Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 27, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND STABILITY OF SOLUTIONS FOR NONLINEAR MECKING-LÜCKE-GRILHÉ EQUATIONS

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Dedicated to Jean Grilhé on his 73-th birthday

Abstract. In this article, we present the nonlinear Mecking-Lücke-Grilhé model describing the temporal evolution for simple and multi-instabilities of plastic deformation of stressed monocristal. This model extends the linear problem considered in [9, 13, 14]. Using a nonlinear analysis, we present some results of existence and stability of the solution with respect to the characteristics of the material and the retarded times. Numerical examples validating the theoretical results are also investigated in this study.

1. Introduction

The field of morphological change of solids has seen a considerable development in metallurgical engineering and materials science in the past few years. The search for materials of properties always more efficient led to many studies of the mechanisms associated to plastic deformation. The concept of the dislocation was introduced by Taylor [28, 29] to understand the mechanical behaviour of materials in plasticity. The dislocations help to explain the phenomena of plastic deformations [6, 15, 22], as well as other properties of solids, such as crystal growth and the electrical properties of semiconductors [16].

Localization of plastic deformation in homogeneous materials can be associated with instabilities of the stress-strain curves. These curves present in several cases some rapid oscillations due to the difficulties of creation or propagation of dislocations. This phenomenon can have very different aspects: Portevin-Le-Chatelier PLC effect, twinning, avalanches of dislocations, thermo-mechanical effect, Piobert-Lüders bands. For Example, the PLC effect is observed during stress rate change test of Al-Mg alloys at room temperature [17]. Kuo et al. [17] show that the occurence of plastic instability is strongly related to the retention time and applied stress rate, and this instability could be justified as the interactions between solid solution element, magnesium, and dislocations. Louchet and Brechet [19] present the different types of dislocations patterning during uniaxial deformation as a function of significant physical parameters such as crystalline structure; they shown

2000 *Mathematics Subject Classification.* 34A34, 34D05, 34D20, 34A45.
Key words and phrases. Mecking-Lücke-Grilhé equation; plastic deformation;
delay differential equations; characteristic equation; dislocation; asymptotic stability.

Submitted July 18, 2010 Published February 15, 2011.

that it is determined by a competition between dislocation production and rearrangements and they have improved that this phenomenon is controlled by strain rate and temperature. Sun et al. [27] investigated the finite element method to simulate the propagation of Lüders band by the level of stress concentration and the reduction of the thickness of corresponding element. Graff et al. [7, 8] propose finite element simulations and experimental observations of PLC effect and Lüders bands propagation in notched and compact tensile specimens of aluminum using the macroscopic PLC constitutive model. Some criteria for localization of plastic deformation and other studies in this field are proposed in [1, 3, 4, 5, 21, 30, 31].

In this paper, we are motivated by the works [9, 13, 14] restricted to the linear model. Consider a crystal subject to a mean stress. Under uniaxial traction (or compression), the interactions between dislocations, and the rotation of the traction-axis led to an activation of other slip systems. Consequently the plastic deformation instabilities are observed and can be explained by a delay time in the system's response to solicitations. Grilhé et al. presented in [9] an experimental study and a graphically analysis of the stability of the solution of this model. Using a linear analysis and Lambert's functions, a complete mathematical study (existence, uniqueness, asymptotic stability) of the model with a single delay is presented in [13]. Hilout et al. [14] present a new linear model describing the temporal evolution for multi-instabilities of plastic deformation of stressed monocristal. Here, we present the nonlinear Mecking-Lücke-Grilhé equation NMLGE. Under some assumptions and using a nonlinear analysis, we deduce a differential equations with one and two delays respectively. In the both cases, we show the theoretical existence and stability of the solution according to the characteristics of the material and the retarded times.

This article is presented as follows: In Section 2 we present the mathematical modelling of the plastic deformation instability. In Sections 3 and 4, we consider the case of NMLGE with a single delay and two delays respectively. We present in the both cases some results on existence and stability of the solution according to the characteristics of the material and the retarded times. Numerical examples for stability and instability of the material close to a mean stress using the MATLAB software are also investigated.

2. Mathematical modelling

Consider a crystal sample subject to a mean stress σ_0. The material is placed between two traverses (the first is fixed and the second is mobile). We apply a variable force $\mathcal{F}$ on the mobile traverse assuming a finite and constant velocity:

$$\dot{\varepsilon}(t) = \dot{\varepsilon}_0 = \text{constant}.$$

The strain rate $\dot{\varepsilon}$ is the sum of the plastic strain rate $\dot{\varepsilon_p}$ of the specimen and of the elastic strain rate $\dot{\varepsilon_e} = \dot{\sigma}/M$ of the combined sample and loading system (with a stiffness M)

$$\dot{\varepsilon}(t) = \dot{\varepsilon}_p(t) + \dot{\varepsilon}_e(t). \tag{2.1}$$

The plastic strain rate may be written as

$$\dot{\varepsilon}_p(t) = b\dot{\Sigma}(t)/V, \tag{2.2}$$

where b is the Burgers vector component along the tensile axis, $\Sigma(t)$ is the area swept by the dislocations and V is the sample volume which is supposed to remain constant. The plastic deformation is controlled by the emission of dislocation loops from Frank-Read type sources model. The equation ((2.2) can be written in the following form [20]:

$$\dot{\varepsilon}_p(t) = bn(t)S \tag{2.3}$$

where $n(t)$ denotes the number of loops arising at time t in the unit volume and during unit time and by S the mean area swept by the loops supposed constant during periods which are long enough compared with the period of instabilities. The area S in (2.3) depends on the instantaneous density of the forest and thus on the previous strain history of the sample. We suppose that S varies slowly. Note that the relation (2.3) is established assuming that the area S is instantaneously swept by each dislocation as soon as it is emitted [20, 9]. Grilhé et al. [9] suppose that the plastic instability can be explained by a phase shift, characterized by a time delay between the nucleation and the propagation of dislocations (see [9, 13, 14] for more details). After the flight-time τ', the mobile dislocation gets pinned or reaches the free surface of the sample having covered a constant area $S(\tau') = S$ since it was emitted. Then only loops generated at a time $t = t'$ with $0 < t' < \tau'$, will contribute to the deformation at a time t. Consequently, the equation (2.3) can be written as follows:

$$\dot{\varepsilon}_p(t) = b\int_0^{\tau'} n(t-s)\dot{S}(s)\,ds. \tag{2.4}$$

To simplify the problem, Grilhé et al. [9] suppose that

$$\dot{S}(t) = S\delta(t-\tau) \tag{2.5}$$

where δ is Dirac's distribution and τ is the delay given by

$$\tau = \frac{\int_0^\infty \dot{S}(t)dt}{S}. \tag{2.6}$$

3. NMLGE with a single delay

The time lag given by relation (2.6) can be interpreted by the phase displacement between the time of loop nucleation and the time at which the main strain is recorded and approximation (2.5) amounts to replacing S(t) by a step function. Under the assumption (2.5), we can rewrite (2.1) in the form

$$\dot{\varepsilon}(t) = bSn(\sigma(t-\tau)) + \frac{\dot{\sigma}(t)}{M}, \tag{3.1}$$

or

$$M\dot{\varepsilon}(t) = MbSn(\sigma(t-\tau)) + \dot{\sigma}(t). \tag{3.2}$$

Using the linear analysis we establish a differential-difference equation with a single delay (see [13]) to describe the plasticity of a solid becoming deformed by loops of dislocations or micro-twinning. For long-time, it is necessary to use the nonlinear analysis to investigate the stability of system strain-stress curves. Then we use Taylor's expansion of second order of the function $n(\sigma-\tau)$ close to the value σ_0:

$$\begin{aligned} n(\sigma(t-\tau)) &= n(\sigma_0) + \frac{\partial n}{\partial\sigma}(\sigma=\sigma_0)(\sigma(t-\tau)-\sigma_0) \\ &\quad + \frac{1}{2}\frac{\partial^2 n}{\partial\sigma^2}(\sigma=\sigma_0)(\sigma(t-\tau)-\sigma_0)^2. \end{aligned} \tag{3.3}$$

Substituting (3.3) in (3.2) we obtain

$$\dot{\sigma}(t) + \beta\sigma^2(t-\tau) + \theta\sigma(t-\tau) + \xi = 0, \tag{3.4}$$

where

$$\theta = \alpha - 2\beta\sigma_0, \quad \xi = \beta\sigma_0^2 - \alpha\sigma_0,$$
$$\alpha = MbS\frac{\partial n}{\partial \sigma}(\sigma_0) > 0, \quad \beta = \frac{1}{2}MbS\frac{\partial^2 n}{\partial \sigma^2}(\sigma_0) < 0.$$

The signs of α and β respectively are justified by the physical experiments [9].

In the sequel we denote the set

$$\mathbb{C}^+ = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq 0\}.$$

3.1. **Existence and uniqueness.** Equation (3.4) is a nonlinear retarded differential difference equation with delay time τ. To define a function σ in (3.4) for $t \geq 0$, we impose an initial data on the interval $[-\tau, 0]$ (e.g., we consider $\phi \equiv 1$ in $[-\tau, 0]$). In fact, let ϕ be a given continuous function on $[-\tau, 0]$ (ϕ is called preshape function) and we consider the problem (3.4) with initial data ϕ:

$$\begin{gathered} \dot\sigma(t) = -\beta\sigma^2(t-\tau) - \theta\sigma(t-\tau) - \xi = f(\sigma_t), \quad t \geq 0, \\ \sigma(t) = \phi(t), \quad t \in [-\tau, 0]. \end{gathered} \tag{3.5}$$

For fixed $c > 0$, consider the region

$$N = \{t : |\sigma(t)| + |\sigma(t-\tau)| \leq c\}.$$

Proposition 3.1. *Equation* (3.5) *admits a unique solution through* $(0, \phi)$ *defined on* $[-\tau, \infty)$.

Proof. Let $\phi_1, \phi_2 \in \mathcal{C} \cap N$. Then

$$\begin{aligned} |f(\phi_1) - f(\phi_2)| &\leq |\beta||\phi_1^2 - \phi_2^2| + |\theta||\phi_1 - \phi_2| \\ &\leq (|\beta||\phi_1 + \phi_2| + |\theta|)|\phi_1 - \phi_2| \\ &\leq (2c|\beta| + |\theta|)|\phi_1 - \phi_2|. \end{aligned}$$

Therefore, f is locally Lipschitz in ϕ, by [12, theorem 2.3 p. 44] there exists a unique solution of (3.5) through $(0, \phi)$ defined on $[-\tau, \infty)$ by

$$\begin{gathered} \sigma(t) = \phi(t) \quad \text{for } t \in [-\tau, 0], \\ \sigma(t) = \phi(0) + \int_0^t f(\sigma_s)ds \quad \text{for } t \geq 0. \end{gathered} \tag{3.6}$$

□

3.2. **Stability.** In this paragraph we study the stability of the solution of (3.5). So we take the associated homogeneous equation of (3.5)

$$\begin{gathered} \dot\sigma(t) + \theta\sigma(t-\tau) = -\beta\sigma^2(t-\tau), \quad t \geq 0, \\ \sigma(t) = \phi(t), \quad t \in [-\tau, 0]. \end{gathered} \tag{3.7}$$

We denote

$$m_\phi = |\phi| = \sup_{-\tau \leq t \leq 0} |\phi(t)|.$$

Theorem 3.2. *For* m_ϕ *is sufficiently small, the solution of* (3.7) *is asymptotically stable.*

Proof. By [12, theorem A.5, p. 416] the solution of the equation

$$\begin{gathered}\dot\sigma(t) = -\theta\sigma(t-\tau), \quad t \geq 0,\\ \sigma(t) = \phi(t), \quad t \in [-\tau, 0],\end{gathered} \tag{3.8}$$

is asymptotically stable if and only if

$$0 < \tau\theta < \frac{\pi}{2}. \tag{3.9}$$

Thus, under the condition (3.9), we have

$$\lim_{t\to\infty} |\sigma^0(t)| = 0, \tag{3.10}$$

where $\sigma^0(t)$ is the solution of (3.8). That is, under the condition (3.9), all roots of the characteristic equation

$$h(\lambda) = \lambda + \theta e^{-\tau\lambda} = 0, \tag{3.11}$$

have negative real parts (cf. [13]); i.e., (3.11) has no zeros in $\mathbb{C}^+$. Then if s is a root of (3.11), since the equation is of retarded type, there is a positive number $\lambda_1 > 0$ such that every characteristic root s satisfies $Re(s) < -\lambda_1$. By [12, theorem 6.1, p. 23], every solution σ^0 of (3.8) can be represented in the form

$$\sigma^0(t) = X(t)\phi(0) - \theta\int_{-\tau}^{0} X(t-\theta-\tau)\phi(\theta)d\theta. \tag{3.12}$$

By [12, theorem 5.2, p. 20], there exists $c_2 > 0$ such that

$$|X(t)| \leq c_2 e^{-\lambda_1 t}, \quad t \geq 0. \tag{3.13}$$

Consequently,

$$|\sigma^0(t)| \leq c_3 m_\phi e^{-\lambda_1 t},\ t \geq 0, \tag{3.14}$$

where

$$c_3 = c_2 + |\theta| c_2 \frac{1}{\lambda_1}(e^{\lambda_1\tau} - 1).$$

We want to show that for m_ϕ sufficiently small then the solution of (3.7) satisfies

$$|\sigma(t)| < 2c_3 m_\phi e^{-\lambda_2 t}, \quad t \geq -\tau, \tag{3.15}$$

where $0 < \lambda_2 < \lambda_1$.

Let t_0 be the first value such that $t_0 > 0$ and (3.15) is not true. Then by the continuity of σ,

$$\sigma(t_0) = 2c_3 m_\phi e^{-\lambda_2 t_0}. \tag{3.16}$$

On the other hand, the function $f(\sigma(t), \sigma(t-\tau)) = -\beta\sigma^2(t-\tau)$ is continuous for $t \leq t_0$ together with $(\sigma(t), \sigma(t-\tau)) \in N$. By ([2, paragraph 11.5]),

$$\sigma(t) = \sigma^0(t) + \int_0^t X(t-s) f(\sigma(s), \sigma(s-\tau))ds, \quad 0 < t \leq t_0. \tag{3.17}$$

Furthermore,

$$\lim_{|\sigma(s-\tau)|\to 0} \frac{|f(\sigma(s), \sigma(s-\tau))|}{|\sigma(s-\tau)|} = \lim_{|\sigma(s-\tau)|\to 0} -\beta|\sigma(s-\tau)| = 0.$$

Therefore,

$$|f(\sigma(s), \sigma(s-\tau))| \leq \epsilon|\sigma(s-\tau)| \leq 2\epsilon c_3 m_\phi e^{\lambda_2\tau} e^{-\lambda_2 s},\ 0 \leq s-\tau \leq t_0$$

and

$$\begin{aligned}|\sigma(t)| &< c_3 m_\phi e^{-\lambda_2 t} + 2c_2 e^{-\lambda_2 t}\int_0^t e^{\lambda_2 s}\epsilon c_3 m_\phi e^{\lambda_2\tau} e^{-\lambda_2 s} ds\\ &< c_3 m_\phi e^{-\lambda_2 t} + 2c_2\epsilon c_3 m_\phi e^{\lambda_2\tau} t_0 e^{-\lambda_2 t}\end{aligned}$$

for ϵ, m_ϕ sufficiently small and $0 < t \le t_0$. We can choose ϵ such that $2c_2\epsilon e^{\lambda_2\tau}t_0 < 1$, then

$$|\sigma(t)| < 2c_3 m_\phi e^{-\lambda_2 t}, \quad 0 < t \le t_0,$$

This contradicts the relation (3.16). Hence for any $t \ge 0$

$$|\sigma(t)| < 2c_3 m_\phi e^{-\lambda_2 t},$$

then $\lim_{t\to\infty} |\sigma(t)| = 0$. □

3.3. **Numerical tests.** The numerical results (see Fig. 1) do not give the exact solution of (3.7), but they show the asymptotic stability and instability of the solution of (3.7) according to the parameter $\tau\theta$. Various calculations are made by using the MATLAB software. These numerical results validate the theoretical result obtained in Theorem 3.2. Figure 1 (a) and (b) show the asymptotic stability of the solution of (3.7) near to σ_0. The beginning of phase instability of the solution of (3.7) is shown in figure 1 (c) and (d).

4. NMLGE with two delays

In most deformation experiments, several slip systems are active and depend on their orientation with respect to the traction-axis. Even when system of deformation is active, the crystal undergoes a rotation and a secondary deformation-mechanisms becomes active. These slip mechanisms with different activation values, correspond to different delays. Our goal in this section is the modelling of the plastic deformation instabilities when several delays are introduced, each corresponding to a system of deformation. Now we take (2.5) and we consider the general case when several deformation-mechanisms occur simultaneously, leading to several delays. We assume that two deformation-mechanisms are active and τ_1, τ_2 are the corresponding delays ($\tau_1 \neq \tau_2$). Then, we can write

$$\dot S(t) = S_1\delta(t-\tau_1) + S_2\delta(t-\tau_2) \qquad textand \quad S = S_1 + S_2. \tag{4.1}$$

Equation (2.1) can be re-written as follows ($\tau' > \max\{\tau_1,\tau_2\}$)

$$\begin{aligned}\dot\varepsilon(t) &= b\int_0^{\tau'} n(\sigma(t-s))\Big(S_1\delta(s-\tau_1) + S_2\delta(s-\tau_2)\Big)\,ds + \frac{\dot\sigma(t)}{M}\\ &= b\Big(S_1 n(\sigma(t-\tau_1)) + S_2 n(\sigma(t-\tau_2))\Big) + \frac{\dot\sigma(t)}{M}.\end{aligned}\tag{4.2}$$

Thus, we deduce the equation

$$M\dot\varepsilon(t) = MbS_1 n(\sigma(t-\tau_1)) + MbS_2 n(\sigma(t-\tau_2)) + \dot\sigma(t). \tag{4.3}$$

To investigate the stability of system strain-stress curves, we take the Taylor's expansion of second order of the function $n(\sigma-\tau_i)$, $i = 1, 2$, close to the value σ_0 for $i = 1, 2$:

$$n(\sigma(t-\tau_i)) = n(\sigma_0) + \frac{\partial n}{\partial\sigma}(\sigma=\sigma_0)(\sigma(t-\tau_i)-\sigma_0) + \frac{1}{2}\frac{\partial^2 n}{\partial\sigma^2}(\sigma=\sigma_0)(\sigma(t-\tau_i)-\sigma_0)^2.$$

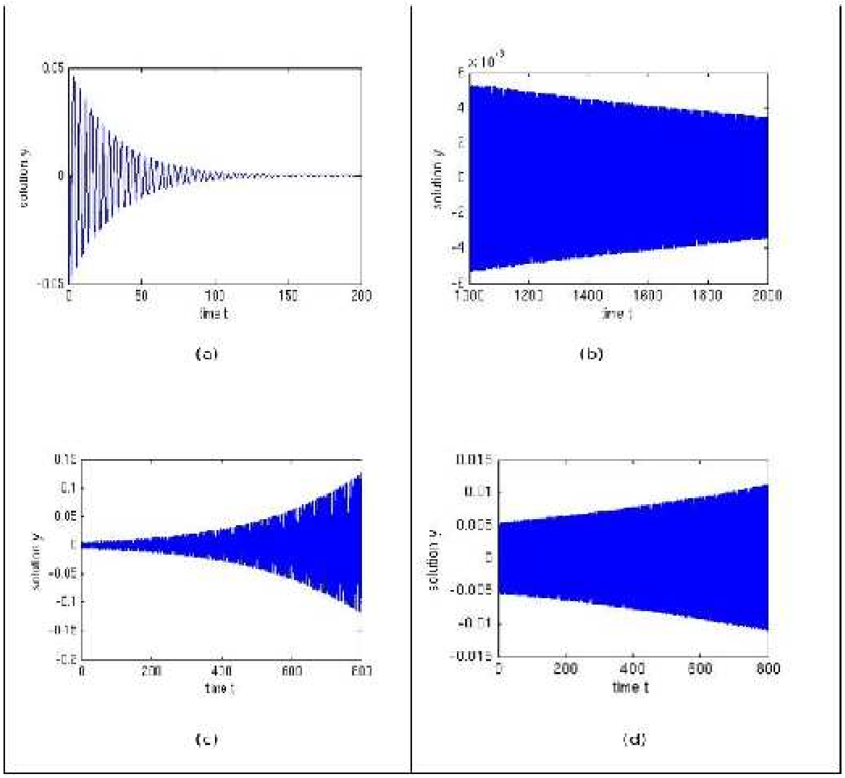

FIGURE 1. (a): $\tau = 1$ $m_\phi = 0.05$, $\beta = -0.25$, $\theta = 1.5$, the solution is stable. (b): $\tau = 1$, $m_\phi = 0.005$, $\beta = -0.5$, $\theta = 1.57$, the solution is stable. (c): $\tau = 1$, $m_\phi = 0.005$, $\beta = -0.5$, $\theta = 1.58$, the solution is unstable. (d): $\tau = 1$, $m_\phi = 0.005$, $\beta = -0.5$, $\theta = 1.573$, the solution is unstable

Substituting in (4.3),

$$
\begin{aligned}
Mbn(\sigma_0)(S_1+S_2) &= MbS_1 n(\sigma_0) + MbS_1\frac{\partial n}{\partial \sigma}(\sigma=\sigma_0)(\sigma(t-\tau_1)-\sigma_0)\\
&\quad + \frac{1}{2}MbS_1\frac{\partial^2 n}{\partial \sigma^2}(\sigma=\sigma_0)(\sigma(t-\tau_1)-\sigma_0)^2 + MbS_2 n(\sigma_0)\\
&\quad + MbS_2\frac{\partial n}{\partial \sigma}(\sigma=\sigma_0)(\sigma(t-\tau_2)-\sigma_0)\\
&\quad + \frac{1}{2}MbS_2\frac{\partial^2 n}{\partial \sigma^2}(\sigma=\sigma_0)(\sigma(t-\tau_2)-\sigma_0)^2 + \dot\sigma(t).
\end{aligned}
$$

Therefore,

$$
\dot\sigma(t) = -\beta_1\sigma^2(t-\tau_1) - \beta_2\sigma^2(t-\tau_2) - \theta_1\sigma(t-\tau_1) - \theta_2\sigma(t-\tau_2) + \gamma. \tag{4.4}
$$

where

$$
\beta_1 = \frac{1}{2}MbS_1\frac{\partial^2 n}{\partial\sigma^2}(\sigma_0) < 0, \quad \beta_2 = \frac{1}{2}MbS_2\frac{\partial^2 n}{\partial\sigma^2}(\sigma_0) < 0, \quad \alpha_1 = MbS_1\frac{\partial n}{\partial\sigma}(\sigma_0) > 0,
$$

$$\alpha_2 = MbS_2\frac{\partial n}{\partial \sigma}(\sigma_0) > 0, \quad \theta_1 = \alpha_1 - 2\beta_1\sigma_0, \quad \theta_2 = \alpha_2 - 2\beta_2\sigma_0, \quad \beta = \beta_1 + \beta_2,$$
$$\alpha = \alpha_1 + \alpha_2, \quad \gamma = \alpha\sigma_0 - \beta\sigma_0^2.$$

Let $\tau = max\{\tau_1, \tau_2\}$, $\phi \in \mathcal{C} = \mathcal{C}([-\tau, 0]; \mathbb{R})$ such that $\sigma(t) = \phi(t)$ for $t \in [-\tau, 0]$. We obtain the system

$$\begin{gathered} \dot{\sigma}(t) = f(\sigma_t(-\tau_1), \sigma_t(-\tau_2)), \quad \text{for } t \geq 0, \\ \sigma(t) = \phi(t), \quad \text{for } t \in [-\tau, 0], \end{gathered} \tag{4.5}$$

where

$$f(x, y) = -\beta_1 x^2 - \beta_2 y^2 - \theta_1 x - \theta_2 y + \gamma.$$

4.1. **Existence and uniqueness.** As in [12, lemma 1.1, p. 39], we have the following result.

Lemma 4.1. *Suppose that $\phi \in \mathcal{C}$, $f : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ is a continuous function. Then finding a solution of equation* (4.5) *is equivalent to solving the integral equation*

$$\begin{gathered} \sigma(t) = \phi(t), \quad t \in [-\tau, 0], \\ \sigma(t) = \phi(0) + \int_0^t f(\sigma_s(-\tau_1), \sigma_s(-\tau_2))ds, \quad t \geq 0. \end{gathered} \tag{4.6}$$

Theorem 4.2. *Problem* (4.5) *admits a unique solution on* $[-\tau, +\infty)$ *through* $(0, \phi)$.

Proof. By [10, theorem 1.1.1], the existence is ensured. Let $t \in I_\alpha = [0, \alpha]$, $\alpha > 0$, and on take the region:

$$N = \{t;\, |\sigma(t)| + |\sigma(t - \tau_1)| + |\sigma(t - \tau_2)| \leq c\}.$$

Let $x, y \in N$ be two solutions of (4.5). Then for $t \geq 0$, we have

$$\begin{aligned} |x(t) - y(t)| &\leq \int_0^t |f(x_s(-\tau_1), x_s(-\tau_2)) - f(y_s(-\tau_1), y_s(-\tau_2))|ds \\ &\leq \int_0^t \Big((-\beta_1|x(s-\tau_1) + y(s-\tau_1)| + \theta_1)|x(s-\tau_1) - y(s-\tau_1)| \\ &\quad + \big(-\beta_2|x(s-\tau_2) + y(s-\tau_2)| + \theta_2\big)|x(s-\tau_2) - y(s-\tau_2)|\Big)ds. \end{aligned}$$

Since $x, y \in N$, then we can write $-\beta_i|x(s-\tau_i) + y(s-\tau_i)| + \theta_i) \leq k_i$, where $k_i = -2c\beta_i + \theta_i$, $i = 1, 2$. Let $k = max\{k_1, k_2\}$, then for $\alpha = \bar{\alpha}$ such that $k\bar{\alpha} < 1$, and $t \in I_{\bar{\alpha}}$, we find

$$|x(t) - y(t)| \leq k\bar{\alpha} \sup_{0 \leq s \leq t} [|x(s-\tau_1) - y(s-\tau_1)| + |x(s-\tau_2) - y(s-\tau_2)|],$$

since $s - \tau_i \in [-\tau, 0]$, $i = 1, 2$; therefore, $x(s - \tau_i) = y(s - \tau_i)$, $i = 1, 2$. Thus, $x(t) = y(t)$ for all $t \in I_{\bar{\alpha}}$. One completes the proof of the theorem by successively stepping intervals of length $\bar{\alpha}$. □

Lemma 4.3. *Consider the associated homogeneous equation with* (4.5)*:*

$$\begin{gathered} \dot{\sigma}(t) = -\theta_1\sigma(t - \tau_1) - \theta_2\sigma(t - \tau_2), \quad t \geq 0, \\ \sigma(t) = \phi(t), \quad t \in [-\tau, 0], \end{gathered} \tag{4.7}$$

The solution of (4.5) *is exponentially bounded; i.e., there exist constants a and b such that*

$$|\sigma(t)| \leq am_\phi e^{bt}, \quad t \geq 0,$$

where $m_\phi = \sup_{-\tau\le t\le 0}|\phi|$.

Proof. We have

$$\sigma(t) = \phi(0) + \int_0^t [-\theta_1\sigma(s-\tau_1) - \theta_2\sigma(s-\tau_2)]ds, \quad t\ge 0.$$

and $\sigma(t) = \phi(t)$ for all $t\in[-\tau,0]$, then for $t\ge 0$ we can write

$$\begin{aligned}
|\sigma(t)| &\le m_\phi + \theta_1\int_0^t |\sigma(s-\tau_1)|ds + \theta_2\int_0^t |\sigma(s-\tau_2)|ds \\
&\le m_\phi + \theta_1 m_\phi\tau_1 + \theta_2 m_\phi\tau_2 + (\theta_1+\theta_2)\int_0^t |\sigma(s)|ds \\
&\le a m_\phi + b\int_0^t |\sigma(s)|ds,
\end{aligned}$$

where $a = 1+\theta_1\tau_1+\theta_2\tau_2$, $b = \theta_1+\theta_2$. By Grönwall's lemma, $|\sigma(t)|\le a m_\phi e^{bt}$, $t\ge 0$. □

In the sequel we use the notation

$$\int_{(c)} = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT},$$

where c is a real number.

4.2. **Stability.** First we define the Fundamental solution. The characteristic equation associated with (4.7) is

$$h(\lambda) = \lambda + \theta_1 e^{-\lambda\tau_1} + \theta_2 e^{-\lambda\tau_2} = 0. \tag{4.8}$$

We are looking for the solution $X(t)$ of (4.7) such that its Laplace transform is $h^{-1}(\lambda)$ with the initial condition

$$X(t) = \begin{cases} 0 & t<0,\\ 1 & t=0.\end{cases}$$

By lemma 4.3 the Laplace transform of $X(t)$ has a sense. We multiply (4.7) by $e^{-\lambda t}$ and we integrate between 0 and ∞:

$$\int_0^\infty e^{-\lambda t}\dot X(t)dt = -\theta_1\int_0^\infty e^{-\lambda t}X(t-\tau_1)dt - \theta_2\int_0^\infty e^{-\lambda t}X(t-\tau_2)dt.$$

An integration by parts gives

$$1 = (-\lambda - \theta_1 e^{-\lambda\tau_1} - \theta_2 e^{-\lambda\tau_2})\int_0^\infty e^{-\lambda t}X(t)dt;$$

therefore,

$$\mathcal{L}(X)(\lambda) = h^{-1}(\lambda). \tag{4.9}$$

The solution of (4.7) which satisfies (4.9) is called *the fundamental solution.* Since $X(t)$ is a function of bounded variation on every compact and is continuous, then the inversion theorem [12] allows us to write

$$X(t) = \int_{(c)} e^{\lambda t}h^{-1}(\lambda)dt.$$

By adapting the proof of [12, Theorem 5.2], we obtain the following result.

Theorem 4.4. *For $\alpha > \alpha_0 = \max\{Re\lambda;\ h(\lambda) = 0\}$, there exists a constant $k > 0$ such that*

$$|X(t)| \le ke^{\alpha t}, \quad t \ge 0.$$

Particularly, if $\alpha_0 < 0$, then we can choose $\alpha_0 < \alpha < 0$ such that $X(t) \to 0$ when $t \to \infty$.

Proof. We have

$$X(t) = \int_{(c)} e^{\lambda t} h^{-1}(\lambda) d\lambda, \tag{4.10}$$

where c is some sufficiently large real number. We may take $c > \alpha$. We first want to prove that

$$X(t) = \int_{(\alpha)} e^{\lambda t} h^{-1}(\lambda) d\lambda. \tag{4.11}$$

We integrate $e^{\lambda t}h^{-1}(\lambda)$ around the boundary of the box $ABCD$ in the complex plane with boundary $L_1M_1L_2M_2$ in the direction indicated (see Fig. 2), where

$$L_1 = \{c + i\tau; -T \le \tau \le T\}, \quad L_2 = \{\alpha + i\tau; -T \le \tau \le T\},$$
$$M_1 = \{\sigma + iT; \alpha \le \sigma \le c\}, \quad M_2 = \{\sigma - iT; \alpha \le \sigma \le c\}.$$

Since $h(\lambda)$ has no zeros in the box, it follows that the integral over the boundary is zero. Therefore, relation (4.11) will be verified if we show that

$$\int_{M_1} e^{\lambda t} h^{-1}(\lambda) d\lambda, \int_{M_2} e^{\lambda t} h^{-1}(\lambda) d\lambda \to 0 \quad \text{as } T \to \infty.$$

FIGURE 2. Γ: inside the rectangle $ABCD$

Choose T_0 such that

$$(1 + \frac{\alpha^2}{T_0^2})^{1/2} - \frac{1}{T_0}(\theta_1 e^{-\tau_1\alpha} + \theta_2 e^{-\tau_2\alpha}) \ge \frac{1}{2}.$$

If $T \ge T_0$ and $\lambda \in M_1$; that is, $\lambda = \sigma + iT,\ \alpha \le \sigma \le c$, and $T \ge T_0$, then

$$|h^{-1}(\lambda)| \le \frac{1}{(\sigma^2 + T^2)^{1/2} - \theta_1 e^{-\tau_1\alpha} - \theta_2 e^{-\tau_2\alpha}} \le \frac{2}{T}.$$

Therefore, by letting $T \to \infty$,

$$|\int_{M_1} e^{\lambda t} h^{-1}(\lambda) d\lambda| \le \frac{2}{T} e^{ct}(c - \alpha) \to 0.$$

The same arguments as previously prove that the integral over M_2 go to 0 by letting $T \to \infty$. This proves the relation (4.11).

Suppose T_0 is as above. If $g(\lambda) = h^{-1}(\lambda) - (\lambda-\alpha)^{-1}$ then for $\lambda = \alpha + iT$; $|T| \geq T_0$, and

$$\begin{aligned} g(\lambda) &= \Big|\frac{1}{\lambda - \theta_1 e^{-\tau_1\alpha} - \theta_2 e^{-\tau_2\alpha}} - \frac{1}{\lambda - \alpha_0}\Big| \\ &= \Big|\frac{\theta_1 e^{-\tau_1\alpha} + \theta_2 e^{-\tau_2\alpha} - \alpha_0}{\lambda - \alpha_0} h^{-1}(\lambda)\Big| \\ &\leq \frac{2}{T^2}(\theta_1 e^{-\tau_1\alpha} + \theta_2 e^{-\tau_2\alpha} + |\alpha_0|). \end{aligned}$$

Then

$$\int_{(\alpha)} |g(\lambda)| d\lambda < \infty, \quad \int_{(\alpha)} |e^{\lambda t} g(\lambda)| d\lambda \leq k_1 e^{\alpha t}, \quad t > 0,$$

where k_1 is a constant. Consequently

$$\int_{(\alpha)} e^{\lambda t}(\lambda - \alpha_0)^{-1} d\lambda \leq k_2 e^{\alpha t}, \quad t > 0,$$

and $|X(t)| \leq k e^{\alpha t}$, $t > 0$, $k = k_1 + k_2$. $\square$

Theorem 4.5. *For $t \geq 0$, the solution of (4.7) is given by*

$$\sigma(\phi, 0)(t) = X(t)\phi(0) - \theta_1 \int_{-\tau_1}^0 X(t - r - \tau_1)\phi(r) dr - \theta_2 \int_{-\tau_2}^0 X(t - r - \tau_2)\phi(r) dr.$$

Proof. Multiply (4.7) by $e^{-\lambda t}$ and we integrate by parts:

$$-\phi(0) + h(\lambda)\mathcal{L}(\sigma)(\lambda) = -\theta_1 e^{-\lambda\tau_1} \int_{-\tau_1}^0 e^{-\lambda r}\phi(r) dr - \theta_2 e^{-\lambda\tau_2} \int_{-\tau_2}^0 e^{-\lambda r}\phi(r) dr.$$

Then, for c is sufficiently large,

$$\sigma(t) = \int_{(c)} h^{-1}(\lambda)[\phi(0) - \theta_1 e^{-\lambda\tau_1} \int_{-\tau_1}^0 e^{-\lambda r}\phi(r) dr - \theta_2 e^{-\lambda\tau_2} \int_{-\tau_2}^0 e^{-\lambda r}\phi(r) dr] d\lambda.$$

For $i = 1, 2$, we consider $w_i : [-\tau_i, \infty) \to [0, 1]$ such that $w_i(r) = 0$ if $r \geq 0$ and $w_i(r) = 1$, if $r < 0$, then we can define ϕ on $[-\tau, \infty)$ by $\phi(r) = \phi(0)$ for $r \geq 0$.

For $i = 1, 2$, we have

$$\begin{aligned} e^{-\lambda\tau_i} \int_{-\tau_i}^0 e^{-\lambda r}\phi(r) dr &= \int_0^\infty e^{-\lambda s}\phi(-\tau_i + s) w_i(-\tau_i + s) ds \\ &= \mathcal{L}(\phi(-\tau_i + \cdot) w_i(-\tau_i + \cdot)). \end{aligned}$$

We can write

$$\begin{aligned} \sigma(t) = X(t)\phi(0) &- \theta_1 \int_0^t X(t-s)\phi(-\tau_1 + s) w(-\tau_1 + s) ds \\ &- \theta_2 \int_0^t X(t-s)\phi(-\tau_2 + s) w(-\tau_2 + s) ds, \end{aligned}$$

and

$$\sigma(t) = X(t)\phi(0) - \theta_1 \int_0^{\tau_1} X(t-s)\phi(-\tau_1 + s) ds - \theta_2 \int_0^{\tau_2} X(t-s)\phi(-\tau_2 + s) ds.$$

Suppose that $r_i = -\tau_i + s$ for $i = 1, 2$. Then

$$\sigma(t) = X(t)\phi(0) - \theta_1 \int_{-\tau_1}^{0} X(t-r-\tau_1)\phi(r)dr - \theta_2 \int_{-\tau_2}^{0} X(t-r-\tau_2)\phi(r)dr.$$

□

Corollary 4.6. *Let* $\alpha_0 = max\{Re(\lambda); h(\lambda) = 0\}$ *and* $\sigma(\phi)(t)$ *is the solution of* (4.7). *Then, for all* $\alpha > \alpha_0$, *there exists a constant* $k = k(\alpha)$ *such that*

$$|\sigma(\phi)(t)| \le k m_\phi e^{\alpha t}, \quad t \ge 0,\ m_\phi = \sup_{-\tau \le r \le 0} |\phi(r)|.$$

Particularly, if $\alpha_0 < 0$, *then we can choose* $\alpha_0 < \alpha < 0$ *such that any solution of* (4.7) *approaches 0, by letting* $t \to \infty$.

Proof. By theorem 4.4, there exists a constant $k_1 > 0$ such that $|X(t)| \le k_1 e^{\alpha t}$. On the other hand, By theorem 4.5 we can write

$$\begin{aligned}
|\sigma(\phi)(t)| &\le |X(t)|m_\phi + \theta_1 m_\phi \int_{-\tau_1}^{0} |X(t-r-\tau_1|dr + \theta_2 m_\phi \int_{-\tau_2}^{0} |X(t-r-\tau_2|dr \\
&\le k_1 m_\phi e^{\alpha t} + \theta_1 k_1 m_\phi \int_{-\tau_1}^{0} e^{\alpha(t-\tau_1-r)}dr + \theta_2 k_1 m_\phi \int_{-\tau_2}^{0} e^{\alpha(t-\tau_2-r)}dr \\
&\le m_\phi e^{\alpha t}[k_1 + \frac{\theta_1}{\alpha} k_1(1+e^{-\alpha\tau_1}) + \frac{\theta_2}{\alpha} k_1(1+e^{-\alpha\tau_2})] \\
&\le k m_\phi e^{\alpha t}.
\end{aligned}$$

□

Remark 4.7. Consider

$$f(\sigma(t-\tau_1), \sigma(t-\tau_2)) = -\beta_1\sigma^2(t-\tau_1) - \beta_2\sigma^2(t-\tau_2),$$

and denote $u(t) = \sigma(t-\tau_1),\ v(t) = \sigma(t-\tau_2)$. Then

$$f(u,v) = -\beta_1 u^2 - \beta_2 v^2, \quad \beta_1 < 0,\ beta_2 < 0.$$

One can easily show that, f is a continuous function, $f(0,0) = 0$, and

$$\begin{aligned}
|f(u_1,v_1) - f(u_2,v_2)| &\le -\beta_1|u_1^2 - u_2^2| - \beta_2|v_1^2 - v_2^2| \\
&\le k(|u_1| + |u_2| + |v_1| + |v_2|)(|u_1 - u_2| + |v_1 - v_2|),
\end{aligned}$$

where $k = \max\{-\beta_1, -\beta_2\}$. We take the region $N = \{t; |\sigma(t)| + |u(t)| + |v(t)| \le c_1\}$, suppose $c_2 = 2c_1 k$, we choose $c_3 = \epsilon c_1 \le c_1$, ($\epsilon$ small enough) such that c_3 satisfies the inequality

$$|u_1 - u_2| + |v_1 - v_2| \le c_3.$$

Then, $c_2 \to 0$ as $c_3 \to 0$. Then f is c_2-Lipschitz on N,

$$|f(u_1,v_1) - f(u_2,v_2)| \le c_2(|u_1 - u_2| + |v_1 - v_2|). \tag{4.12}$$

Remark 4.8. By [14, proposition 3.2] (see also [18]), if

$$\tau_1 \ne \frac{\pi}{2\theta_1} + \frac{2j\pi}{\theta_1}, \quad (j \in \mathbb{N}),\ \tau_1 > \frac{\pi}{2\theta_1},$$

then, for $\tau_2 > 0$, there exists a constant $\delta > 0$ such that the solution of (4.7) is unstable when $\frac{\theta_2}{\theta_1} < \delta$.

Remark 4.9. By [14, propositions 3.1 et 3.3] (see also [18]), we have the stability of the solution of (4.7) under the following conditions:

(1)
$$\theta_2 < \theta_1, \quad \tau_1 \le \frac{1}{\theta_1+\theta_2}, \quad \tau_2 > 0. \tag{4.13}$$
(2)
$$\theta_2 > \theta_1, \quad \frac{\pi}{2\tau_1} < (\theta_1^2+\theta_2^2)^{1/2} < \frac{3\pi}{2\tau_1}$$
and for all $\tau_2 \in [0, \tau_{2,c}]$ such that $\tau_{2,c}$ is the critical value which given as
$$\tau_{2,c} = \frac{1}{\omega_0} \arccos(-\frac{\theta_1 \cos \omega_0 \theta_1 \tau_1}{\theta_2}),$$
where ω_0 is the unique solution of the equation
$$\frac{\omega^2 + 1 - \theta_2^2/\theta_1^2}{2\omega} = \sin \omega\theta_1\tau_1.$$
(3) $\tau_1 \in [\frac{1}{\theta_1+\theta_2}, \frac{\pi}{2\theta_1}]$, in this case the stability depends only on the critical value τ_2.
(4) For τ_1 as fixed $\tau_1 > \frac{\pi}{2\theta_1}$, there exists a value $\tau_{0,c}$ such that the solution of (4.7) is stable for all $\tau_2 \le \tau_{0,c}$.

For each root s of $h(\lambda)$ (see [14, 18]), there exists $\lambda_0 > 0$ such that $Re(s) < -\lambda_0$. By theorem 4.4, there exists a constant c_4 such that
$$|X(t)| \le c_4 e^{-\lambda_0 t}, \quad t \le 0. \tag{4.14}$$
By Corollary 4.6, we can find a constant c_5 such that
$$|\sigma_0(t)| \le c_5 m_\phi e^{-\lambda_0 t}, \quad t \ge 0, \tag{4.15}$$
with $\sigma_0(t)$ is the solution of (4.7).

Using the notation of Remark 4.7, we consider
$$\begin{gathered} \dot\sigma(t) = -\theta_1\sigma(t-\tau_1) - \theta_2\sigma(t-\tau_2) + f(u(t), v(t)), \quad t \ge 0, \\ \sigma(t) = \phi(t), \quad t \in [-\tau, 0]. \end{gathered} \tag{4.16}$$
We have the following result.

Theorem 4.10. *Suppose that m_ϕ is sufficiently small. Then the solution of (4.16) is a continuous function on $[-\tau, \infty)$, given by*
$$\begin{gathered} \dot\sigma(t) = \sigma_0(t) + \int_0^t f(u(s), v(s)) X(t-s) ds, \quad t \ge 0, \\ \sigma(t) = \phi(t), \quad t \in [-\tau, 0], \end{gathered} \tag{4.17}$$
where $\sigma_0(t)$ is the solution of linear equation (4.7), and $X(t)$ is the fundamental solution of (4.7). Therefore if m_ϕ is sufficiently small, then $\lim_{t\to\infty} |\sigma(t)| = 0$.

Proof. We use ideas from [2, Chapter 11]. Let $\{\sigma_n(t)\}_{n\ge 0}$ is a sequence defined by
$$\begin{gathered} \sigma_{n+1}(t) = \sigma_0(t) + \int_0^t f(u_n(s), v_n(s)) X(t-s) ds, \quad t \ge 0, \\ \sigma_{n+1}(t) = \phi(t), \quad t \in [-\tau, 0], \end{gathered} \tag{4.18}$$
where $u_n(s) = \sigma_n(s-\tau_1)$, $v_n(s) = \sigma_n(s-\tau_2)$. We will show that this sequence is well defined; i.e.,
$$|\sigma_n(t)| \le 2c_5 m_\phi, \quad n = 0, 1, \dots, \ t \ge -\tau. \tag{4.19}$$

For $n = 0$, (4.19) is verified for all $t \in [-\tau, 0]$, if we take $c_5 > 1/2$. We proceed by recurrence. Let $t \geq 0$, suppose that (4.19) is verified. We will show that

$$|\sigma_{n+1}(t)| \leq 2c_5 m_\phi, \quad n = 0, 1, \dots, \ t \geq 0. \tag{4.20}$$

For m_ϕ is sufficiently small, we can take $c_3 = 8c_5 m_\phi$; therefore,

$$|\sigma_n(s-\tau_1)| + |\sigma_n(s-\tau_2)| \leq 4c_5 m_\phi \leq \frac{c_3}{2}, \quad s \geq 0.$$

By (4.12), we find that

$$\begin{aligned} |f(\sigma_n(s-\tau_1), \sigma_n(s-\tau_2))| &\leq c_2[|\sigma_n(s-\tau_1)| + |\sigma_n(s-\tau_2)|] \\ &\leq \frac{1}{2} c_2 c_3 = 4c_2 c_5 m_\phi. \end{aligned}$$

Then

$$\begin{aligned} |\sigma_{n+1}(t)| &\leq c_5 m_\phi e^{-\lambda_0 t} + 4c_2 c_4 c_5 m_\phi \int_0^t e^{-\lambda_0 (t-s)} ds \\ &\leq c_5 m_\phi + 4c_2 c_4 c_5 m_\phi \int_0^t e^{-\lambda_0 r} dr \\ &\leq c_5 m_\phi + 4c_2 c_4 c_5 m_\phi / \lambda_0. \end{aligned}$$

Since $c_2 \to 0$ as $m_\phi \to 0$, we can choose m_ϕ such that $4c_2 c_4/\lambda_0 < 1$. Then

$$|\sigma_{n+1}(t)| \leq 2c_5 m_\phi, \quad n = 0, 1, \dots, \ t \geq -\tau.$$

The sequence $\{\sigma_n(t)\}_{n \geq 0}$ is well defined for $t \geq -\tau$, and it is bounded uniformly.

Now we prove that $\{\sigma_n(t)\}_{n \geq 0}$ converges. For $n \geq 1$, we find that

$$\begin{aligned} |\sigma_{n+1}(t) - \sigma_n(t)| \leq \int_0^t &|f(\sigma_n(s-\tau_1), \sigma_n(s-\tau_2)) \\ &- f(\sigma_{n-1}(s-\tau_1), \sigma_{n-1}(s-\tau_2))| X(t-s) ds. \end{aligned}$$

By (4.19), we have

$$|\sigma_n(t-\tau_1) - \sigma_{n-1}(t-\tau_1)| + |\sigma_n(t-\tau_2) - \sigma_{n-1}(t-\tau_2)| \leq 8c_5 m_\phi = c_3.$$

Using (4.12), we find that

$$\begin{aligned} |\sigma_{n+1}(t) - \sigma_n(t)| \leq c_2 c_4 \int_0^t &[|\sigma_n(s-\tau_1) - \sigma_{n-1}(s-\tau_1)| \\ &+ |\sigma_n(s-\tau_2) - \sigma_{n-1}(s-\tau_2)|] e^{-\lambda_0 (t-s)} ds. \end{aligned}$$

Let

$$m_n(t) = \sup_{-\tau \leq s \leq t} |\sigma_n(s) - \sigma_{n-1}(s)|, \ n \geq 1.$$

For $t \geq -\tau$, $n \geq 1$, we have

$$|\sigma_{n+1}(t) - \sigma_n(t)| \leq 2c_2 c_4 m_n(t) \int_0^t e^{-\lambda_0 (t-s)} ds. \tag{4.21}$$

Since $\sigma_{n+1}(t) = \sigma_n(t)$ for $t \in [-\tau, 0]$, we obtain

$$m_{n+1}(t) \leq c_6 m_n(t), \quad t \geq -\tau, \tag{4.22}$$

where $c_6 = 2c_2c_4\int_0^t e^{-\lambda_0(t-s)}ds$. For m_ϕ is sufficiently small, we can take $c_6 < 1$, because that $c_2 \to 0$ as $c_3 \to 0$. Consequently,

$$\sum_{n=0}^{\infty} \sup_{-\tau\le s\le t} |\sigma_{n+1}(s) - \sigma_n(s)|, \tag{4.23}$$

is convergent, since it is bounded by $m_1(t)\sum_{n=0}^{\infty} c_6^n$, where

$$|m_1(t)| \le \sup_{-\tau\le s\le t} |\sigma_1(s)| + \sup_{-\tau\le s\le t} |\sigma_0(t)| \le 4c_5m_\phi.$$

The convergence of (4.23) is uniform, then $\{\sigma_n(t)\}_{n\ge 0}$ converges uniformly to $\sigma(t)$. By (4.18), $\sigma(t)$ satisfies the condition $\sigma(t) = \phi(t)$ for $t \in [-\tau, 0]$. It also satisfies (4.17). $\sigma(t)$ is a continuous function for all $t \ge -\tau$. By (4.17), we have

$$|\sigma(t)| \le c_5m_\phi e^{-\lambda_0 t} + c_2c_4\int_0^t [|\sigma(s-\tau_1)| + |\sigma(s-\tau_2)|]|X(t-s)|ds,$$

$$\begin{aligned}|\sigma(t)| \le c_5m_\phi e^{-\lambda_0 t} &+ c_2c_4\int_{-\tau_1}^{t-\tau_1} |\sigma(r)||X(t-r-\tau_1)|dr\\ &+ c_2c_4\int_{-\tau_2}^{t-\tau_2} |\sigma(r)||X(t-r-\tau_2)|dr,\end{aligned}$$

Suppose that $k = 2c_2c_4(e^{\lambda_0\tau} - 1)/\lambda_0$, then

$$|\sigma(t)|e^{\lambda_0 t} \le c_5m_\phi + km_\phi + c_2c_4e^{\lambda_0\tau_1}\int_0^t |\sigma(r)|e^{\lambda_0 r}dr + c_2c_4e^{\lambda_0\tau_2}\int_0^t |\sigma(r)|e^{\lambda_0 r}dr.$$

Therefore,

$$|\sigma(t)|e^{\lambda_0 t} \le c_5m_\phi + km_\phi + 2c_2c_4e^{\lambda_0\tau}\int_0^t |\sigma(r)|e^{\lambda_0 r}dr.$$

By Grönwall's lemma,

$$|\sigma(t)|e^{\lambda_0 t} \le (c_5 + k)m_\phi \exp{(2c_2c_4e^{\lambda_0\tau})t},$$

and

$$|\sigma(t)| \le (c_5 + k)m_\phi \exp{(-\lambda_0 + 2c_2c_4e^{\lambda_0\tau})t}.$$

Since $c_2 \to 0$ as $m_\phi \to 0$, for m_ϕ is sufficiently small, we obtain $\lim_{t\to\infty} |\sigma(t)| = 0$. □

4.3. **Numerical tests.** As in the previous section (Section 3) we present some numerical results using MATLAB to show asymptotic stability and instability of solution of (4.16) according to the physical parameters α_1, α_2, β_1, β_2, τ_1 and τ_2; see Table 1.

References

[1] G. Ananthakrishna, C. Fressengeas, M. Grosbras and J. Vergnol, *On the existence of chaos in jerky flow.* Scripta Metallurgica et Materialia 32(11) (1995) 1731-1737.

[2] R. E. Belman and K. L. Cooke, *Differential-difference equations*, Mathematical in Sciences and Engineering, Academic Press, 1963.

[3] Y. Brechet and F. Louchet, *Localization of plastic deformation*, Solid State Phenomena vol. 3 & 4 (1988) 347-356.

[4] A. Coujou, A. Beneteau and N. Clement, *Observation in situ de l'interception d'un champ homogène de dislocations parfaites mobiles par une micromacle: Compétition entre micromaclage et glissement*, Scripta Metallurgica 19(7) (1985) 891-895.

Table 1. Table of stability/instability/Hopf-bifurcation of the material

$\theta_1 > \theta_2$, $\tau_1 \leq \frac{1}{\theta_1+\theta_2}$, $\tau_2 > 0$	Stability	Fig. 3 (a)
$\theta_1 < \theta_2$, $\tau_1 \leq \frac{1}{\theta_1+\theta_2}$, $\tau_2 > 0$	Instability	Fig. 3 (b)
$\theta_1 > \theta_2$, $\tau_1 \leq \frac{1}{\theta_1+\theta_2}$, $\tau_2 > 0$	Hopf-bifurcation	Fig. 3 (c)
$\frac{\pi}{2\tau_1} < (\theta_1^2+\theta_2^2)^{1/2} < \frac{3\pi}{2\tau_1}$, $\theta_2 > \theta_1$, $\tau_2 \in [0, \tau_{2,c}]$	Stability	Fig. 3 (d)
$\frac{\pi}{2\tau_1} < (\theta_1^2+\theta_2^2)^{1/2} < \frac{3\pi}{2\tau_1}$, $\theta_2 > \theta_1$	Instability	Fig. 3 (e)
$\frac{\pi}{2\tau_1} < (\theta_1^2+\theta_2^2)^{1/2} < \frac{3\pi}{2\tau_1}$, $\theta_2 > \theta_1$	Hopf-bifurcation	Fig. 3 (f)
$\tau_1 \in [\frac{1}{\theta_1+\theta_2}, \frac{\pi}{2\theta_1}]$, $\tau_2 \in [0, \tau_{2,c}]$	Stability	Fig. 4 (b)
$\tau_1 \in [\frac{1}{\theta_1+\theta_2}, \frac{\pi}{2\theta_1}]$	Instability	Fig. 4 (a)
$\tau_1 \in [\frac{1}{\theta_1+\theta_2}, \frac{\pi}{2\theta_1}]$	Hopf-bifurcation	Fig. 4 (c)
$\tau_1 > \frac{\pi}{2\theta_1}$, $\tau_2 \in [0, \tau_{0,c}]$	Stability	Fig. 4 (d)
$\tau_1 > \frac{\pi}{2\theta_1}$	Instability	Fig. 4 (e)
$\tau_1 > \frac{\pi}{2\theta_1}$	Hopf-bifurcation	Fig. 4 (f)

[5] S. Farenc, A. Coujou and A. Couret, *Twin propagation in TiAl*, Materials Science and Engineering: A 164(1-2) (1993) 438-442.

[6] J. Friedel, *Dislocations*, Oxford, Pergamon Press, 1964.

[7] S. Graff, S. Forest, J. L. Strudel, C. Prioul, P. Pilvin and J.L. Béchade, *Finite element simulations of dynamic strain ageing effects at V-notches and crak tips*, Scripta Materialia 52 (2005) 1181-1186.

[8] S. Graff, S. Forest, J. L. Strudel, C. Prioul, P. Pilvin and J. L. Béchade, *Strain localization phenomena associated with static and dynamic strain ageing in notched specimens: expriments and finite element simulations*, Materials Sciences and Engineering A (2004) 181-185.

[9] J. Grilhé, N. Junqua, F. Tranchant and J. Vergnol, *Model for instabilities during plastic deformation at constant cross-head velocity*, J. Physique 45 (1984) 939-943.

[10] I. Győri and G. Ladas, *Oscillation theory of delay differential equations*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1991.

[11] J. H. Hale and W. Huang, *Global geometry of the stable regions for two delays differential equations*, J. Math. Anal. Appl. 178 (1993) 344-362.

[12] J. H. Hale and S. M. V. Lunel, *Introduction to functional differential equations*, Springer, 1993.

[13] S. Hilout, M. Boutat and J. Grilhé, *Plastic deformation instabilities: Lambert solutions of Mecking-Lücke equation with delay*, Mathematical Problems in Engineering 2007 Article ID 45951.

[14] S. Hilout, M. Boutat, I. Laadnani and J. Grilhé, *mathematical modelling of plastic deformation instabilities with two delays*, Applied Mathematical Modelling 34(9) (2010) 2484-2492.

[15] J. P. Hirth and J. Lothe, *Theory of dislocations*, Mc Graw-Hill, 1968.

[16] B. Jouffrey, Historique de *la notion de dislocation*, Ecole d'été d'Yravals, 3-14 septembre, Diffusion les éditions de Physique, Publié sous la direction de : P. Groh, L. P. Kubin, J. L. Martin (1979) 1-16.

[17] C. M. Kuo, C. H. Tso and C. H. Lin, *Plastic deformation of Al-Mg alloys during stress rate change test*, Materials Science and Engineering A 519 (2009) 32-37.

[18] X. Li, S. Ruan and J. Wei, *Stability and bifurcation in delay-differential equations with two delays*, J. Math. Anal. Appl. 236 (1999) 254-280.

[19] F. Louchet and Y. Brechet, *Dislocation patterning in uniaxial deformation*, Solid State Phenomena 3/4 (1988) 335–346.

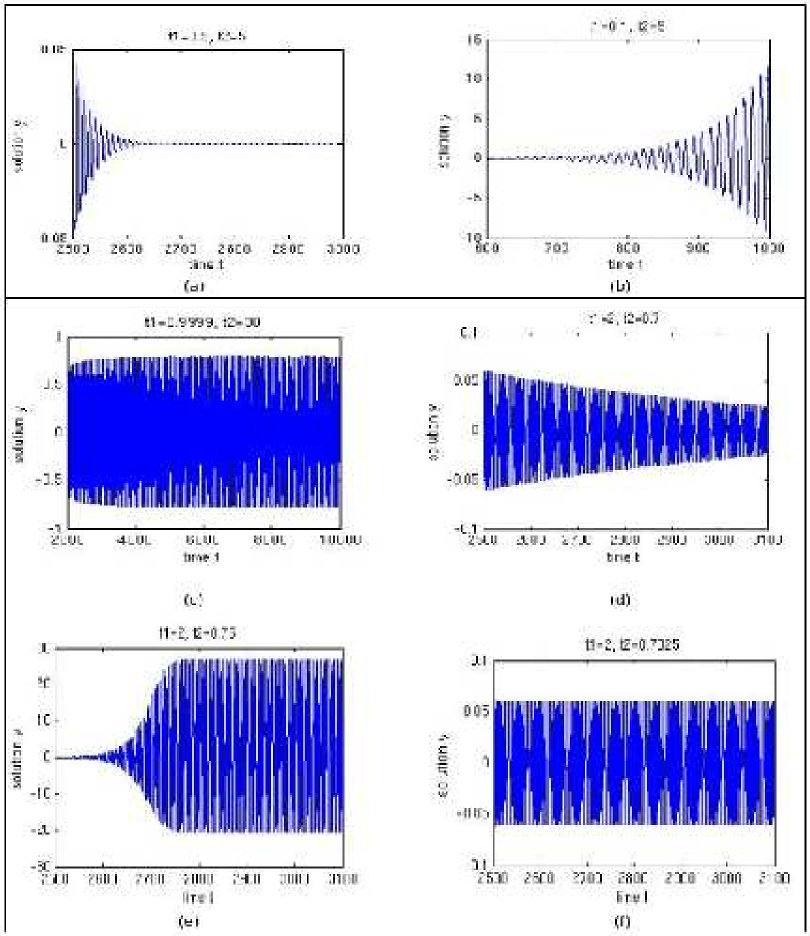

Figure 3. (a): $(\theta_1, \theta_2) = (1.1, 0.9)$, $(\beta_1, \beta_2) = (-0.01, -0.001)$, $m_\phi = 0.05$. (b): $(\theta_1, \theta_2) = (0.9, 1.1)$, $(\beta_1, \beta_2) = (-0.01, -0.01)$, $m_\phi = 0.005$. (c): $(\theta_1, \theta_2) = (0.5, 0.4999)$, $(\beta_1, \beta_2) = (-0.01, -0.01)$, $m_\phi = 0.6$. (d) and (e): $(\theta_1, \theta_2) = (0.8, 1.1)$, $(\beta_1, \beta_2) = (-0.01, -0.01)$, $m_\phi = 0.05$. (f): $(\theta_1, \theta_2) = (0.9, 1.1)$, $(\beta_1, \beta_2) = (-0.01, -0.01)$, $m_\phi = 0.05$

[20] H. Mecking and K. Lücke, *A new aspect of the theory of flow stress of metals*, Scripta Metallurgica 4 (1970) 427–432.

[21] M. C. Miguel, A. Vespignani, S. Zapperi, J. Weiss and J.R. Grasso, *Complexity in dislocation dynamics: model*, Materials Sciences and Engineering A (2001) 324–327.

[22] F. R. N. Nabarro, *Theory of crystal dislocations*, Clarendon, Oxford, 1967.

[23] L. S. Pontryagin, *On the zeros of some elementary transcendental functions*, [Russian] Izv. Akad. Nauk SSSR Ser. Mat. 6 (1942) 541-561; [English translation] Amer. Math. Soc. Transl. 2(1) (1955) 95–110.

[24] A. Portevin and F. Le Chatelier, *Heat treatment of aluminum-copper alloys*, Transactions of the American Society of Steel Treating (1924) 457–478.

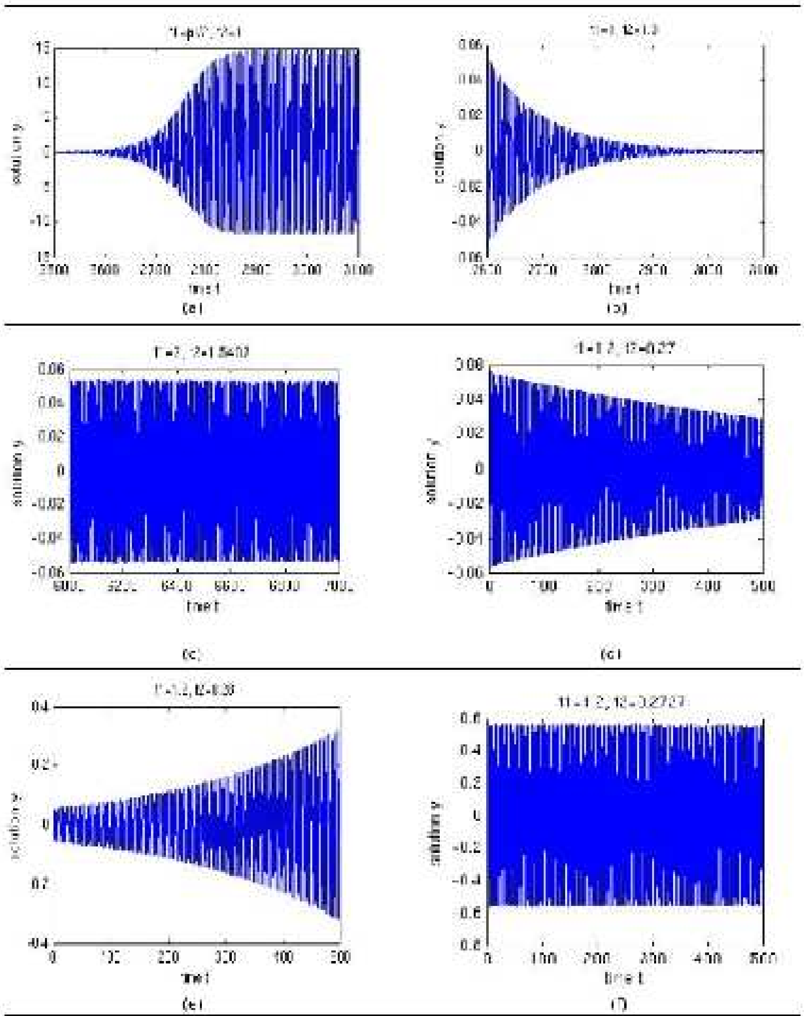

FIGURE 4. (a): $(\theta_1, \theta_2) = (0.6, 0.8)$, $(\beta_1, \beta_2) = (-0.01, -0.01)$, $m_\phi = 0.05$. (b): $(\theta_1, \theta_2) = (0.8, 0.6)$, $(\beta_1, \beta_2) = (-0.01, -0.01)$, $m_\phi = 0.05$. (c): $(\theta_1, \theta_2) = (0.1, 0.9)$, $(\beta_1, \beta_2) = (-0.01, -0.01)$, $m_\phi = 0.05$. (d), (e) and (f): $(\theta_1, \theta_2) = (1.5, 0.7)$, $(\beta_1, \beta_2) = (-0.02, -0.01)$, $m_\phi = 0.5$.

[25] S. Ruan and J. Wei, *On the zeros of transcendental functions with applications to stability of delay differential equations with two delays*, Dyn. of Cont., Disc. and Imp. Systems, Serie A: Math. Analy. 10 (2003) 863–874.

[26] L. F. Shampine and S. Thompson, *Solving DDEs in MATLAB*, Appl. Numer. Math. 37 (2001) 441–458.

[27] H. B. Sun, F. Yoshida, X. Max, T. Kamei and M. Ohmori, *Finite element simulation on the propagation of Lüders band and effect of stress concentration*, Materials Letters 57(21) (2003) 3206–3210.

[28] G. I. Taylor, *The mechanism of plastic deformation of crystals. Part I. Theoretical*, Pro. Roy. Soc. Lond. A 145 (1934) 362–387.

[29] G. I. Taylor, *The mechanism of plastic deformation of crystals. Part II. Comparison with observations*, Pro. Roy. Soc. Lond. A 145 (1934) 388–404.

[30] F. Tranchant, J. Vergnol and P. Franciosi, *On the twinning initiation criterion in Cu–Al alpha single crystal–I. Experimental and numerical analysis of slip and dislocation patterns up to the onset of twinning*, Acta Metallurgica et Materialia 41(5) (1993) 1531-1541.

[31] S. Y. Yang and W. Tong, *Interaction between dislocations and alloying elements and its implication on crystal plasticity of aluminum alloys*, Materials Sciences and Engineering A 309/310 (2001) 300-303.

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