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POSITIVE SOLUTIONS FOR A SECOND-ORDER SYSTEM WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. This article concerns the existence of positive solutions to a second-order system with integral boundary conditions. By applying Krasnoselskii fixed point theorem, we show the existence of solutions under certain conditions.

1. Introduction

In this article, we investigate the existence of positive solutions to the following system of second order ordinary differential equations with integral boundary conditions:

$$\begin{gathered} x''(t) = -f(t, x(t), y(t)), \quad (t, x, y) \in (0,1) \times [0,+\infty) \times [0,+\infty), \\ y''(t) = -g(t, x(t), y(t)), \quad (t, x, y) \in (0,1) \times [0,+\infty) \times [0,+\infty), \\ x(0) - ax'(0) = \int_0^1 \varphi_0(s)y(s)ds, \quad x(1) + bx'(1) = \int_0^1 \varphi_1(s)y(s)ds, \\ y(0) - ay'(0) = \int_0^1 \psi_0(s)x(s)ds, \quad y(1) + by'(1) = \int_0^1 \psi_1(s)x(s)ds, \end{gathered} \tag{1.1}$$

where $f, g \in C([0,1]\times[0,+\infty)\times[0,+\infty), [0,+\infty))$, $\varphi_0, \varphi_1, \psi_0, \psi_1 \in C([0,1], [0,+\infty))$, a and b are positive real parameters.

Boundary value problems with positive solutions describe many phenomena in the applied sciences found in the theory of nonlinear diffusion generated by nonlinear sources, thermal ignition of gases, and concentration in chemical or biological problems. Readers may refer to [3, 4, 7] for details. In the past few years, much effort has been devoted to the study of the existence of positive solutions to ordinary differential equations or systems with different kinds of boundary conditions, see [1, 10, 11, 12, 13].

On the other hand, problems with integral boundary conditions arise naturally in thermal conduction problems [5], semiconductor problems [9] and hydrodynamic problems [6]. Many authors have investigated scalar problems with integral boundary conditions; see for instance [2, 8, 15, 16]. Particularly, in [2], Boucherif discussed

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the following boundary value problem with integral boundary condition:

$$\begin{gathered} y''(t)=f(t,y(t)),\quad 0<t<1,\\ y(0)-ay'(0)=\int_0^1 g_0(s)y(s)ds,\\ y(1)-by'(1)=\int_0^1 g_1(s)y(s)ds. \end{gathered} \tag{1.2}$$

He obtained the existence of positive solutions of Problem (1.2) by applying Krasnoselskii fixed point theorem in a cone.

However, to the best of our knowledge, there seem to be quite few works on ordinary differential systems of second order with integral boundary conditions. In 2005, Yang [14] studied the following system with integral boundary condition:

$$\begin{gathered} -u''(t)=f(t,u,v),\qquad -v''(t)=g(t,u,v),\\ u(1)=H_1\Big(\int_0^1 u(\tau)d\alpha(\tau)\Big),\quad v(1)=H_2\Big(\int_0^1 v(\tau)d\beta(\tau)\Big),\\ u(0)=v(0)=0, \end{gathered} \tag{1.3}$$

where α and β are increasing nonconstant functions defined on $[0,1]$ with $\alpha(0)=0=\beta(0)$; $f\in C([0,1]\times R^+\times R^+,R^+)$ and $g\in C([0,1]\times R^+\times R^+,R^+)$; and $H_i\in C(R^+,R^+)(i=1,2)$. Here $\int_0^1 u(\tau)d\alpha(\tau)$ and $\int_0^1 v(\tau)d\beta(\tau)$ denote the Riemann-Stieltjes integrals. By using the fixed point index theory in a cone and a priori estimates as the main tools in the proofs, he proved the existence of positive solutions to (1.3).

Motivated by the works mentioned above, we intend to study the existence of positive solutions of Problem (1.1). Compared with the scalar case (1.2) and Problem (1.3), the characteristic of (1.1) is that the two exponents x and y are coupled not only in the equations, but also on the boundary conditions, which make the resolvent kernel $R(t,s)$ in our system much more complicated. This in turn brings substantial difficulties in proving the complete continuity of the operator T (see Section 2 for its definition). Therefore our results cannot be routinely deduced from the ones of (1.2) and (1.3) in the above literature. The outline of this paper is as follows. We present some preliminaries in Section 2 and the main results are proved in Section 3. In Section 4 we will give two examples to illustrate our results.

2. Preliminaries

In this section, we present some propositions and lemmas that will be used in the proof of our main results.

We shall denote by $C[0,1]$ the Banach Space consisting of all continuous functions on $[0,1]$ equipped with the standard norm

$$\|u\|=\max_{0\le t\le 1}|u(t)|,$$

and equip the Banach space $C[0,1]\times C[0,1]$ with the standard norm

$$\|(u,v)\|=\|u\|+\|v\|=\max_{0\le t\le 1}|u(t)|+\max_{0\le t\le 1}|v(t)|.$$

We will use the following assumptions:

((H0) $f,g\in C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$, $\varphi_i,\psi_i\in C([0,1],[0,+\infty))$, $i=1,2$, a and b are positive real parameters.

(H1) φ_0, φ_1 are continuous, positive and the auxiliary function

$$\Phi(t,s) = \frac{1}{1+a+b}[(1+b-t)\varphi_0(s) + (a+t)\varphi_1(s)], \quad t,s \in [0,1],$$

satisfies

$$0 \le m_\Phi := \min\{\Phi(t,s) : t,s \in [0,1]\} \le M_\Phi := \max\{\Phi(t,s) : t,s \in [0,1]\} < 1.$$

(H2) ψ_0, ψ_1 are continuous, positive functions on $[0,1]$ and the auxiliary function

$$\Psi(t,s) = \frac{1}{1+a+b}[(1+b-t)\psi_0(s) + (a+t)\psi_1(s)], \quad t,s \in [0,1],$$

satisfies

$$0 \le m_\Psi := \min\{\Psi(t,s) : t,s \in [0,1]\} \le M_\Psi := \max\{\Psi(t,s) : t,s \in [0,1]\} < 1.$$

Evidently, $(x,y) \in C^2(0,1) \times C^2(0,1)$ is a solution of Problem (1.1) if and only if $(x,y) \in C[0,1] \times C[0,1]$ is a solution to the system of integral equations

$$\begin{aligned} x(t) &= \int_0^1 G(t,s) f(s,x(s),y(s))ds + \frac{1+b-t}{1+a+b}\int_0^1 \varphi_0(s)y(s)ds \\ &\quad + \frac{a+t}{1+a+b}\int_0^1 \varphi_1(s)y(s)ds, \quad t \in [0,1], \\ y(t) &= \int_0^1 G(t,s) g(s,x(s),y(s))ds + \frac{1+b-t}{1+a+b}\int_0^1 \psi_0(s)x(s)ds \\ &\quad + \frac{a+t}{1+a+b}\int_0^1 \psi_1(s)x(s)ds, \quad t \in [0,1], \end{aligned} \tag{2.1}$$

where for $(t,s) \in [0,1] \times [0,1]$,

$$G(t,s) = \begin{cases} k_1(t)k_2(s), & 0 \le t \le s, \\ k_1(s)k_2(t), & 0 \le s \le t\,, \end{cases}$$

where

$$k_1(t) = a+t, \quad k_2(t) = \frac{1+b-t}{1+a+b}.$$

It is clear that $k_1(t) > 0$ and $k_2(t) > 0$ for all $t \in [0,1]$, and $G(t,s) > 0$ for all $(t,s) \in [0,1] \times [0,1]$. Moreover, we have the following propositions:

Proposition 2.1. *There exists a positive continuous function $\gamma : [0,1] \to \mathbb{R}$ such that $G(t,s) \ge \gamma(t)G(s,s)$ for all $t,s \in [0,1]$. Moreover, $\gamma_0 := \min\{\gamma(t) : t \in [0,1]\} > 0$.*

The proof of the above proposition is similar to [2, Lemma 2], and we omit it here.

Proposition 2.2. *Under assumption* (H0)*, for all $t,s \in [0,1]$, we have $G(t,s) \le G(s,s)$.*

The of the above proposition follows standard argument, it is omitted here.

Let us denote two operators $A, B := C[0,1] \times C[0,1] \to C[0,1]$ as follows:

$$\begin{aligned} A(x,y)(t) &= \int_0^1 G(t,s)f(s,x(s),y(s))ds + \int_0^1 \Phi(t,s)y(s)ds, \\ B(x,y)(t) &= \int_0^1 G(t,s)g(s,x(s),y(s))ds + \int_0^1 \Psi(t,s)x(s)ds. \end{aligned}$$

Then we define an operator $T : C[0,1] \times C[0,1] \to C[0,1] \times C[0,1]$ as

$$Tz(t) = \int_0^1 H(t,s)F(s,x(s),y(s))ds + \int_0^1 K(t,s)z(s)ds = \begin{pmatrix} A(x,y)(t) \\ B(x,y)(t) \end{pmatrix}, \quad (2.2)$$

where

$$\begin{gathered} z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \ (x,y) \in C[0,1] \times C[0,1], \quad H(t,s) = \begin{pmatrix} G(t,s) & 0 \\ 0 & G(t,s) \end{pmatrix}, \\ F(s,x(s),y(s)) = \begin{pmatrix} f(s,x(s),y(s)) \\ g(s,x(s),y(s)) \end{pmatrix} \quad K(t,s) = \begin{pmatrix} 0 & \Phi(t,s) \\ \Psi(t,s) & 0 \end{pmatrix}. \end{gathered}$$

It is clear that the existence of a positive solution for (2.1) is equivalent to the existence of a nontrivial fixed point of T in $C[0,1] \times C[0,1]$. To obtain a positive solution of (1.1), we need the following lemma.

Lemma 2.3. *Assume* (H0)–(H2) *hold. Then* $T : C[0,1] \times C[0,1] \to C[0,1] \times C[0,1]$ *is a completely continuous operator.*

Proof. Firstly, we prove that T is a compact operator. That is, for any bounded subset $D \subset C[0,1] \times C[0,1]$, we show that $T(D)$ is relatively compact in $C[0,1] \times C[0,1]$. Since $D \subset C[0,1] \times C[0,1]$ is a bounded subset, there exists a constant $\overline{M} > 0$ such that $\|z\| = \|x\| + \|y\| \le \overline{M}$ for any $z \in D$.

By applying (H0), (H1) and Proposition 2.2, we obtain

$$\begin{aligned} \|A(x,y)\| &= \max_{0 \le t \le 1} \Big| \int_0^1 G(t,s)f(s,x(s),y(s))ds + \int_0^1 \Phi(t,s)y(s)ds \Big| \\ &\le L \int_0^1 |G(s,s)|ds + M_\Phi \overline{M} < +\infty. \end{aligned}$$

Here $L = \max\{f(t,x,y) : 0 \le t \le 1, |x| \le \overline{M}, |y| \le \overline{M}\} + \max\{g(t,x,y) : 0 \le t \le 1, |x| \le \overline{M}, |y| \le \overline{M}\}$. Similarly, we can obtain

$$\|B(x,y)\| \le L \int_0^1 |G(s,s)|ds + M_\Psi \overline{M} < +\infty.$$

Then from the definition of the norm of the product space $C[0,1] \times C[0,1]$, we have

$$\begin{aligned} \|T(z)\| &= \|A(x,y)\| + \|B(x,y)\| \\ &\le 2L \int_0^1 |G(s,s)|ds + (M_\Phi + M_\Psi)\overline{M} < +\infty. \end{aligned}$$

Therefore, $T(D)$ is uniformly bounded with the norm of $C[0,1]\times C[0,1]$. Moreover, for any $t\in(0,1)$, we have

$$
\begin{aligned}
&|\frac{d}{dt}A(x,y)(t)|\\
&=\Big|\Big(\int_0^t G(t,s)f(s,x(s),y(s))ds+\int_t^1 G(t,s)f(s,x(s),y(s))ds\Big)'\\
&\quad -\frac{1}{1+a+b}\int_0^1\varphi_0(s)y(s)ds+\frac{1}{1+a+b}\int_0^1\varphi_1(s)y(s)ds\Big|\\
&=\Big|\frac{1}{1+a+b}\Big[-\int_0^1 sf(s,x(s),y(s))ds-a\int_0^t f(s,x(s),y(s))ds\\
&\quad +(b+1)\int_t^1 f(s,x(s),y(s))ds-\int_0^1\varphi_0(s)y(s)ds+\int_0^1\varphi_1(s)y(s)ds\Big]\Big|\\
&\le\frac{1}{1+a+b}[(2+a+b)L+2K\overline{M}]<+\infty,
\end{aligned}
$$

where

$$
\begin{aligned}
K&=\max\{\varphi_0(t):0\le t\le 1\}+\max\{\varphi_1(t):0\le t\le 1\}\\
&\quad+\max\{\psi_0(t):0\le t\le 1\}+\max\{\psi_1(t):0\le t\le 1\}.
\end{aligned}
$$

Thus, it is easy to prove that $A(D)$ is equicontinuous. This together with the Arzelá-Ascoli theorem guarantees that $A(D)$ is relatively compact in $C[0,1]$.

Similarly, we can prove that $B(D)$ is relatively compact in $C[0,1]$. Therefore, $T(D)$ is relatively compact in $C[0,1]\times C[0,1]$. On the other hand, according to the definition of T, it is easily seen that T is continuous. We obtain that T is completely continuous. The proof is complete. □

We shall discuss the existence of a positive solution of (1.1) by using the following fixed point theorem of cone expansion and compression.

Lemma 2.4 ([2, Theorem 4]). *Let E be a Banach space and $K\subset E$ be a cone. Suppose Ω_1 and Ω_2 are two bounded open sets in Banach space E such that $\theta\in\Omega_1$, $\overline{\Omega_1}\subset\Omega_2$ and suppose that the operator $T:K\cap(\overline{\Omega_2}\setminus\Omega_1)\to K$ is completely continuous such that*

- (A1) *$\|Tx\|\le\|x\|$ for all $x\in K\cap\partial\Omega_1$ and $\|Tx\|\ge\|x\|$ for all $x\in K\cap\partial\Omega_2$ or*
- (A2) *$\|Tx\|\ge\|x\|$ for all $x\in K\cap\partial\Omega_1$ and $\|Tx\|\le\|x\|$ for all $x\in K\cap\partial\Omega_2$.*

Then T has a fixed point in $K\cap(\overline{\Omega_2}\setminus\Omega_1)$.

To use Lemma 2.4, let $E=C[0,1]\times C[0,1]$,

$$P=\{u\in C[0,1],\ u(t)\ge 0,\ t\in[0,1]\},$$

and

$$P_0=\{(u,v)\in P\times P,\min_{0\le t\le 1}((u,v))=\min_{0\le t\le 1}(u(t)+v(t))\ge\frac{1-M}{1-m}\gamma_0\|(u,v)\|\},$$

where

$$M=\max\{M_\Phi,M_\Psi\},\quad m=\min\{m_\Phi,m_\Psi\}.$$

It is easy to see that P_0 is a cone in E.

Lemma 2.5. *Under Assumptions* (H0)–(H2)*, the operator $T:P_0\to P_0$ is a completely continuous.*

Proof. By Lemma 2.3, we only need to prove that $T(P_0) \subset P_0$. Define an operator $N : C[0,1] \times C[0,1] \to C[0,1] \times C[0,1]$ by $N(z)(t) = \int_0^1 K(t,s)z(s)ds$. Then $N(P \times P) \subset P \times P$. Noting that

$$\begin{aligned}\|Nz(t)\| &= \max_{0\le t\le 1} |\int_0^1 \Phi(t,s)y(s)ds| + \max_{0\le t\le 1} |\int_0^1 \Psi(t,s)x(s)ds| \\ &\le \max\{M_\Phi, M_\Psi\}\|z\|,\end{aligned}$$

one has $\|N\| \le \max\{M_\Phi, M_\Psi\} < 1$. Then $I - N$ is invertible. Similarly to [2, Lemma 3], we have

$$Tz(t) = \int_0^1 H(t,s)F(s,x(s),y(s))ds + \int_0^1 R(t,s)\int_0^1 H(s,\tau)F(\tau,x(\tau),y(\tau))d\tau ds,$$

where $R(t,s)$ is the resolvent kernel by $R(t,s) = \sum_{j=1}^\infty K_j(t,s)$ and $K_j(t,s) = \int_0^1 K(t,\tau)K_{j-1}(\tau,s)d\tau$, $j = 2,3,\dots$, and $K_1(t,s) = K(t,s)$. Let

$$R(t,s) = \begin{pmatrix} R_1(t,s) & R_2(t,s) \\ R_3(t,s) & R_4(t,s)\end{pmatrix}.$$

It can be easily verified that

$$\begin{gathered}\frac{m^2}{1-m^2} \le R_1(t,s),\ R_4(t,s) \le \frac{M^2}{1-M^2}, \\ \frac{m}{1-m^2} \le R_2(t,s),\ R_3(t,s) \le \frac{M}{1-M^2}.\end{gathered}$$

From Propositions 2.1, 2.2 and (H1), we find that

$$\begin{aligned}\|Tz(t)\| \le \frac{1}{1-M}[&\int_0^1 G(\tau,\tau)f(\tau,x(\tau),y(\tau))d\tau \\ &+ \int_0^1 G(\tau,\tau)g(\tau,x(\tau),y(\tau))d\tau],\end{aligned} \tag{2.3}$$

$$\begin{aligned}\min_{0\le t\le 1} Tz(t) \ge \frac{\gamma_0}{1-m}[&\int_0^1 G(\tau,\tau)f(\tau,x(\tau),y(\tau))d\tau \\ &+ \int_0^1 G(\tau,\tau)g(\tau,x(\tau),y(\tau))d\tau].\end{aligned} \tag{2.4}$$

By (2.3) and (2.4), we have

$$\min_{0\le t\le 1} Tz(t) \ge \frac{1-M}{1-m}\gamma_0\|Tz(t)\|.$$

Therefore, $T(P_0) \subset P_0$. $\square$

3. Main results

In this section, we show the existence of positive solutions to (1.1). Firstly, we introduce some notation.

$$\begin{aligned} f_\beta &= \liminf_{|x|+|y|\to\beta} \min_{0\le t\le 1} \frac{f(t,x,y)}{|x|+|y|}, \quad f^\beta = \limsup_{|x|+|y|\to\beta} \max_{0\le t\le 1} \frac{f(t,x,y)}{|x|+|y|}, \\ g_\beta &= \liminf_{|x|+|y|\to\beta} \min_{0\le t\le 1} \frac{g(t,x,y)}{|x|+|y|}, \quad g^\beta = \limsup_{|x|+|y|\to\beta} \max_{0\le t\le 1} \frac{g(t,x,y)}{|x|+|y|},\end{aligned}$$

where $\beta = 0$ or ∞.

Theorem 3.1. *Assume that* (H0)–(H2) *hold. If*
$$f^0, g^0 < \frac{1-M}{2\int_0^1 G(s,s)ds} \quad \text{and} \quad f_\infty, g_\infty > \frac{(1-m)^2}{2\gamma_0^2(1-M)\int_0^1 G(s,s)ds},$$
then Problem (1.1) *has at least one positive solution.*

Proof. Since $f^0, g^0 < \frac{1-M}{2\int_0^1 G(s,s)ds}$, there exists an $r > 0$, such that $f(t,x,y) \le (f^0+\varepsilon_1)(|x|+|y|)$, and $g(t,x,y) \le (g^0+\varepsilon_1)(|x|+|y|)$ for $t \in [0,1]$, $|x|+|y| \le r$, where ε_1 satisfies $f^0+\varepsilon_1 \le \frac{1-M}{2\int_0^1 G(s,s)ds}$ and $g^0+\varepsilon_1 \le \frac{1-M}{2\int_0^1 G(s,s)ds}$.

Let $\Omega_1 = \{z=(x,y) \in P \times P,\ \|z\| < r\}$. For any $z=(x,y) \in \partial\Omega_1 \cap P_0$, we have
$$\begin{aligned}\|Tz\| &\le \frac{1}{1-M}\int_0^1 G(s,s)ds \cdot (f^0+\varepsilon_1+g^0+\varepsilon_1)\cdot \|(x,y)\| \\ &\le \|z\|.\end{aligned} \tag{3.1}$$
On the other hand, since $f_\infty, g_\infty > \frac{(1-m)^2}{2\gamma_0^2(1-M)\int_0^1 G(s,s)ds}$, there exists an $R > r > 0$, such that $f(t,x,y) \ge (f_\infty-\varepsilon_2)(|x|+|y|)$ and $g(t,x,y) \ge (g_\infty-\varepsilon_2)(|x|+|y|)$ for $t \in [0,1]$, $|x|+|y| \ge R$, where ε_2 satisfies $f_\infty-\varepsilon_2 \ge \frac{(1-m)^2}{2\gamma_0^2(1-M)\int_0^1 G(s,s)ds}$ and $g_\infty-\varepsilon_2 \ge \frac{(1-m)^2}{2\gamma_0^2(1-M)\int_0^1 G(s,s)ds}$.

Let $\Omega_2 = \{z=(x,y) \in P \times P,\ \|z\| < R_1\}$, where $R_1 = \frac{1-m}{(1-M)\gamma_0}R$. For any $z=(x,y) \in \partial\Omega_2 \cap P_0$, we have
$$\begin{aligned}\|Tz(t)\| \ge \min_{0\le t\le 1} Tz(t) &\ge \frac{\gamma_0^2(1-M)}{(1-m)^2}\int_0^1 G(s,s)ds \cdot (f_\infty-\varepsilon_2+g_\infty-\varepsilon_2)\cdot\|z\| \\ &\ge \|z\|.\end{aligned} \tag{3.2}$$
Applying Lemma 2.4 to (3.1) and (3.2) yields that T has a fixed point $z^* \in P_0 \cap (\overline{\Omega_2}\backslash\Omega_1)$ and hence z^* is a positive solution of (1.1). $\square$

From the proof of Theorem 3.1, we can also obtain the following result.

Theorem 3.2. *Assume that* (H0)–(H2) *hold. If* $f^\infty, g^\infty < \frac{1-M}{2\int_0^1 G(s,s)ds}$ *and* $f_0, g_0 > \frac{(1-m)^2}{2\gamma_0^2(1-M)\int_0^1 G(s,s)ds}$, *then* (1.1) *has at least one positive solution.*

Next we discuss the multiplicity of positive solutions for Problem (1.1). We obtain the following results.

Theorem 3.3. *Assume that* (H0)–(H2) *hold, and*

(i) $f_0 > \frac{(1-m)^2}{\gamma_0^2(1-M)\int_0^1 G(s,s)ds}$ *and* $g_\infty > \frac{(1-m)^2}{\gamma_0^2(1-M)\int_0^1 G(s,s)ds}$;

(ii) *There exists an* $l > 0$ *such that* $\max_{0\le t\le 1,(x,y)\in\partial\Omega_1} f(t,x,y) < \frac{1-M}{2\int_0^1 G(s,s)ds}l$ *and* $\max_{0\le t\le 1,(x,y)\in\partial\Omega_1} g(t,x,y) < \frac{1-M}{2\int_0^1 G(s,s)ds}l$, *where* $\Omega_1 := \{z=(x,y) \in P \times P, \|z\| < l\}$.

Then Problem (1.1) *has at least two positive solutions.*

Proof. Since $f_0 > \frac{(1-m)^2}{\gamma_0^2(1-M)\int_0^1 G(s,s)ds}$, we can choose $\varepsilon_3 > 0$ such that $f_0-\varepsilon_3 \ge \frac{(1-m)^2}{\gamma_0^2(1-M)\int_0^1 G(s,s)ds}$, and also there exists an $0 < l_1 < l$, such that $f(t,x,y) \ge (f_0 -$

$\varepsilon_3)(|x|+|y|)$ for $t \in [0,1], |x|+|y| \le l_1$. Let $\Omega_{l_1} := \{z=(x,y) \in P \times P, \|z\| < l_1\}$, For any $z=(x,y) \in \partial\Omega_{l_1} \cap P_0$, we have

$$\begin{aligned} \|Tz\| &\ge \frac{\gamma_0}{1-m}\int_0^1 G(s,s)ds \cdot (f_0-\varepsilon_3)(\frac{1-M}{1-m}\gamma_0\|z\|) \\ &\ge \|z\|. \end{aligned} \tag{3.3}$$

Again, by using $g_\infty \ge \frac{(1-m)^2}{\gamma_0^2(1-M)\int_0^1 G(s,s)ds}$, we can choose $\varepsilon_4 > 0$, such that $g_\infty - \varepsilon_4 \ge \frac{(1-m)^2}{\gamma_0^2(1-M)\int_0^1 G(s,s)ds}$, and also there exists an $l_2 > l$, such that $g(t,x,y) \ge (g_\infty - \varepsilon_4)(|x|+|y|)$ for $t \in [0,1], |x|+|y| \ge l_2$. Let $\Omega_{\widetilde{l_2}} := \{z=(x,y) \in P \times P, \|z\| < \widetilde{l_2}\}$, where $\widetilde{l_2} = \frac{1-m}{(1-M)\gamma_0} l_2$. For any $z=(x,y) \in \partial\Omega_{\widetilde{l_2}} \cap P_0$, we have

$$\begin{aligned} \|Tz\| &\ge \frac{\gamma_0}{1-m}\int_0^1 G(s,s)ds \cdot (g_\infty-\varepsilon_4)(\frac{1-M}{1-m}\gamma_0\|z\|) \\ &\ge \|z\|. \end{aligned} \tag{3.4}$$

By (ii), for $z=(x,y) \in \partial\Omega_1 \cap P_0$, we have

$$\begin{aligned} \|Tz(t)\| &\le \frac{1}{1-M}[\int_0^1 G(\tau,\tau)f(\tau,x(\tau),y(\tau))d\tau + \int_0^1 G(\tau,\tau)g(\tau,x(\tau),y(\tau))d\tau] \\ &< \frac{1}{1-M}\int_0^1 G(s,s)ds \cdot \frac{1-M}{\int_0^1 G(s,s)ds} l = l. \end{aligned} \tag{3.5}$$

Therefore, from (3.3), (3.5) and Lemma 2.4, it follows that (1.1) has at least one positive solution $z_1 \in P_0$ with $l_1 \le \|z_1\| < l$. Similarly, from (3.4), (3.5) and Lemma 2.4, it follows that (1.1) has at least one positive solution $z_2 \in P_0$ with $l < \|z_2\| \le \widetilde{l_2}$. Therefore, (1.1) has at least two positive solutions. The proof is complete. □

Similarly, we have the following results.

Theorem 3.4. *Assume* (H0)–(H2), *and*

(i) $f_0, f_\infty > \frac{(1-m)^2}{\gamma_0^2(1-M)\int_0^1 G(s,s)ds}$ *or* $g_0, f_\infty > \frac{(1-m)^2}{\gamma_0^2(1-M)\int_0^1 G(s,s)ds}$ *or* $g_0, g_\infty > \frac{(1-m)^2}{\gamma_0^2(1-M)\int_0^1 G(s,s)ds}$;

(ii) *There exists an* $l > 0$ *such that* $\max_{0\le t\le 1,(x,y)\in\partial\Omega_1} f(t,x,y) < \frac{1-M}{2\int_0^1 G(s,s)ds} l$ *and* $\max_{0\le t\le 1,(x,y)\in\partial\Omega_1} g(t,x,y) < \frac{1-M}{2\int_0^1 G(s,s)ds} l$, *where* $\Omega_1 := \{z=(x,y) \in P \times P, \|z\| < l\}$.

Then Problem (1.1) *has at least two positive solutions.*

4. Examples

Example 4.1. Set $f(t,x,y) = \sqrt{\frac{1+t}{8}}(x^2+y^2)$, $g(t,x,y) = \sqrt{1-\frac{t}{4}}[(x^2+y^2)^2+(x^2+y^2)e^{-(x^2+y^2)}]$, $a=1$, $b=1$, $\varphi_i = \psi_i = 1/3$, $i=0,1$. Then $\int_0^1 G(s,s)ds = 13/6$, $\gamma_0 = 1/6$, $M=m=1/3$,

$$\frac{1-M}{2\int_0^1 G(s,s)ds} = \frac{2}{13}, \quad \frac{(1-m)^2}{2\gamma_0^2(1-M)\int_0^1 G(s,s)ds} = \frac{72}{13}.$$

Then conditions of Theorem 3.1 are satisfied. We obtain that Problem (1.1) has at least one positive solution.

Example 4.2. Set $f(t,x,y) = \sqrt{\frac{1+t}{32}} \cdot \sqrt[3]{x^2+y^2}$, $g(t,x,y) = \frac{2-t}{208}(x^2+y^2)(1+e^{-(x^2+y^2)})$, $a=1$, $b=1$, $l=6$, $\varphi_i = \psi_i = 1/3$, $i=0,1$. Then $\int_0^1 G(s,s)ds = 13/6$, $\gamma_0 = 1/6$, $M = m = 1/3$,

$$\frac{1-M}{2\int_0^1 G(s,s)ds} = \frac{2}{13}, \quad \frac{(1-m)^2}{\gamma_0^2(1-M)\int_0^1 G(s,s)ds} = \frac{144}{13}.$$

Then the conditions of Theorem 3.3 are satisfied. We obtain that Problem (1.1) has at least two positive solution.

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