and 1.3.

2. Preliminaries

We list here a few integral inequalities. The first one that we need is the Hardy inequality with cylindrical weights [7]. It states that

$$\bar{\mu}_{a,k} \int_{\mathbb{R}^N} |y|^{-2(a+1)} v^2 \, dx \leq \int_{\mathbb{R}^N} |y|^{-2a} |\nabla v|^2 \, dx, \quad \text{for all } v \in \mathcal{H}_\mu,$$

The starting point for studying (1.1) is the Hardy-Sobolev-Maz'ya inequality that is particular to the cylindrical case $k < N$ and that was proved by Maz'ya in [6]. It states that there exists positive constant $C_{a,2_*}$ such that

$$C_{a,2_*} \Big(\int_{\mathbb{R}^N} |y|^{-2_* b} |v|^{2_*} \, dx \Big)^{2/2_*} \leq \int_{\mathbb{R}^N} (|y|^{-2a} |\nabla v|^2 - \mu |y|^{-2(a+1)} v^2) \, dx,$$

for any $v \in C_c^\infty((\mathbb{R}^k \backslash \{0\}) \times \mathbb{R}^{N-k})$.

Proposition 2.1 ([6]). *The value*

$$S_{\mu,2_*} = S_{\mu,2_*}(k, 2_*) := \inf_{v \in \mathcal{H}_\mu \backslash \{0\}} \frac{\int_{\mathbb{R}^N} (|y|^{-2a} |\nabla v|^2 - \mu |y|^{-2(a+1)} v^2) \, dx}{(\int_{\mathbb{R}^N} |y|^{-2_* b} |v|^{2_*} \, dx)^{2/2_*}} \tag{2.1}$$

is achieved on $\mathcal{H}_\mu$, for $2 \leq k < N$ and $\mu \leq \bar{\mu}_{a,k}$.

Definition 2.2. Let $c \in \mathbb{R}$, E be a Banach space and $I \in C^1(E, \mathbb{R})$.

(i) $(u_n)_n$ is a Palais-Smale sequence at level c (in short $(PS)_c$) in E for I if $I(u_n) = c + o_n(1)$ and $I'(u_n) = o_n(1)$, where $o_n(1) \to 0$ as $n \to \infty$.

(ii) We say that I satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence in E for I has a convergent subsequence.

2.1. **Nehari manifold.** It is well known that I is of class C^1 in $\mathcal{H}_\mu$ and the solutions of (1.1) are the critical points of I which is not bounded below on $\mathcal{H}_\mu$. Consider the Nehari manifold

$$\mathcal{N} = \{u \in \mathcal{H}_\mu \backslash \{0\} : \langle I'(u), u \rangle = 0\},$$

Thus, $u \in \mathcal{N}$ if and only if

$$\|u\|_{a,\mu}^2 - \int_{\mathbb{R}^N} h |y|^{-2_* b} |u|^{2_*} \, dx - \lambda \int_{\mathbb{R}^N} g u \, dx = 0. \tag{2.2}$$

Note that $\mathcal{N}$ contains every nontrivial solution of (1.1). Moreover, we have the following results.

Lemma 2.3. *The functional I is coercive and bounded from below on $\mathcal{N}$.*

Proof. If $u \in \mathcal{N}$, then by ((2.2) and the Hölder inequality, we deduce that

$$\begin{aligned} I(u) &= ((2_* - 2)/2_* 2) \|u\|_{a,\mu}^2 - \lambda (1 - (1/2_*)) \int_{\mathbb{R}^N} g u \, dx \\ &\geq ((2_* - 2)/2_* 2) \|u\|_{a,\mu}^2 - \lambda (1 - (1/2_*)) \|u\|_{a,\mu} \|g\|_{\mathcal{H}'_\mu} \\ &\geq -\lambda^2 C_0, \end{aligned} \tag{2.3}$$

where

$$C_0 := C_0(\|g\|_{\mathcal{H}'_\mu}) = [(2_* - 1)^2 / 2_* 2 (2_* - 2)] \|g\|_{\mathcal{H}'_\mu}^2 > 0.$$

Thus, I is coercive and bounded from below on $\mathcal{N}$. $\square$

Define

$$\Psi_\lambda(u) = \langle I'(u), u\rangle.$$

Then, for $u \in \mathcal{N}$,

$$\begin{aligned}\langle \Psi'_\lambda(u), u\rangle &= 2\|u\|^2_{a,\mu} - 2_* \int_{\mathbb{R}^N} h|y|^{-2_* b}|u|^{2_*}\,dx - \lambda \int_{\mathbb{R}^N} gu\,dx \\ &= \|u\|^2_{a,\mu} - (2_* - 1)\int_{\mathbb{R}^N} h|y|^{-2_* b}|u|^{2_*}\,dx \\ &= \lambda(2_* - 1)\int_{\mathbb{R}^N} gu\,dx - (2_* - 2)\|u\|^2_{a,\mu}.\end{aligned} \tag{2.4}$$

Now, we split $\mathcal{N}$ in three parts:

$$\mathcal{N}^+ = \{u \in \mathcal{N} : \langle \Psi'_\lambda(u), u\rangle > 0\}, \quad \mathcal{N}^0 = \{u \in \mathcal{N}\langle \Psi'_\lambda(u), u\rangle = 0\},$$
$$\mathcal{N}^- = \{u \in \mathcal{N} : \langle \Psi'_\lambda(u), u\rangle < 0\}$$

We have the following results.

Lemma 2.4. *Suppose that there exists a local minimizer u_0 for I on $\mathcal{N}$ and $u_0 \notin \mathcal{N}^0$. Then, $I'(u_0) = 0$ in $\mathcal{H}'_\mu$.*

Proof. If u_0 is a local minimizer for I on $\mathcal{N}$, then there exists $\theta \in \mathbb{R}$ such that

$$\langle I'(u_0), \varphi\rangle = \theta\langle \Psi'_\lambda(u_0), \varphi\rangle$$

for any $\varphi \in \mathcal{H}_\mu$.

If $\theta = 0$, then the lemma is proved. If not, taking $\varphi \equiv u_0$ and using the assumption $u_0 \in \mathcal{N}$, we deduce

$$0 = \langle I'(u_0), u_0\rangle = \theta\langle \Psi'_\lambda(u_0), u_0\rangle.$$

Thus

$$\langle \Psi'_\lambda(u_0), u_0\rangle = 0,$$

which contradicts that $u_0 \notin \mathcal{N}^0$. □

Let

$$\Lambda_1 := (2_* - 2)(2_* - 1)^{-(2_*-1)/(2_*-2)}[(h_0)^{-1}S_{\mu,2_*}]^{2_*/2(2_*-2)}\|g\|^{-1}_{\mathcal{H}'_\mu}. \tag{2.5}$$

Lemma 2.5. *We have $\mathcal{N}^0 = \emptyset$ for all $\lambda \in (0, \Lambda_1)$.*

Proof. Let us reason by contradiction. Suppose $\mathcal{N}^0 \neq \emptyset$ for some $\lambda \in (0, \Lambda_1)$. Then, by (2.4) and for $u \in \mathcal{N}^0$, we have

$$\begin{aligned}\|u\|^2_{a,\mu} &= (2_* - 1)\int_{\mathbb{R}^N} h|y|^{-2_* b}|u|^{2_*}\,dx \\ &= \lambda((2_* - 1)/(2_* - 2))\int_{\mathbb{R}^N} gu\,dx.\end{aligned} \tag{2.6}$$

Moreover, by (G), the Hölder inequality and the Sobolev embedding theorem, we obtain

$$\Big[\big((h_0)^{-1}S_{\mu,2_*}\big)^{2_*/2}/(2_*-1)\Big]^{1/(2_*-2)} \le \|u\|_{a,\mu} \le \big[\lambda\big((2_*-1)\|g\|_{\mathcal{H}'_\mu}/(2_*-2)\big)\big]. \tag{2.7}$$

This implies that $\lambda \ge \Lambda_1$, which is a contradiction to $\lambda \in (0, \Lambda_1)$. □

Thus $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$ for $\lambda \in (0, \Lambda_1)$. Define

$$c := \inf_{u\in\mathcal{N}} I(u), \quad c^+ := \inf_{u\in\mathcal{N}^+} I(u), \quad c^- := \inf_{u\in\mathcal{N}^-} I(u).$$

We need also the following Lemma.

Lemma 2.6. *(i) If $\lambda \in (0, \Lambda_1)$, then $c \le c^+ < 0$.*
(ii) If $\lambda \in (0, (1/2)\Lambda_1)$, then $c^- > C_1$, where

$$\begin{aligned} C_1 = C_1(\lambda, S_{\mu,2_*}\|g\|_{\mathcal{H}'_\mu}) = \big((2_* - 2)/2_*2\big)(2_* - 1)^{2/(2_*-2)}(S_{\mu,2_*})^{2_*/(2_*-2)} \\ - \lambda(1 - (1/2_*))(2_* - 1)^{2/(2_*-2)}\|g\|_{\mathcal{H}'_\mu}. \end{aligned}$$

Proof. (i) Let $u \in \mathcal{N}^+$. By (2.4),

$$[1/(2_* - 1)]\|u\|^2_{a,\mu} > \int_{\mathbb{R}^N} h|y|^{-2_*b}|u|^{2_*}\,dx$$

and so

$$\begin{aligned} I(u) &= (-1/2)\|u\|^2_{a,\mu} + (1 - (1/2_*))\int_{\mathbb{R}^N} h|y|^{-2_*b}|u|^{2_*}\,dx \\ &< [(-1/2) + (1 - (1/2_*))(1/(2_* - 1))]\|u\|^2_{a,\mu} \\ &= -((2_* - 2)/2_*2)\|u\|^2_{a,\mu}; \end{aligned}$$

we conclude that $c \le c^+ < 0$.
(ii) Let $u \in \mathcal{N}^-$. By (2.4),

$$[1/(2_* - 1)]\|u\|^2_{a,\mu} < \int_{\mathbb{R}^N} h|y|^{-2_*b}|u|^{2_*}\,dx.$$

Moreover, by Sobolev embedding theorem, we have

$$\int_{\mathbb{R}^N} h|y|^{-2_*b}|u|^{2_*}\,dx \le (S_{\mu,2_*})^{-2_*/2}\|u\|^{2_*}_{a,\mu}.$$

This implies

$$\|u\|_{a,\mu} > [(2_* - 1)]^{-1/(2_*-2)}(S_{\mu,2_*})^{2_*/2(2_*-2)}, \quad \text{for all } u \in \mathcal{N}^-.$$

By (2.3),

$$I(u) \ge ((2_* - 2)/2_*2)\|u\|^2_{a,\mu} - \lambda(1 - (1/2_*))\|u\|_{a,\mu}\|g\|_{\mathcal{H}'_\mu}.$$

Thus, for all $\lambda \in (0, (1/2)\Lambda_1)$, we have $I(u) \ge C_1$. □

For each $u \in \mathcal{H}_\mu$, we write

$$t_m := t_{\max}(u) = \Big[\frac{\|u\|_{a,\mu}}{(2_* - 1)\int_{\mathbb{R}^N} h|y|^{-2_*b}|u|^{2_*}\,dx}\Big]^{1/(2_*-2)} > 0.$$

Lemma 2.7. *Let $\lambda \in (0, \Lambda_1)$. For each $u \in \mathcal{H}_\mu$, one has the following:*
(i) If $\int_{\mathbb{R}^N} g(x)u\,dx \le 0$, then there exists a unique $t^- > t_m$ such that $t^-u \in \mathcal{N}^-$ and

$$I(t^-u) = \sup_{t\ge0} I(tu).$$

(ii) If $\int_{\mathbb{R}^N} g(x)u\,dx > 0$, then there exist unique t^+ and t^- such that $0 < t^+ < t_m < t^-$, $t^+u \in \mathcal{N}^+$, $t^-u \in \mathcal{N}^-$,

$$I(t^+u) = \inf_{0\le t\le t_m} I(tu) \text{ and } I(t^-u) = \sup_{t\ge0} I(tu).$$

The proof of the above lemma follows from a proof in [5], with minor modifications.

3. Proof of Theorem 1.2

For the proof we need the following results.

Proposition 3.1 ([5]). *(i) If $\lambda \in (0, \Lambda_1)$, then there exists a minimizing sequence $(u_n)_n$ in $\mathcal{N}$ such that*

$$I(u_n) = c + o_n(1), \quad I'(u_n) = o_n(1) \quad \text{in } \mathcal{H}'_\mu, \tag{3.1}$$

where $o_n(1)$ tends to 0 as n tends to ∞.

(ii) if $\lambda \in (0, (1/2)\Lambda_1)$, then there exists a minimizing sequence $(u_n)_n$ in $\mathcal{N}^-$ such that

$$I(u_n) = c^- + o_n(1), \quad I'(u_n) = o_n(1) \quad \text{in } \mathcal{H}'_\mu.$$

Now, taking as a starting point the work of Tarantello [8], we establish the existence of a local minimum for I on $\mathcal{N}^+$.

Proposition 3.2. *If $\lambda \in (0, \Lambda_1)$, then I has a minimizer $u_1 \in \mathcal{N}^+$ and it satisfies*

(i) *$I(u_1) = c = c^+ < 0$,*
(ii) *u_1 is a solution of (1.1).*

Proof. (i) By Lemma 2.3, I is coercive and bounded below on $\mathcal{N}$. We can assume that there exists $u_1 \in \mathcal{H}_\mu$ such that

$$\begin{gathered} u_n \rightharpoonup u_1 \quad \text{weakly in } \mathcal{H}_\mu, \\ u_n \rightharpoonup u_1 \quad \text{weakly in } L^{2_*}(\mathbb{R}^N, |y|^{-2_* b}), \\ u_n \to u_1 \quad \text{a.e in } \mathbb{R}^N. \end{gathered} \tag{3.2}$$

Thus, by (3.1) and (3.2), u_1 is a weak solution of (1.1) since $c < 0$ and $I(0) = 0$. Now, we show that u_n converges to u_1 strongly in $\mathcal{H}_\mu$. Suppose otherwise. Then $\|u_1\|_{a,\mu} < \liminf_{n\to\infty} \|u_n\|_{a,\mu}$ and we obtain

$$\begin{aligned} c \le I(u_1) &= ((2_* - 2)/2_* 2)\|u_1\|^2_{a,\mu} - \lambda(1 - (1/2_*)) \int_{\mathbb{R}^N} g u_1 \, dx \\ &< \liminf_{n\to\infty} I(u_n) = c. \end{aligned}$$

We have a contradiction. Therefore, u_n converges to u_1 strongly in $\mathcal{H}_\mu$. Moreover, we have $u_1 \in \mathcal{N}^+$. If not, then by Lemma 2.7, there are two numbers t_0^+ and t_0^-, uniquely defined so that $t_0^+ u_1 \in \mathcal{N}^+$ and $t_0^- u_1 \in \mathcal{N}^-$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt} I(tu_1)\Big|_t = t_0^+ = 0, \quad \frac{d^2}{dt^2} I(tu_1)\Big| t = t_0^+ > 0,$$

there exists $t_0^+ < t^- \le t_0^-$ such that $I(t_0^+ u_1) < I(t^- u_1)$. By Lemma 2.7,

$$I(t_0^+ u_1) < I(t^- u_1) < I(t_0^- u_1) = I(u_1),$$

which is a contradiction. □

4. Proof of Theorem 1.3

In this section, we establish the existence of a second solution of (1.1). For this, we require the following Lemmas, with C_0 is given in (2.3).

Lemma 4.1. *Assume that* (G) *holds and let* $(u_n)_n \subset \mathcal{H}_\mu$ *be a* $(PS)_c$ *sequence for* I *for some* $c \in \mathbb{R}$ *with* $u_n \rightharpoonup u$ *in* $\mathcal{H}_\mu$. *Then,* $I'(u) = 0$ *and*

$$I(u) \geq -C_0\lambda^2.$$

Proof. It is easy to prove that $I'(u) = 0$, which implies that $\langle I'(u), u\rangle = 0$, and

$$\int_{\mathbb{R}^N} h|y|^{-2_* b}|u|^{2_*}\,dx = \|u\|^2_{a,\mu} - \lambda \int_{\mathbb{R}^N} gu\,dx.$$

Therefore,

$$I(u) = ((2_* - 2)/2_* 2)\|u\|^2_{a,\mu} - \lambda(1 - (1/2_*)) \int_{\mathbb{R}^N} gu\,dx.$$

Using (2.3), we obtain

$$I(u) \geq -C_0\lambda^2.$$

□

Lemma 4.2. *Assume that* (G) *holds and for any* $(PS)_c$ *sequence with* c *is a real number such that* $c < c^*_\lambda$. *Then, there exists a subsequence which converges strongly. Here* $c^*_\lambda := ((2_* - 2)/2_* 2)(h_0)^{-2/(2_*-2)}(S_{\mu,2_*})^{2_*/(2_*-2)} - C_0\lambda^2$.

Proof. Using standard arguments, we get that $(u_n)_n$ is bounded in $\mathcal{H}_\mu$. Thus, there exist a subsequence of $(u_n)_n$ which we still denote by $(u_n)_n$ and $u \in \mathcal{H}_\mu$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } \mathcal{H}_\mu,$$
$$u_n \rightharpoonup u \quad \text{weakly in } L^{2_*}(\mathbb{R}^N, |y|^{-2_* b}).$$
$$u_n \to u \quad \text{a.e in } \mathbb{R}^N.$$

Then, u is a weak solution of (1.1). Let $v_n = u_n - u$, then by Brézis-Lieb [4], we obtain

$$\|v_n\|^2_{a,\mu} = \|u_n\|^2_{a,\mu} - \|u\|^2_{a,\mu} + o_n(1) \tag{4.1}$$

and

$$\int_{\mathbb{R}^N} h|y|^{-2_* b}|v_n|^{2_*}\,dx = \int_{\mathbb{R}^N} h|y|^{-2_* b}|u_n|^{2_*}\,dx - \int_{\mathbb{R}^N} h|y|^{-2_* b}|u|^{2_*}\,dx + o_n(1). \tag{4.2}$$

On the other hand, by using the assumption (H), we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} h(x)|y|^{-2_* b}|v_n|^{2_*}\,dx = h_0 \lim_{n\to\infty} \int_{\mathbb{R}^N} |y|^{-2_* b}|v_n|^{2_*}\,dx. \tag{4.3}$$

Since $I(u_n) = c + o_n(1)$, $I'(u_n) = o_n(1)$ and by (4.1), (4.2), and (4.3) we deduce that

$$\begin{gathered}(1/2)\|v_n\|^2_{a,\mu} - (1/2_*) \int_{\mathbb{R}^N} h|y|^{-2_* b}|v_n|^{2_*}\,dx = c - I(u) + o_n(1), \\ \|v_n\|^2_{a,\mu} - \int_{\mathbb{R}^N} h|y|^{-2_* b}|v_n|^{2_*}\,dx = o_n(1).\end{gathered} \tag{4.4}$$

Hence, we may assume that

$$\|v_n\|^2_{a,\mu} \to l, \quad \int_{\mathbb{R}^N} h|y|^{-2_* b}|v_n|^{2_*}\,dx \to l. \tag{4.5}$$

Sobolev inequality gives $\|v_n\|_{a,\mu}^2 \geq (S_{\mu,2_*}) \int_{\mathbb{R}^N} h|y|^{-2_*b}|v_n|^{2_*}\,dx$. Combining this inequality with (4.5), we obtain

$$l \geq S_{\mu,2_*}(l^{-1}h_0)^{-2/2_*}.$$

Either $l = 0$ or $l \geq (h_0)^{-2/(2_*-2)}(S_{\mu,2_*})^{2_*/(2_*-2)}$. Suppose that

$$l \geq (h_0)^{-2/(2_*-2)}(S_{\mu,2_*})^{2_*/(2_*-2)}.$$

Then, from (4.4), (4.5) and Lemma 4.1, we obtain

$$c \geq ((2_* - 2)/2_*2)l + I(u) \geq c_\lambda^*,$$

which is a contradiction. Therefore, $l = 0$ and we conclude that u_n converges to u strongly in $\mathcal{H}_\mu$. □

Lemma 4.3. *Assume that* (G) *and* (H) *hold. Then, there exist* $v \in \mathcal{H}_\mu$ *and* $\Lambda_* > 0$ *such that for* $\lambda \in (0, \Lambda_*)$, *one has*

$$\sup_{t\geq 0} I(tv) < c_\lambda^*.$$

In particular, $c^- < c_\lambda^*$ *for all* $\lambda \in (0, \Lambda_*)$.

Proof. Let φ_ε be such that

$$\varphi_\varepsilon(x) = \begin{cases} \omega_\varepsilon(x) & \text{if } g(x) \geq 0 \text{ for all } x \in \mathbb{R}^N \\ \omega_\varepsilon(x - x_0) & \text{if } g(x_0) > 0 \text{ for } x_0 \in \mathbb{R}^N \\ -\omega_\varepsilon(x) & \text{if } g(x) \leq 0 \text{ for all } x \ \in \mathbb{R}^N \end{cases}$$

where ω_ε satisfies (2.1). Then, we claim that there exists $\varepsilon_0 > 0$ such that

$$\lambda \int_{\mathbb{R}^N} g(x)\varphi_\varepsilon(x)\,dx > 0 \quad \text{for any } \varepsilon \in (0, \varepsilon_0). \tag{4.6}$$

In fact, if $g(x) \geq 0$ or $g(x) \leq 0$ for all $x \in \mathbb{R}^N$, (4.6) obviously holds. If there exists $x_0 \in \mathbb{R}^N$ such that $g(x_0) > 0$, then by the continuity of $g(x)$, there exists $\eta > 0$ such that $g(x) > 0$ for all $x \in B(x_0, \eta)$. Then by the definition of $\omega_\varepsilon(x - x_0)$, it is easy to see that there exists an ε_0 small enough such that

$$\lambda \int_{\mathbb{R}^N} g(x)\omega_\varepsilon(x - x_0)\,dx > 0, \quad \text{for any } \varepsilon \in (0, \varepsilon_0).$$

Now, we consider the functions

$$f(t) = I(t\varphi_\varepsilon), \quad \tilde{f}(t) = (t^2/2)\|\varphi_\varepsilon\|_{a,\mu}^2 - (t^{2_*}/2_*) \int_{\mathbb{R}^N} h|y|^{-2_*b}|\varphi_\varepsilon|^{2_*}\,dx.$$

Then, for all $\lambda \in (0, \Lambda_1)$,

$$f(0) = 0 < c_\lambda^*.$$

By the continuity of f, there exists $t_0 > 0$ small enough such that

$$f(t) < c_\lambda^*, \quad \text{for all } t \in (0, t_0).$$

On the other hand,

$$\max_{t\geq 0} \tilde{f}(t) = ((2_* - 2)/2_*2)(h_0)^{-2/(2_*-2)}(S_{\mu,2_*})^{2_*/(2_*-2)}.$$

Then, we obtain

$$\sup_{t\geq 0} I(t\varphi_\varepsilon) < ((2_* - 2)/2_*2)(h_0)^{-2/(2_*-2)}(S_{\mu,2_*})^{2_*/(2_*-2)} - \lambda t_0 \int_{\mathbb{R}^N} g\varphi_\varepsilon\,dx.$$

Now, taking $\lambda > 0$ such that

$$-\lambda t_0 \int_{\mathbb{R}^N} g\varphi_\varepsilon \, dx < -C_0\lambda^2,$$

and by (4.6), we obtain

$$0 < \lambda < (t_0/C_0)\Big(\int_{\mathbb{R}^N} g\varphi_\varepsilon\Big), \quad \text{for } \varepsilon << \varepsilon_0.$$

Set

$$\Lambda_* = \min\{\Lambda_1,\ (t_0/C_0)(\int_{\mathbb{R}^N} g\varphi_\varepsilon)\}.$$

We deduce that

$$\sup_{t\geq 0} I(t\varphi_\varepsilon) < c_\lambda, \quad \text{for all } \lambda \in (0, \Lambda_*). \tag{4.7}$$

Now, we prove that

$$c^- < c_\lambda^*, \quad \text{for all } \lambda \in (0, \Lambda_*).$$

By (G) and the existence of w_n satisfying (2.1), we have

$$\lambda \int_{\mathbb{R}^N} g w_n \, dx > 0.$$

Combining this with Lemma 2.7 and from the definition of c^- and (4.7), we obtain that there exists $t_n > 0$ such that $t_n w_n \in \mathcal{N}^-$ and for all $\lambda \in (0, \Lambda_*)$,

$$c^- \leq I(t_n w_n) \leq \sup_{t\geq 0} I(t w_n) < c_\lambda^*.$$

□

Now we establish the existence of a local minimum of I on $\mathcal{N}^-$.

Proposition 4.4. *There exists $\Lambda_2 > 0$ such that for $\lambda \in (0, \Lambda_2)$, the functional I has a minimizer u_2 in $\mathcal{N}^-$ and satisfies*

(i) *$I(u_2) = c^-$,*
(ii) *u_2 is a solution of (1.1) in $\mathcal{H}_\mu$,*

where $\Lambda_2 = \min\{(1/2)\Lambda_1, \Lambda_*\}$ with Λ_1 defined as in (2.5) and Λ_* defined as in the proof of Lemma 4.3.

Proof. By Proposition 3.1 (ii), there exists a $(PS)_{c^-}$ sequence for I, $(u_n)_n$ in $\mathcal{N}^-$ for all $\lambda \in (0, (1/2)\Lambda_1)$. From Lemmas 4.2, 4.3 and 2.6 (ii), for $\lambda \in (0, \Lambda_*)$, I satisfies $(PS)_{c^-}$ condition and $c^- > 0$. Then, we get that $(u_n)_n$ is bounded in $\mathcal{H}_\mu$. Therefore, there exist a subsequence of $(u_n)_n$ still denoted by $(u_n)_n$ and $u_2 \in \mathcal{N}^-$ such that u_n converges to u_2 strongly in $\mathcal{H}_\mu$ and $I(u_2) = c^-$ for all $\lambda \in (0, \Lambda_2)$. Finally, by using the same arguments as in the proof of the Proposition 3.2, for all $\lambda \in (0, \Lambda_1)$, we have that u_2 is a solution of (1.1). □

Now, we complete the proof of Theorem 1.3. By Propositions 3.2 and 4.4, we obtain that (1.1) has two solutions u_1 and u_2 such that $u_1 \in \mathcal{N}^+$ and $u_2 \in \mathcal{N}^-$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$, this implies that u_1 and u_2 are distinct.

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