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PICONE'S IDENTITY FOR THE P-BIHARMONIC OPERATOR WITH APPLICATIONS

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Abstract. In this article, a Picone-type identity for the weighted p-biharmonic operator is established and comparison results for a class of half-linear partial differential equations of fourth order based on this identity are derived.

1. Introduction

The purpose of this article is to present a Picone-type identity for the weighted p-biharmonic operator extending the known formula for a pair of ordinary iterated Laplacians with positive weights a and A which says that if $u, v, a\Delta u$ and $A\Delta v$ are twice continuously differentiable functions with $v(x) \neq 0$, then

$$\begin{aligned} &\operatorname{div}\big[u\nabla(a\Delta u) - a\Delta u\nabla u - \frac{u^2}{v}\nabla(A\Delta v) + A\Delta v\nabla\big(\frac{u^2}{v}\big)\big] \\ &= -\frac{u^2}{v}\Delta(A\Delta v) + u\Delta(a\Delta u) + (A-a)(\Delta u)^2 \\ &\quad - A\big(\Delta u - u\frac{\Delta v}{v}\big)^2 + 2A\frac{\Delta v}{v}|\nabla u - \frac{u}{v}\nabla v|^2 \end{aligned} \tag{1.1}$$

(see [3]). Here $\operatorname{div}, \nabla, \Delta$ are the usual divergence, nabla and Laplace operators and $|\cdot|$ denotes the Euclidean length of a vector in $\mathbb{R}^n$. In [3], the integrated form of (1.1) was used to obtain a variety of qualitative results (including Sturmian comparison theorems, integral inequalities of the Wirtinger type and lower bounds for eigenvalues) for a pair of linear elliptic partial differential equations of the form

$$\Delta(a(x)\Delta u) - c(x)u = 0, \tag{1.2}$$

$$\Delta(A(x)\Delta v) - C(x)v = 0 \tag{1.3}$$

(or for the inequalities $u[\Delta(a(x)\Delta u) - c(x)u] \le 0$ and $\Delta(A(x)\Delta v) - C(x)v \ge 0$) considered in a bounded domain $G \subset \mathbb{R}^n$ with a piecewise smooth boundary ∂G.

We extend the formula (1.1) to the case where $\Delta(a\Delta u)$ and $\Delta(A\Delta v)$ are replaced by the more general weighted p-biharmonic operators $\Delta(a|\Delta u|^{p-2}\Delta u)$ and

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$\Delta(A|\Delta v|^{p-2}\Delta v), p > 1$, respectively, and show that some of results in [3] remain valid for half-linear partial differential equations

$$\Delta(a(x)|\Delta u|^{p-2}\Delta u) - c(x)|u|^{p-2}u = 0, \tag{1.4}$$

$$\Delta(A(x)|\Delta v|^{p-2}\Delta v) - C(x)|v|^{p-2}v = 0 \tag{1.5}$$

which reduce to (1.2) and (1.3) when $p = 2$.

This article is organized as follows. In Section 2, we establish several forms of the desired generalization of Picone-Dunninger formula. Next, in Section 3, we illustrate applications of the basic identities by deriving Sturmian comparison theorems and other qualitative results concerning differential equations and inequalities involving the weigthed p-bilaplacian.

For related results in the particular case $n = 1$ see [4] (general $p > 1$) and [5] ($p = 2$). Picone identities for various kinds of half-linear partial differential equations of the second order and their applications can be found in the monographs [1, 6].

2. Picone's Identity

Let G be a bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary ∂G and let $a \in \mathrm{C}^2(\bar{G}, \mathbb{R}_+)$, $A \in \mathrm{C}^2(\bar{G}, \mathbb{R}_+)$, $c \in \mathrm{C}(\bar{G}, \mathbb{R})$ and $C \in \mathrm{C}(\bar{G}, \mathbb{R})$ where $\mathbb{R}_+ = (0, \infty)$. For a fixed $p > 1$ define the function $\varphi_p : \mathbb{R} \to \mathbb{R}$ by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$, and consider partial differential operators of the form

$$l[u] = \Delta(a(x)\varphi_p(\Delta u)) - c(x)\varphi_p(u),$$
$$L[v] = \Delta(A(x)\varphi_p(\Delta v)) - C(x)\varphi_p(v)$$

with the domains $\mathcal{D}_l(G)$ (resp. $\mathcal{D}_L(G)$) defined to be the sets of all functions u (resp. v) of class $\mathrm{C}^2(\bar{G}, \mathbb{R})$ such that $a(x)\varphi_p(u)$ (resp. $A(x)\varphi_p(v)$) are in $\mathrm{C}^2(G, \mathbb{R}) \cap \mathrm{C}(\bar{G}, \mathbb{R})$.

Also, denote by Φ_p the form defined for $X, Y \in \mathbb{R}$ and $p > 1$ by

$$\Phi_p(X, Y) := X\varphi_p(X) + (p-1)Y\varphi_p(Y) - pX\varphi_p(Y).$$

From the Young inequality it follows that $\Phi_p(X, Y) \geq 0$ for all $X, Y \in \mathbb{R}$ and the equality holds if and only if $X = Y$.

We begin with the following lemma which can be verified by a routine computation. We call it a *weaker form of Picone's identity* because of the relative weak hypothesis that u is an arbitrary twice continuously differentiable function which does not need to satisfy any differential equation or inequality nor even to be in the domain of the operator l.

Lemma 2.1. *If $u \in \mathrm{C}^2(\bar{G}, \mathbb{R}), v \in \mathcal{D}_L(G)$ and v does not vanish in G, then*

$$\begin{aligned} &\operatorname{div}\big[-\frac{|u|^p}{\varphi_p(v)}\nabla(A\varphi_p(\Delta v)) + A\varphi_p(\Delta v)\nabla\big(\frac{|u|^p}{\varphi_p(v)}\big)\big] \\ &= -\frac{|u|^p}{\varphi_p(v)}L[v] + A|\Delta u|^p - C|u|^p - A\Phi_p\big(\Delta u, u\frac{\Delta v}{v}\big) \\ &\quad + p(p-1)A|u|^{p-2}\varphi_p\big(\frac{\Delta v}{v}\big)\big|\nabla u - \frac{u}{v}\nabla v\big|^2. \end{aligned} \tag{2.1}$$

An integration of (2.1) with the use of the divergence theorem gives the Picone identity in the integral form

$$\begin{aligned} &-\int_{\partial G}\frac{|u|^p}{\varphi_p(v)}\frac{\partial(A\varphi_p(\Delta v))}{\partial\nu}ds+\int_{\partial G}(p-1)A\varphi_p\big(\frac{\Delta v}{v}\big)\big[\frac{\varphi_p}{v}\big(v\frac{\partial u}{\partial\nu}-u\frac{\partial v}{\partial\nu}\big)\big]ds\\ &+\int_{\partial G}A\varphi_p\big(\frac{\Delta v}{v}\big)\varphi_p(u)\frac{\partial u}{\partial\nu}ds\\ &=-\int_G\frac{|u|^p}{\varphi_p(v)}L[v]dx+\int_G\big[A|\Delta u|^p-C|u|^p\big]dx\\ &\quad+\int_G\big[p(p-1)A|u|^{p-2}\varphi_p\big(\frac{\Delta v}{v}\big)\big|\nabla u-\frac{u}{v}\nabla v\big|^2-A\Phi_p\big(\Delta u,u\frac{\Delta v}{v}\big)\big]dx, \end{aligned} \tag{2.2}$$

where $\partial/\partial\nu$ denotes the exterior normal derivative, which extends the formula in [3, Theorem 2.1].

Adding to (2.1) the obvious identity

$$\operatorname{div}\big[u\nabla(a\varphi_p(\Delta u))-a\varphi_p(\Delta u)\nabla u\big]=ul[u]-a|\Delta u|^p+c|u|^p,$$

which holds for any $u\in\mathcal{D}_l(G)$, yields the following stronger form of Picone's formula.

Lemma 2.2. *If $u\in\mathcal{D}_l(G),v\in\mathcal{D}_L(G)$ and $v(x)\neq0$ in G, then*

$$\begin{aligned} &\operatorname{div}\big[u\nabla(a\varphi_p(\Delta u))-a\varphi_p(\Delta u)\nabla u-\frac{|u|^p}{\varphi_p(v)}\nabla(A\varphi_p(\Delta v))\\ &\quad+A\varphi_p(\Delta v)\nabla\big(\frac{|u|^p}{\varphi_p(v)}\big)\big]\\ &=-\frac{|u|^p}{\varphi_p(v)}L[v]+ul[u]+(A-a)|\Delta u|^p+(c-C)|u|^p-A\Phi_p\big(\Delta u,u\frac{\Delta v}{v}\big)\\ &\quad+p(p-1)A|u|^{p-2}\varphi_p\big(\frac{\Delta v}{v}\big)\big|\nabla u-\frac{u}{v}\nabla v\big|^2. \end{aligned} \tag{2.3}$$

Again, integrating (2.3) and using the divergence theorem we easily obtain the following integral version of the formula which generalizes the result from Dunninger [3, Theorem 2.2]:

$$\begin{aligned} &\int_{\partial G}\frac{u}{\varphi_p(v)}\big[\varphi_p(v)\frac{\partial(a\varphi_p(\Delta u))}{\partial\nu}-\varphi_p(u)\frac{\partial(A\varphi_p(\Delta v))}{\partial\nu}\big]ds\\ &\quad+\int_{\partial G}(p-1)A\varphi_p\big(\frac{\Delta v}{v}\big)\big[\frac{\varphi_p(u)}{v}\big(v\frac{\partial u}{\partial\nu}-u\frac{\partial v}{\partial\nu}\big)\big]ds\\ &\quad+\int_{\partial G}\frac{1}{\varphi_p(v)}\frac{\partial u}{\partial\nu}\big[A\varphi_p(u)\varphi_p(\Delta v)-a\varphi_p(v)\varphi_p(\Delta u)\big]ds\\ &=\int_G\frac{u}{\varphi_p(v)}\{\varphi_p(v)l[u]-\varphi_p(u)L[v]\}dx\\ &\quad+\int_G\big[(A-a)|\Delta u|^p+(c-C)|u|^p\big]dx\\ &\quad+\int_G\big[p(p-1)A|u|^{p-2}\varphi_p\big(\frac{\Delta v}{v}\big)\big|\nabla u-\frac{u}{v}\nabla v\big|^2-A\Phi_p\big(\Delta u,u\frac{\Delta v}{v}\big)\big]dx. \end{aligned} \tag{2.4}$$

3. Applications

As a first application of identity (2.2) we prove the following result.

Theorem 3.1. *If there exists a nontrivial function $u \in \mathrm{C}^2(\bar{G}, \mathbb{R})$ such that*

$$u = 0 \quad \text{on } \partial G, \tag{3.1}$$

$$M_p[u] \equiv \int_G \big[A(x)|\Delta u|^p - C(x)|u|^p\big]dx \le 0, \tag{3.2}$$

then there does not exist a $v \in \mathcal{D}_L(G)$ which satisfies

$$L[v] \ge 0 \quad \text{in } G, \tag{3.3}$$

$$v > 0 \quad \text{on } \partial G, \tag{3.4}$$

$$\Delta v < 0 \quad \text{in } G. \tag{3.5}$$

Proof. Suppose to the contrary that there exists a $v \in \mathcal{D}_L(G)$ satisfying (3.3)-(3.5). Since $v > 0$ on ∂G and $\Delta u < 0$ in G, the maximum principle implies that $v > 0$ on $\bar{G}$. Thus, the integral identity (2.2) is valid and it implies, in view of the hypotheses (3.1)-(3.5), that

$$\begin{aligned} 0 &\ge M_p[u] - \int_G \frac{|u|^p}{\varphi_p(v)} L[v]dx \\ &= -\int_G \big[p(p-1)A|u|^{p-2}\varphi_p\big(\frac{\Delta v}{v}\big)\big|\nabla u - \frac{u}{v}\nabla v\big|^2 - A\Phi_p\big(\Delta u, u\frac{\Delta v}{v}\big)\big]dx \\ &\ge -\int_G p(p-1)A|u|^{p-2}\varphi_p\big(\frac{\Delta v}{v}\big)\big|\nabla u - \frac{u}{v}\nabla v\big|^2 dx \ge 0. \end{aligned}$$

It follows that $\nabla u - \frac{u}{v}\nabla v = 0$ in G and therefore $u/v = k$ in $\bar{G}$ for some nonzero constant k. Since $u = 0$ on ∂G and $v > 0$ on ∂G, we have a contradiction. Hence no v satisfying (3.3)-(3.5) can exist. □

Theorem 3.2. *If there exists a nontrivial $u \in \mathrm{C}^2(\bar{G}, \mathbb{R})$ which satisfies* (3.1) *and* (3.2)*, then every solution $v \in \mathcal{D}_L(G)$ of the inequality* (3.3) *satisfying* (3.5) *and*

$$v(x) > 0 \quad \text{for some } x \in G \tag{3.6}$$

has a zero in $\bar{G}$.

Proof. If the function v satisfies (3.3), (3.5) and (3.6), then either $v(x) < 0$ for some $x \in \partial G$, and so v must vanish somewhere in G, or $v \ge 0$ on ∂G. In the latter case, however, Theorem 3.1 implies that $v(x) = 0$ for some $x \in \partial G$, and the proof is complete. □

As an immediate consequence of Theorem 3.2 we obtain the following integral inequality of the Wirtinger type.

Corollary 3.3. *If there exists a $v \in \mathcal{D}_L(G)$ such that $L[v] = 0$, $v > 0$ and $\Delta v < 0$ in G, then for any nontrivial function $u \in \mathrm{C}^2(\bar{G}, \mathbb{R})$ satisfying $u = 0$ on ∂G, we have*

$$\int_G A(x)|\Delta u|^p dx \ge \int_G C(x)|u|^p dx\,.$$

As a further application of Picone's identities established in Section 2 we derive the Sturmian comparison theorem. It belongs to weak comparison results in the sense that the conclusion with respect to v applies (similarly as in Theorem 3.2) to $\bar{G}$ rather than G.

Theorem 3.4. *If there exists a nontrivial $u \in \mathcal{D}_l(G)$ such that*

$$\int_G ul[u]dx \le 0, \tag{3.7}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial G, \tag{3.8}$$

$$V_p[u] \equiv \int_G \big[(a-A)|\Delta u|^p + (C-c)|u|^p\big]dx \ge 0, \tag{3.9}$$

then every $v \in \mathcal{D}_L(G)$ which satisfies (3.3), (3.5), (3.6) *has a zero in $\bar{G}$.*

Proof. Suppose that $v(x) \neq 0$ in $\bar{G}$. Then, condition (3.6) implies that $v(x) > 0$ for all $x \in \bar{G}$ and from the integral Picone's identity (2.4) we obtain, in view of (3.3),(3.5) and (3.7)-(3.9), that

$$\begin{aligned} 0 &= V_p[u] + \int_G ul[u]dx - \int_G \frac{|u|^p}{v^{p-1}} L[v]dx \\ &\quad - \int_G \big[p(p-1)A|u|^{p-2}\frac{|\Delta v|^{p-1}}{v^{p-1}}\big|\nabla u - \frac{u}{v}\nabla v\big|^2 - A\Phi_p\big(\Delta u, u\frac{\Delta v}{v}\big)\big]dx \\ &\le -\int_G p(p-1)A|u|^{p-2}\frac{|\Delta v|^{p-1}}{v^{p-1}}\big|\nabla u - \frac{u}{v}\nabla v\big|^2 dx \le 0. \end{aligned}$$

Consequently, $\nabla(u/v) = 0$ in G; that is, $u/v = k$ in G, and hence on $\bar{G}$ by continuity, for some nonzero constant k. However, this cannot happen since $u = 0$ on ∂G whereas $v > 0$ on ∂G. This contradiction shows that v must vanish somewhere in $\bar{G}$. □

As a final application of the Picone identity (2.4) we obtain a lower bound for the first eigenvalue of the nonlinear eigenvalue problem

$$\Delta(|\Delta u|^{p-2}\Delta u) = \lambda |u|^{p-2}u \quad \text{in } G, \tag{3.10}$$

$$u = \Delta u = 0 \quad \text{on } \partial G \tag{3.11}$$

investigated by Drábek and Ôtani [2]. They proved that for any $p > 1$ the Navier eigenvalue problem (3.10)-(3.11) considered on a bounded domain $G \in \mathbb{R}^n$ with a smooth boundary ∂G, has a principal eigenvalue λ_1 which is simple and isolated and that there exists strictly positive eigenfunction u_1 in G associated with λ_1 and satisfying $\partial u_1/\partial \nu < 0$ on ∂G.

Actually, our technique based on the identity (2.4) allows to consider more general nonlinear eigenvalue problem

$$l[u] = \lambda |u|^{p-2}u \quad \text{in } G, \tag{3.12}$$

$$u = 0, \quad \Delta u + \sigma \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial G, \tag{3.13}$$

where $0 \le \sigma \le +\infty$ (the case $\sigma = +\infty$ corresponds to the boundary condition $\partial u/\partial \nu = 0$) and $lu \equiv \Delta(a\varphi_p(\Delta u)) - c\varphi_p(u)$ as before.

Theorem 3.5. *Let λ_1 be the first eigenvalue of* (3.12)-(3.13) *and $u_1 \in \mathcal{D}_l(G)$ be the corresponding eigenfunction. If there exists a function $v \in \mathcal{D}_L(G)$ such that*

$$v > 0 \quad \text{in } \bar{G}, \tag{3.14}$$

$$\Delta v \le 0 \quad \text{in } G \tag{3.15}$$

and if $V_p[u_1] \ge 0$, *then*

$$\lambda_1 \ge \inf_{x\in G} \big[\frac{L[v]}{v^{p-1}}\big].$$

Proof. The identity (2.4), in view of the above hypotheses, implies that

$$\begin{aligned}
&\lambda_1 \int_G |u_1|^p dx - \int_G |u_1|^p \frac{L[v]}{v^{p-1}} dx \\
&= V_p[u_1] + \int_G \big[p(p-1)A|u_1|^{p-2}\frac{|\Delta v|^{p-1}}{v^{p-1}}\big|\nabla u_1 - \frac{u_1}{v}\nabla v\big|^2 + A\Phi_p(\Delta u_1, u_1\Delta v/v)\big]dx \\
&\quad + \int_{\partial G} \sigma^{p-1} a\big|\frac{\partial u_1}{\partial \nu}\big|^p ds \ge 0,
\end{aligned}$$

from which the conclusion readily follows. □

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