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L^2-WELL-POSED CAUCHY PROBLEM FOR FOURTH-ORDER DISPERSIVE EQUATIONS ON THE LINE

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Abstract. Mizuhara [2] obtained conditions for the Cauchy problem of a fourth-order dispersive operator to be well posed in the L^2 sense. Two of those conditions were shown to be necessary under additional assumptions. In this article, we prove the necessity without the additional assumptions.

1. Introduction

Let L be a fourth-order dispersive operator given by

$$L = D_t - D_x^4 - a(x)D_x^3 - b(x)D_x^2 - c(x)D_x - d(x) \tag{1.1}$$

where $D_t = \frac{1}{i}\partial_t$, $D_x = \frac{1}{i}\partial_x$. We consider the Cauchy problem

$$Lu = f(x,t), \quad (x,t) \in \mathbb{R}^2$$

with the initial data on the line $t = 0$, $u(x,0) = g(x)$.

Mizuhara [2], extending the arguments on [3], obtained the following result.

> The above Cauchy problem is L^2-well-posed if the coefficients $a(x)$, $b(x)$, $c(x)$ satisfy:
>
> $$\Big|\int_{x_0}^{x_1} \Im a(y)\,dy\Big| \le C, \tag{1.2}$$
>
> $$\Big|\int_{x_0}^{x_1} \Im(b(y) - 3a(y)^2/8)\,dy\Big| \le C|x_1 - x_0|^{1/3}, \tag{1.3}$$
>
> $$\Big|\int_{x_0}^{x_1} \Im(c(y) - 2a(y)b(y) + a(y)^3/8)\,dy\Big| \le C|x_1 - x_0|^{2/3} \tag{1.4}$$
>
> for any $x_0, x_1 \in \mathbb{R}$, where $\Im(\cdot)$ is the imaginary part of a complex number.

In the same article, it was shown that (1.2) is necessary for the L^2-well-posedness. While the necessity of conditions (1.3) and (1.4) is shown under the additional assumption that there exist a constant μ such that

$$\Big|\int_{x_0}^{x_1} \Re(b(y) - 3a(y)^2/8 - \mu)\,dy\Big| \le C|x_1 - x_0|^{1/2}, \tag{1.5}$$

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where $\Re(\cdot)$ is the real part of a complex number.

In this article, we show that the conditions (1.3) and (1.4) are necessary for the L^2-well-posedness, without using the additional assumption (1.5).

The method of proof is almost same as that in [2]; that is, under the assumption that the conditions are not satisfied, we construct the sequences of oscillating solutions that are not consistent with the estimates required to be L^2-well-posed. In our construction, we use "time independent" phases. We remark that the idea of the above method has its origin in Mizohata's works on Schrödinger type equations (see for example [1]).

To make our method clear, we consider dispersive operators

$$L[u] = D_t u - D_x^k u - \sum_{j=1}^{k} a_j(x) D_x^{k-j} u$$

with $k \geq 3$. In the next section we draw some necessary conditions for L^2-well-posedness. As for the case $k = 4$, we show the necessity of the conditions (1.3) and (1.4).

In the following, we denote by $B^\infty(\mathbb{R})$ the space of infinitely differentiable functions on $\mathbb{R}$ that are bounded on $\mathbb{R}$ together with all their derivatives of any order. We denote by $\|f(\cdot)\|$ L^2-norm of $f(x)$ given by $\|f(\cdot)\| = \big(\int_{\mathbb{R}} |f(x)|^2\,dx\big)^{1/2}$. We use C or C with some subindex to denote positive constants that may be different, line by line.

2. Main Result

Let L be a dispersive operator given by

$$L[u] = D_t u - D_x^k u - \sum_{j=1}^{k} a_j(x) D_x^{k-j} u \tag{2.1}$$

with $k \geq 3$ and $a_j(x) \in B^\infty(\mathbb{R})$.

Let T be a positive number. Consider the Cauchy problem forward and backward for L;

$$L[u] = f(x,t) \quad (x,t) \in \mathbb{R} \times (-T,T) \tag{2.2}$$

with the initial condition

$$u(x,0) = g(x) \quad x \in \mathbb{R}. \tag{2.3}$$

We say that the Cauchy problem (2.2)–(2.3) is L^2-well-posed, if for any $f(x,t) \in L^1([-T,T], L^2(\mathbb{R}))$ and any $g(x) \in L^2(\mathbb{R})$, there exists one and only one solution $u(x,t)$ in $C^0([-T,T], L^2(\mathbb{R}))$ to the above problem satisfying the following two estimates: for any $t \in [0,T]$,

$$\|u(\cdot,t)\| \leq C\Big(\|g(\cdot)\| + \int_0^t \|f(\cdot,s)\|\,ds\Big), \tag{2.4}$$

$$\|u(\cdot,-t)\| \leq C\Big(\|g(\cdot)\| + \int_{-t}^0 \|f(\cdot,s)\|\,ds\Big), \tag{2.5}$$

where the constant C does not depend on t, $f(x,t)$, or $g(x)$.

We consider the behaviour of the oscillating solution $u(x,t) = e^{i(\xi x + \xi^k t)} U(x,t,\xi)$ to the equation $L[u] = 0$.

Define the operator L_0 by

$$L_0[U] = e^{-i(\xi x+\xi^k t)} L[e^{i(\xi x+\xi^k t)} U].$$

Then we see that

$$L_0 = D_t - \xi^{k-1}(kD_x + a_1(x)) - \sum_{j=2}^{k} \xi^{k-j}\Big(\binom{k}{k-j} D_x^j + \sum_{l=1}^{j} a_l(x)\binom{k-l}{k-j} D_x^{j-l}\Big).$$

Setting $d_1(x) = -a_1(x)/k$ and multiplying $e^{iS_1(x)}$ with $S_1(x) = \int_{x_0}^{x} d_1(y)\,dy$, we eliminate the term $-\xi^{k-1}a_1(x)$ from L_0. That is, defining the operator L_1 by

$$L_1[U] = e^{-iS_1(x)} L_0[e^{iS_1(x)} U],$$

we obtain

$$L_1 = D_t - \xi^{k-1} kD_x - \sum_{j=2}^{k} \xi^{k-j} P_{1,j}(x, D_x)$$

where

$$P_{1,j}(x, D_x) = \sum_{l=0}^{j} b_{j,l}(x) D_x^l.$$

Next, we eliminate the term $-\xi^{k-2} b_{2,0}(x)$ from L_1 by multiplying $e^{iS_2(x)/\xi}$ with $S_2(x) = \int_{x_0}^{x} d_2(y)\,dy$ with $d_2(x) = -b_{2,0}(x)/k$. That is, defining the operator L_2 by

$$L_2[U] = e^{-iS_2(x)/\xi} L_1[e^{iS_2(x)/\xi} U],$$

we see that L_2 satisfies

$$L_2 = D_t - \xi^{k-1} kD_x - \sum_{j=2}^{2k} \xi^{k-j} P_{2,j}(x, D_x)$$

where

$$P_{2,2}(x, D_x) = \sum_{l=1}^{2} c_{2,l}(x) D_x^l.$$

and, for $j > 2$

$$P_{2,j}(x, D_x) = \sum_{l=0}^{\min\{j,k\}} c_{j,l}(x) D_x^l.$$

Repeating this process, we obtain the following result.

Proposition 2.1. *There exist the functions $d_1(x), d_2(x), \dots, d_k(x) \in B^\infty(\mathbb{R})$, such that with $S(x, x_0, \xi)$ defined by*

$$S(x, x_0, \xi) = \sum_{j=1}^{k} \frac{1}{\xi^{j-1}} \int_{x_0}^{x} d_j(y)\,dy$$

the operator L_{00} defined by

$$L_{00}[U] = e^{-iS(x,x_0,\xi)} L_0[e^{iS(x,x_0,\xi)} U],$$

which has the form

$$L_{00} = D_t - \xi^{k-1} kD_x - \sum_{j=2}^{k+k(k-1)} \xi^{k-j} P_j(x, D_x) \tag{2.6}$$

where $P_j(x, D_x)$ is a differential operator of order at most k. In particular for $j = 2, \dots, k$,

$$P_j(x, D_x) = \sum_{q=1}^{j} p_{j,q}(x) D_x^q \,. \tag{2.7}$$

Here the functions $d_j(x)$ are uniquely determined by the coefficients of L.

Remark 2.2. We see from (2.6) and (2.7) that $L_{00}[1] = \sum_{j=1}^{k(k-1)} \xi^{-j} r_j(x)$ with some $r_j(x)$.

Proof of Proposition 2.1. We have to show only the uniqueness. Assume that there exist some $\tilde{d}_j(x)$ $(1 \le j \le k)$ such that the operator $\tilde{L}_{00}$ given by

$$\tilde{L}_{00}[U] = e^{-i\tilde{S}(x,x_0,\xi)} L_0[e^{i\tilde{S}(x,x_0,\xi)} U],$$

where $\tilde{S}(x, x_0, \xi) = \sum_{j=1}^{k} \frac{1}{\xi^{j-1}} \int_{x_0}^{x} \tilde{d}_j(y)\, dy$, has the form similar to L_{00}, that is, $\tilde{L}_{00}[1] = \sum_{j=1}^{k(k-1)} \xi^{-j} \tilde{r}_j(x)$ with some $\tilde{r}_j(x)$.

Since $L_0[U] = e^{iS(x,x_0,\xi)} L_{00}[e^{-iS(x,x_0,\xi)} U]$, we obtain

$$\tilde{L}_{00}[U] = e^{-i(\tilde{S}(x,x_0,\xi) - S(x,x_0,\xi)} L_{00}[e^{i(\tilde{S}(x,x_0,\xi) - S(x,x_0,\xi)} U].$$

Then

$$\sum_{j=1}^{k(k-1)} \xi^{-j} \tilde{r}_j(x) = e^{-i(\tilde{S}(x,x_0,\xi) - S(x,x_0,\xi)} L_{00}[e^{i(\tilde{S}(x,x_0,\xi) - S(x,x_0,\xi)}].$$

Comparing the coefficient of ξ^{k-j} $(j = 1, 2, \dots, k)$, we see that $\tilde{d}_j(x) = d_j(x)$ by the induction on j. $\square$

Note that for the fourth-order operator in (1.1), we have the following: (see also [2])

$$d_1(x) = \frac{-a(x)}{4} \tag{2.8}$$

$$d_2(x) = \frac{-1}{4}(b(x) - \frac{3}{8}a(x)^2 - \frac{3}{2}D_x a(x)) \tag{2.9}$$

$$d_3(x) = \frac{-1}{4}\Big(c(x) + \frac{a(x)^3}{8} - \frac{a(x)b(x)}{2} + D_x(4D_x d_1(x) + 6d_2(x))\Big). \tag{2.10}$$

In this note, we show the following result.

Theorem 2.3. *If the Cauchy problem (2.2)–(2.3) is L^2-well-posed, then the functions $d_1(x), d_2(x), \dots, d_{k-1}(x)$ given in Proposition 2.1, satisfy: For $1 \le j \le k-1$ and any $x_0, x_1 \in \mathbb{R}$,*

$$\Big| \int_{x_0}^{x_1} \Im d_j(y)\, dy \Big| \le C|x_1 - x_0|^{\frac{j-1}{k-1}}. \tag{2.11}$$

By Theorem 2.3, it follows from (2.8), (2.9) and (2.10) that it is necessary that (1.2), (1.3) and (1.4) hold for the Cauchy problem, for the operator given by (1.1), to be L^2-well-posed.

To prove Theorem 2.3, we prepare following propositions.

Proposition 2.4. *If the Cauchy problem* (2.2)-(2.3) *is* L^2*-well-posed, then we have*

$$\Big|\int_{x_0}^{x_1} \Im d_1(y)\,dy\Big| \le C \tag{2.12}$$

for any $x_0, x_1 \in \mathbb{R}$.

Proof. Assuming that $\int_{x_0}^{x_1} \Im d_1(y)\,dy$ is not bounded, we construct the sequence of solutions $u_n(x,t)$ that are not consistent with the estimates (2.4) or (2.5). Indeed, if $\int_{x_0}^{x_1} \Im d_1(y)\,dy$ is not bounded, for any positive integer n we can find $x_{0,n}, x_{1,n} \in \mathbb{R}$ satisfying

$$\Big|\int_{x_{0,n}}^{x_{1,n}} \Im d_1(y)\,dy\Big| > n.$$

Here, we may assume that

$$-\int_{x_{0,n}}^{x_{1,n}} \Im d_1(y)\,dy > n$$

by exchanging $x_{0,n}$ and $x_{1,n}$ if necessary. Now we set $\xi_n = n|x_{1,n} - x_{0,n}|$. We remark that the boundedness of $d_1(x)$ implies that $|x_{1,n} - x_{0,n}| \to \infty$ as $n \to \infty$. Hence $\xi_n \to \infty$ as $n \to \infty$. We choose t_n so that $x_{1,n} = x_{0,n} - kt_n\xi_n^{k-1}$. That is, $t_n = -(x_{1,n} - x_{0,n})/(kn|x_{1,n} - x_{0,n}|\xi_n^{k-2})$. We note that $|t_n\xi_n^{k-2}| = 1/(kn)$ and $t_n \to 0$ as $n \to \infty$.

Since $\xi_n = n|x_{1,n} - x_{0,n}|$, it follows that, if $j \ge 2$,

$$\Big|\frac{1}{\xi_n^{j-1}}\int_{x_{0,n}}^{x_{1,n}} d_j(y)\,dy\Big| \le C.$$

Then, by setting $x_0 = x_{0,n}$ and $\xi = \xi_n$ in $S(x, x_0, \xi)$; that is, $S(x, x_{0,n}, \xi_n) = \sum_{j=1}^{k} \frac{1}{\xi_n^{j-1}} \int_{x_{0,n}}^{x} d_j(y)\,dy$, we have, for large n,

$$|S(x_{1,0}, x_{0,n}, \xi_n) - \int_{x_{0,n}}^{x_{1,n}} d_1(y)\,dy| \le C, \quad -\Im S(x_{1,n}, x_{0,n}, \xi_n) \ge \frac{n}{2}.$$

Consider the case where there exist infinitely many n's such that $t_n > 0$. Then, by choosing a subsequence, we may assume $t_n > 0$ for all $n > 0$. Let $s_n \in [0, t_n]$ be a number satisfying

$$-\Im S(x_{0,n} - ks_n\xi_n^{k-1}, x_{0,n}, \xi_n) = \max_{0\le t\le t_n} -\Im S(x_{0,n} - kt\xi_n^{k-1}, x_{0,n}, \xi_n).$$

Since $x_{0,n} - kt_n\xi_n^{k-1} = x_{1,n}$, we see that $-\Im S(x_{0,n} - ks_n\xi_n^{k-1}, x_{0,n}, \xi_n) \ge n/2$. Pick a non-negative function $g(x) \in C^\infty(\mathbb{R})$ satisfying:

$$g(x) = 0 \quad \text{for } |x| \ge 1, \tag{2.13}$$

$$\int_{\mathbb{R}} g(x)^2\,dx = 1. \tag{2.14}$$

Set

$$u_n(x,t) = e^{i(x\xi_n + t\xi_n^k + S(x, x_{0,n}, \xi_n))} g(x + tk\xi_n^{k-1} - x_{0,n}).$$

Then

$$L[u_n(x,t)] = e^{i(x\xi_n + t\xi_n^k + S(x, x_{0,n}, \xi_n))} L_{00}[g(x + tk\xi_n^{k-1} - x_{0,n})].$$

Noting $(D_t - k\xi_n^{k-1}D_x)g(x + tk\xi_n^{k-1} - x_{0,n}) = 0$, we see that

$$L_{00}[g(x + tk\xi_n^{k-1} - x_{0,n})] = \sum_{0\le j\le k,\ 0\le q\le k^2-2} \xi_n^{k-2-q} r_{q,j}(x) g^{(j)}(x + tk\xi_n^{k-1} - x_{0,n})$$

and

$$\begin{aligned}L[u_n(x,t)] = e^{i(x\xi_n + t\xi_n^k + S(x,x_{0,n},\xi_n))} \\ \times \sum_{0\le j\le k,\ 0\le q\le k^2-2} \xi_n^{k-2-q} r_{q,j}(x) g^{(j)}(x+tk\xi_n^{k-1}-x_{0,n}).\end{aligned}$$

On the support of $g^{(j)}(x+tk\xi_n^{k-1}-x_{0,n})$, where $|x-(x_{0,n}-kt\xi_n^{k-1})|\le 1$, we have

$$|S(x,x_{0,n},\xi_n) - S(x_{0,n}-kt\xi_n^{k-1},x_{0,n},\xi_n)| \le C. \tag{2.15}$$

By the definition of s_n, if $0\le t\le s_n$, $-\Im S(x_{0,n}-kt\xi_n^{k-1},x_{0,n},\xi_n) \le -\Im S(x_{0,n}-ks_n\xi_n^{k-1},x_{0,n},\xi_n)$. Then, if $0\le t\le s_n$, we obtain

$$|L[u_n(x,t)]| \le Ce^{-\Im S(x_{0,n}-ks_n\xi_n^{k-1},x_{0,n},\xi_n)}\xi_n^{k-2}\sum_{j=0}^{k}|g^{(j)}(x+tk\xi_n^{k-1}-x_{0,n})|,$$

from which we obtain

$$\begin{aligned}\int_0^{s_n}\|L[u_n(\cdot,t)]\|\,dt &\le Cs_n\xi_n^{k-2}e^{-\Im S(x_{0,n}-ks_n\xi_n^{k-1},x_{0,n},\xi_n)} \\ &\le C\frac{1}{kn}e^{-\Im S(x_{0,n}-ks_n\xi_n^{k-1},x_{0,n},\xi_n)}.\end{aligned} \tag{2.16}$$

While we obtain

$$\|u_n(\cdot,0)\| \le C \tag{2.17}$$

from

$$u_n(x,0) = e^{i(x\xi_n+S(x,x_{0,n},\xi_n))}g(x-x_{0,n})$$

and (2.15). Here we remark $S(x_{0,n},x_{0,n},\xi_n)=0$.

On the other hand, from

$$u_n(x,s_n) = e^{i(x\xi_n+S(x,x_{0,n},\xi_n))}g(x+ks_n\xi_n^{k-1}-x_{0,n})$$

and (2.15), it follows that

$$\|u_n(\cdot,s_n)\| \ge C_0e^{-\Im S(x_{0,n}-ks_n\xi_n^{k-1},x_{0,n},\xi_n)}. \tag{2.18}$$

If the Cauchy problem is L^2-well-posed, we have estimate (2.4):

$$\|u_n(\cdot,s_n)\| \le C(\|u(\cdot,0)]\| + \int_0^{s_n}\|L[u(\cdot,t)]\|\,dt).$$

Hence estimates (2.16),(2.17) and (2.18) imply

$$e^{-\Im S(x_{0,n}-ks_n\xi_n^{k-1},x_{0,n},\xi_n)} \le C_0^{-1}C(1+\frac{1}{n}e^{-\Im S(x_{0,n}-ks_n\xi_n^{k-1},x_{0,n},\xi_n)}).$$

But since $-\Im S(x_{0,n}-ks_n\xi_n^{k-1},x_{0,n},\xi_n)\to\infty$ as $n\to\infty$, the above estimate is impossible for large n. Then (2.12) has to hold. In the case where there exists an N such that $t_n<0$ for $n>N$, we can construct similarly to the previous case, a sequence of functions $u_n(x,t)$ that are not consistent with estimate (2.5). $\square$

Proposition 2.5. *Let $l\in\{1,2,\dots,k-2\}$. Assume that, for any $j\in\{1,2,\dots,l\}$ and any $x,\xi\in\mathbb{R}$,*

$$\Big|\int_x^{x+\xi^l}\Im d_j(y)\,dy\Big| \le C|\xi|^{j-1}. \tag{2.19}$$

If the Cauchy problem (2.2)–(2.3) *is L^2-well-posed, then*

$$\Big|\sum_{j=1}^{l+1}\frac{1}{\xi^{j-1}}\int_x^{x+\xi^{l+1}}\Im d_j(y)\,dy\Big|\le C \tag{2.20}$$

for any $x,\xi\in\mathbb{R}$ with $\xi\neq 0$.

Proof. Similarly to the proof of Proposition 2.4, assuming that (2.20) is not valid, we construct the sequence of solutions $u_n(x,t)$ that are not consistent with the estimates (2.4) or (2.5). Indeed, if $\sum_{j=1}^{l+1}\frac{1}{\xi^{j-1}}\int_x^{x+\xi^{l+1}}\Im d_j(y)\,dy$ is not bounded, for any positive integer n we can find $x_n\in\mathbb{R}$ and $\xi_n\in\mathbb{R}\setminus\{0\}$ such that

$$\Big|\sum_{j=1}^{l+1}\frac{1}{\xi_n^{j-1}}\int_{x_n}^{x_n+\xi_n^{l+1}}\Im d_j(y)\,dy\Big|>n^2.$$

We note that the boundedness of $d_j(x)$ implies that $|\xi_n|\to\infty$ as $n\to\infty$. We set $y_p=x_n+\frac{p}{n}\xi_n^{l+1}$ ($p=0,1,2,\dots,n$). Then, noting

$$\sum_{p=1}^{n}\int_{y_{p-1}}^{y_p}d_j(y)\,dy=\int_{x_n}^{x_n+\xi_n^{l+1}}d_j(y)\,dy,$$

we see that there exists some p such that

$$\Big|\sum_{j=1}^{l+1}\frac{1}{\xi_n^{j-1}}\int_{y_{p-1}}^{y_p}\Im d_j(y)\,dy\Big|>n.$$

Then, redefining x_n by $x_n=y_{p-1}$, we have

$$\Big|\sum_{j=1}^{l+1}\frac{1}{\xi_n^{j-1}}\int_{x_n}^{x_n+\frac{\xi_n^{l+1}}{n}}\Im d_j(y)\,dy\Big|>n.$$

First we consider the case where for infinitely many n, we have

$$-\sum_{j=1}^{l+1}\frac{1}{\xi_n^{j-1}}\int_{x_n}^{x_n+\frac{\xi_n^{l+1}}{n}}\Im d_j(y)\,dy>n.$$

Then we consider only such n.

We define t_n by $kt_n\xi_n^{k-1}=-\frac{\xi_n^{l+1}}{n}$; that is, $t_n=\frac{-1}{n\xi_n^{k-2-l}}$. We see that $t_n\to 0$ as $n\to\infty$. Similarly to the proof of Proposition 2.4, using the phase function $S(x,x_n,\xi_n)=\sum_{j=1}^{k}\frac{1}{\xi_n^{j-1}}\int_{x_n}^{x}d_j(y)\,dy$ and a non-negative function $g(x)\in C^\infty(\mathbb{R})$ satisfying (2.13) and (2.14), we consider $u_n(x,t)$ given by

$$u_n(x,t)=e^{i(\xi x+t\xi^k+S(x,x_n,\xi_n))}g\Big(\frac{x+kt\xi_n^{k-1}-x_n}{\xi_n^l}\Big)|\xi_n|^{-l/2}.$$

We note that, if $|x+kt\xi_n^{k-1}-x_n|\le|\xi_n|^l$ and $|t|\le|t_n|$,

$$|x-x_n|\le|\xi_n|^l+|kt\xi_n^{k-1}|\le|\xi_n|^l+|\xi_n^{l+1}/n|$$

from which we obtain, on the support of $u_n(x,t)$,

$$\Big|\frac{1}{\xi_n^{j-1}}\int_{x_n}^{x}d_j(y)\,dy\Big|\le C$$

for $j \geq l+2$. Hence, on the support of $u_n(x,t)$,

$$|S(x,x_n,\xi_n) - \sum_{j=1}^{l+1} \frac{1}{\xi_n^{j-1}} \int_{x_n}^{x} d_j(y)\,dy| \leq C \tag{2.21}$$

On the other hand, if $|x + kt\xi_n^{k-1} - x_n| \leq |\xi_n|^l$, the assumption (2.19) on $d_j(x)$ ($j = 1,\dots,l$) of Proposition 2.5 implies that

$$\Big| \int_{x_n}^{x} \Im d_j(y)\,dy - \int_{x_n}^{x_n - kt\xi_n^{k-1}} \Im d_j(y)\,dy\Big| \leq C|\xi_n|^{j-1}$$

which implies that

$$\Big| \sum_{j=1}^{l+1} \frac{1}{\xi_n^{j-1}} \int_{x_n}^{x} \Im d_j(y)\,dy - \sum_{j=1}^{l+1} \frac{1}{\xi_n^{j-1}} \int_{x_n}^{x_n - kt\xi_n^{k-1}} \Im d_j(y)\,dy\Big| \leq C \tag{2.22}$$

on the support of $u_n(x,t)$.

Similarly to the proof of Proposition 2.4, we assume $t_n > 0$ and choose $s_n \in [0,t_n]$ so that

$$-\sum_{j=1}^{l+1} \frac{1}{\xi_n^{j-1}} \int_{x_n}^{x_n - ks_n\xi_n^{k-1}} \Im d_j(y)\,dy = \max_{0\leq t\leq t_n} \Big(-\sum_{j=1}^{l+1} \frac{1}{\xi_n^{j-1}} \int_{x_n}^{x_n - kt\xi_n^{k-1}} \Im d_j(y)\,dy\Big).$$

We have

$$L[u_n(x,t)] = e^{i(\xi x + t\xi^k + S(x,x_n,\xi_n))} L_{00}[g(\frac{x + kt\xi_n^{k-1} - x_n}{\xi_n^l})|\xi_n|^{-l/2}].$$

Note that $(D_t - k\xi_n^{k-1}D_x)g(\frac{x+kt\xi_n^{k-1}-x_n}{\xi_n^l}) = 0$ and

$$D_x^j g(\frac{x + kt\xi_n^{k-1} - x_n}{\xi_n^l}) = g^{(j)}(\frac{x + kt\xi_n^{k-1} - x_n}{\xi_n^l})\xi_n^{-jl}.$$

Then we see from (2.6), (2.7) and $l+2 \leq k$ that

$$\begin{aligned}
&L_{00}[g(\frac{x + kt\xi_n^{k-1} - x_n}{\xi_n^l})|\xi_n|^{-l/2}] \\
&= \sum_{k\geq j\geq 0, k^2-2-l\geq p\geq 0} \xi_n^{k-2-l-p} r_{p,j}(x) g^{(j)}(\frac{x + kt\xi_n^{k-1} - x_n}{\xi_n^l})|\xi_n|^{-l/2}.
\end{aligned}$$

On the support of $g(\frac{x+kt\xi_n^{k-1}-x_n}{\xi_n^l})$ with $0 \leq t \leq s_n$, we have

$$\Big|\Im S(x,x_n,\xi_n) - \sum_{j=1}^{l+1} \frac{1}{\xi_n^{j-1}} \int_{x_n}^{x_n - kt\xi_n^{k-1}} \Im d_j(y)\,dy\Big| \leq C$$

Then if $0 \leq t \leq s_n$, we have

$$\|L[u_n(x,t)]\| \leq C|\xi_n|^{k-2-l} e^{-\Im S(x_n - ks_n\xi_n^{k-1}, x_n, \xi_n)}.$$

Hence

$$\begin{aligned}
&\int_0^{s_n} \|L[u_n(x,t)]\|\,dt \leq Cs_n|\xi_n|^{k-2-l} e^{-\Im S(x_n - ks_n\xi_n^{k-1}, x_n, \xi_n)} \\
&\leq \frac{C}{n} e^{-\Im S(x_n - ks_n\xi_n^{k-1}, x_n, \xi_n)}.
\end{aligned} \tag{2.23}$$

Noting $u_n(x,0) = e^{i(\xi x+S(x,x_n,\xi_n))}g(\frac{x-x_n}{\xi_n^l})|\xi_n|^{-l/2}$, we obtain $|\Im S(x,x_n,\xi_n)| \le C$ on the support of $u_n(x,0)$ from (2.21) and (2.22). Then we have

$$\|u_n(x,0)\| \le C. \tag{2.24}$$

Finally we see from (2.21) and (2.22) that, on the support of $u_n(x,s_n)$,

$$-\Im S(x,x_n,\xi_n) \ge -\Im S(x_n - ks_n\xi_n^{k-1}, x_n, \xi_n) + C,$$

from which we obtain

$$\|u_n(x,s_n)\| \ge C_0 e^{-\Im S(x_n-ks_n\xi_n^{k-1},x_n,\xi_n)}. \tag{2.25}$$

If the Cauchy problem is L^2-well-posed, we have the estimate (2.4), to which we apply (2.23), (2.24) and (2.25). Then we obtain the inequality that is not valid for large n. Hence the estimate (2.20) has to hold.

In the case where there exists some integer $N > 0$ such that $t_n < 0$ for $n > N$. Then we can construct the series of functions $u_n(x,t)$ for which the estimate (2.5) is not valid for large n.

If there exists some integer $N > 0$ such that, for $n > N$,

$$\sum_{j=1}^{l+1} -\frac{1}{\xi_n^{j-1}} \int_{x_n}^{x_n+\xi_n^{l+1}} \Im d_j(y)\,dy < -n,$$

then, by setting, $y_n = x_n + \xi_n^{l+1}$, we have

$$\sum_{j=1}^{l+1} -\frac{1}{\xi_n^{j-1}} \int_{y_n}^{y_n-\xi_n^{l+1}} \Im d_j(y)\,dy > n.$$

By setting $t_n = -\xi_n^{l-k}/n$, as the above argument, we can construct the series of functions $u_n(x,t)$ which are not consistent with the estimates (2.4) or (2.5). $\square$

Remark 2.6. If the coefficient $a_1(x)$ of L is zero, we can obtain the oscillating solutions $u_n(x,t)$ having smaller $L[u_n(x,t)]$ in the power of ξ_n by solving the transport equation. We note that, if $a_1(x) = 0$, the operator $P_2(x,D_x)$ appearing in (2.6) is $P_2(x,D_x) = \binom{k}{2}D_x^2$.

Note that $L_{00}[g(\frac{x-(x_n-kt\xi_n^{k-1})}{\xi_n^l})|\xi_n|^{-l/2}]$ is a sum of

$$\xi_n^p r_{p,j}(x) g^{(j)}(\frac{x-(x_n-kt\xi_n^{k-1})}{\xi_n^l})|\xi_n|^{-l/2}$$

with $0 \le j \le k$ and $-k(k-1) \le p \le k-2-l$. We choose

$$g_{j,p}(x,t) = \xi_n^p \frac{-i}{k\xi_n^{k-1}} \int_{x_n}^{x} r_{p,j}(y)\,dy\, g^{(j)}(\frac{x-(x_n-kt\xi_n^{k-1})}{\xi_n^l})|\xi_n|^{-l/2}$$

as a solution of the transport equation

$$D_t g - k\xi_n^{k-1} D_x g = \xi_n^p r_{p,j}(x) g^{(j)}(\frac{x-(x_n-kt\xi_n^{k-1})}{\xi_n^l})|\xi_n|^{-l/2}.$$

We have

$$
\begin{aligned}
&\xi_n^{k-2} D_x^2 g_{j,p}(x,t) \\
&= \xi_n^p \frac{-1}{k\xi_n} D_x r_{p,j}(x) g^{(j)}\big(\frac{x-(x_n-kt\xi_n^{k-1})}{\xi_n^l}\big)|\xi_n|^{-l/2} \\
&\quad + \xi_n^p \frac{2i}{k\xi_n^{1+l}} r_{p,j}(x) g^{(j+1)}\big(\frac{x-(x_n-kt\xi_n^{k-1})}{\xi_n^l}\big)|\xi_n|^{-l/2} \\
&\quad + \xi_n^p \frac{i\xi_n^{k-2-2l}}{k\xi_n^{k-1}} \int_{x_n}^x r_{p,j}(y)\,dy\, g^{(j+2)}\big(\frac{x-(x_n-kt\xi_n^{k-1})}{\xi_n^l}\big)|\xi_n|^{-l/2}
\end{aligned} \tag{2.26}
$$

and

$$
\begin{aligned}
&\xi_n^{k-3} D_x g_{j,p}(x,t) \\
&= \xi_n^p \frac{-1}{k\xi_n^2} r_{p,j}(x) g^{(j)}\big(\frac{x-(x_n-kt\xi_n^{k-1})}{\xi_n^l}\big)|\xi_n|^{-l/2} \\
&\quad + \xi_n^p \frac{-\xi_n^{k-3-l}}{k\xi_n^{k-1}} \int_{x_n}^x r_{p,j}(y)\,dy\, g^{(j+1)}\big(\frac{x-(x_n-kt\xi_n^{k-1})}{\xi_n^l}\big)|\xi_n|^{-l/2}.
\end{aligned} \tag{2.27}
$$

Then it follows from $1 \le l \le k-2$, $|x-x_n| \le |\xi_n|^l + k|t|\xi_n^{k-1}$ on the support of $g^{(j)}(\frac{x-(x_n-kt\xi_n^{k-1})}{\xi_n^l})$, and $|s_n\xi_n^{k-2-l}| \le 1/n$, that, if $0 \le t \le s_n$ and n is large, L^2 norm of $L_{00}[g(x,t) - \sum g_{j,p}(x,t)]$ is smaller than that of $L_{00}[g(x,t)]$ where $g(x,t) = g(\frac{x-(x_n-kt\xi_n^{k-1})}{\xi_n^l})|\xi_n|^{-l/2}$ and we assume $s_n > 0$. We see also that L^2 norm of $g_{j,p}(x,0)$ is smaller than that of $g(x,0)$ for large n. Taking into account of (2.26) and (2.27), we see that $L_{00}[g(x,t) - \sum g_{j,p}(x,t)]$ is also a linear combination of terms like: $\xi_n^p r_{p,j}(x) g^{(j)}(\frac{x-(x_n-kt\xi_n^{k-1})}{\xi_n^l})|\xi_n|^{-l/2}$. Then we can repeat this process.

Proposition 2.7. *Let $l \in \{1,2,\dots,k-2\}$. Assume that the estimate (2.20) holds for any $x_0, \xi \in \mathbb{R}$ with $\xi \neq 0$.*

Then we see that, for $j = 1,2,\dots,l+1$,

$$
\big|\int_{x_0}^{x_1} \Im d_j(y)\,dy\big| \le C|x_1-x_0|^{(j-1)/(l+1)}. \tag{2.28}
$$

Proof. Indeed, for any integer $p \ge 1$, any $y \in \mathbb{R}$ and any $\eta \in \mathbb{R}\setminus\{0\}$, we see that

$$
\frac{2^{p(j-1)}}{\eta^{j-1}} \int_y^{y+\eta^{l+1}} d_j(y)\,dy = \sum_{q=1}^{2^{p(l+1)}} \frac{1}{(2^{-p}\eta)^{j-1}} \int_{y+(2^{-p}\eta)^{l+1}(q-1)}^{(y+(2^{-p}\eta)^{l+1}(q-1))+(2^{-p}\eta)^{l+1}} d_j(y)\,dy.
$$

Then from (2.20) we obtain

$$
\big|\sum_{j=1}^{l+1} \frac{2^{p(j-1)}}{\eta^{j-1}} \int_y^{y+\eta^{l+1}} \Im d_j(y)\,dy\big| \le C_p. \tag{2.29}
$$

Here the constant C_p may depend on p, but not on y or on η. Hence, by setting $X_j = \frac{1}{\eta^{j-1}} \int_y^{y+\eta^{l+1}} \Im d_j(y)\,dy$ $(j = 1,2,\dots,l+1)$, for $p = 0,1,\dots,l$, we have

$$
\sum_{j=1}^{l+1} 2^{p(j-1)} X_j = K_p
$$

with $|K_p| \le C_p$.

Since the $l+1$-th order matrix whose (i,j) element is $2^{(i-1)(j-1)}$ is invertible, we see that $X_j = \frac{1}{\eta^{j-1}} \int_y^{y+\eta^{l+1}} \Im d_j(y)\,dy$ is bounded on $\mathbb{R}_y \times \mathbb{R}_\eta \setminus \{0\}$. Hence

$$\Big| \int_y^{y+\eta^{l+1}} \Im d_j(y)\,dy \Big| \le C|\eta|^{j-1}$$

which implies

$$\Big| \int_y^{w} \Im d_j(y)\,dy \Big| \le C|w-y|^{(j-1)/(l+1)}$$

for any $y, w \in \mathbb{R}$, where $j = 1, 2, \dots, l+1$. The proof is complete. □

Proof of Theorem 2.3. Using Proposition 2.4, 2.5 and 2.7, we see obviously that the assertion of Theorem 2.3 is valid. □

Remark 2.8. For the operator L, defined in the Introduction, Mizuhara [2] proved Proposition 2.4, Proposition 2.5 in the case of $l = 1$, and Proposition 2.7 in the case of $l = 2$.

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