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S-ASYMPTOTICALLY PERIODIC SOLUTIONS FOR PARTIAL DIFFERENTIAL EQUATIONS WITH FINITE DELAY

WILLIAM DIMBOUR, GISÈLE MOPHOU, GASTON M. N'GUÉRÉKATA

Abstract. In this article, we give some sufficient conditions for the existence and uniqueness of S-asymptotically periodic (mild) solutions for some partial functional differential equations. To illustrate our main result, we study a diffusion equation with delay.

1. Introduction

The main purpose of this work is to study the existence and uniqueness of S-asymptotically periodic solutions in the α-norm for the partial differential equation

$$\begin{gathered} \frac{d}{dt}u(t) = -Au(t) + L(u_t) + f(t, u(t)) \quad \text{for } t \geq 0, \\ u_0 = \varphi \end{gathered} \tag{1.1}$$

where $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$, $t \geq 0$ on a Banach space $\mathbb{X}$.

For $0 < \alpha \leq 1$, let A^α be the fractional power of A with domain $D(A^\alpha)$, which endowed with the norm $|x|_\alpha = \|A^\alpha x\|$ forms a Banach space $\mathbb{X}_\alpha$. Let $\mathcal{C}_\alpha = C([-r, 0], \mathbb{X}_\alpha)$ be the Banach space of all continuous functions from $[-r, 0]$ to $\mathbb{X}_\alpha$ endowed with the norm

$$|\phi|_{\mathcal{C}_\alpha} = \sup_{-r \leq \theta \leq 0} |\phi(\theta)|_\alpha.$$

Let L be a bounded linear operator from $\mathcal{C}_\alpha$ to $\mathbb{X}_\alpha$, and $f : \mathbb{R} \times \mathbb{X}_\alpha \to \mathbb{X}_\alpha$ a continuous function. As usual the history function $x_t \in \mathcal{C}_\alpha$ is defined by

$$x_t(\theta) = x(t + \theta) \quad \text{for } \theta \in]-r, 0].$$

The theory of partial functional differential equations and its applications are an active are of research; see for instance [16, 17, 29] and the references therein. Several articles study the existence and uniqueness of almost periodic, almost automorphic, and weighted pseudo almost periodic solutions of various differential

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equations. In [11], the author deals with the existence of $C^{(n)}$-almost periodic and $C^{(n)}$-automorphic solution of the equation

$$\begin{gathered}\frac{d}{dt}u(t) = -Au(t) + L(u_t) + f(t) \quad \text{for } t \geq 0, \\ u_0 = \varphi\end{gathered} \tag{1.2}$$

To achieve his goal, the author uses the the variation of constants formula and the reduction method developed by Adimy et al. [1]. Ezzinbi and Boukli-Hacene [13] studied the existence and uniqueness of weighted pseudo-almost automorphic solution for (1.2), using the variation of constants formula developed by Ezzinbi and N'Guérékata [14].

The literature relative to S-asymptotically periodic functions remains limited due to the novelty of the concept. Qualitative properties of such functions are discussed for instance in [4, 18, 21]. In [4], the authors present a new composition theorem for such functions. Various properties of S-asymptotically periodic functions are also investigated in a general study of classes of bounded continuous functions taking values in a Banach space $\mathcal{X}$. In [6], a new concept of weighted S-asymptotically periodic functions is introduced generalizing in a natural way the one studied here. There are some papers dealing with the existence of S-asymptotically periodic solutions of differential equations and fractional differential equations in finite as well as infinite dimensional spaces; see [4, 18, 19, 21, 25]. In this paper, motivated by all these works, we first reconsider (1.2) and prove that if f is an S-asymptotically periodic function in the α-norm then its has a unique solution on $[-r, +\infty[$. Moreover, the restriction of the solution on $\mathbb{R}^+$ is S-asymptotically periodic solutions in the α-norm. This allow us to study the existence and uniqueness of an S-asymptotically periodic solution in the α-norm, for (1.1).

This work is organized as follows. In Section 2, we recall some fundamental properties of S-asymptotically periodic functions and fractional powers of a closed operator. Section 3 is devoted to the main result. We illustrate our main result in Section 4 by examining the existence and uniqueness of S-asymptotically periodic (mild) solutions for some diffusion equations with delay.

2. PRELIMINARIES

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space. Denote by $C(\mathbb{R}^+, \mathbb{X})$, the space of all continuous functions from $\mathbb{R}^+$ to $\mathbb{X}$, and by $BC(\mathbb{R}^+, \mathbb{X})$ the space of all bounded continuous functions $\mathbb{R}^+ \to \mathbb{X}$. The space $BC(\mathbb{R}^+, \mathbb{X})$ endowed with the supremum norm $\|f\|_\infty := \sup_{t\geq 0} ||f(t)||$ is a Banach space.

S-asymptotically periodic functions.

Definition 2.1. For a function f in $BC(\mathbb{R}^+, \mathbb{X})$, we say that f belongs to $C_0(\mathbb{R}^+, \mathbb{X})$ if $\lim_{t\to\infty} \|f(t)\| = 0$.

Let ω be a fixed positive number and $f \in BC(\mathbb{R}^+, \mathbb{X})$. We say that f is ω-periodic, denoted by $f \in P_\omega(\mathbb{X})$, if f has period ω. Note that $P_\omega(\mathbb{X})$ is a Banach subspace of $BC(\mathbb{R}^+, \mathbb{X})$ under the supremum norm.

Definition 2.2 ([4, 21])**.** Let $f \in BC(\mathbb{R}^+, \mathbb{X})$ and $\omega > 0$. We say that f is asymptotically ω-periodic if $f = g + h$ where $g \in P_\omega(\mathbb{X})$ and $h \in C_0(\mathbb{R}^+, \mathbb{X})$.

We denote by $AP_\omega(\mathbb{X})$ the set of all asymptotically ω-periodic functions from $\mathbb{R}^+$ to $\mathbb{X}$. Note that $AP_\omega(\mathbb{X})$ is a Banach space under the supremum norm.

From the above definitions, it follows that $AP_\omega(\mathbb{X}) = P_\omega(\mathbb{X}) \bigoplus C_0(\mathbb{R}^+, \mathbb{X})$; cf. [21].

Definition 2.3 ([18])**.** A function $f \in BC(\mathbb{R}^+, \mathbb{X})$ is called S-asymptotically ω-periodic if there exists ω such that $\lim_{t\to\infty}(f(t+\omega) - f(t)) = 0$. In this case we say that ω is an asymptotic period of f and that f is S-asymptotically ω-periodic.

We will denote by $SAP_\omega(\mathbb{X})$, the set of all S-asymptotically ω-periodic functions from $\mathbb{R}^+ to \mathbb{X}$. Then we have

$$AP_\omega(\mathbb{X}) \subset SAP_\omega(\mathbb{X}).$$

Note that the inclusion above is strict. Consider the function $f : \mathbb{R}^+ \to c_0$ where $c_0 = \{x = (x_n)_{n\in\mathbb{N}} : \lim_{n\to\infty} x_n = 0\}$ equipped with the norm $\|x\| = \sup_{n\in\mathbb{N}} |x(n)|$, and $f(t) = (\frac{2nt^2}{t^2+n^2})_{n\in\mathbb{N}}$. Then $f \in SAP_\omega(c_0)$ but $f \notin AP_\omega(c_0)$; see [18, Example 3.1].

The following result is due to Henriquez-Pierri-Tàboas; [18, Proposition 3.5].

Theorem 2.4. *The space $SAP_\omega(\mathbb{X})$ endowed with the norm $\|\cdot\|_\infty$ is a Banach space.*

Theorem 2.5 ([4, Theorem 3.7])**.** *Let $\phi : \mathbb{X} \to \mathbb{Y}$ be a function which is uniformly continuous on bounded subsets of $\mathbb{X}$ and such that ϕ maps bounded subsets of $\mathbb{X}$ into bounded subsets of $\mathbb{Y}$. Then for all $f \in SAP_\omega(\mathbb{X})$, the composition $\phi \circ f := [t \to \phi(f(t))] \in SAP_\omega(\mathbb{X})$.*

Corollary 2.6 ([4, Corollary 3.10])**.** *Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces, and denote by $\mathbb{B}(\mathbb{X}, \mathbb{Y})$, the space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$. Let $A \in \mathbb{B}(\mathbb{X}, \mathbb{Y})$. Then when $f \in SAP_\omega(\mathbb{X})$, we have $Af := [t \to Af(t)] \in SAP_\omega(\mathbb{Y})$.*

Next we consider asymptotically ω-periodic functions with parameters.

Definition 2.7 ([18])**.** *A continuous function $f : [0, \infty[\times\mathbb{X} \to \mathbb{X}$ is said to be uniformly S-asymptotically ω-periodic on bounded sets if for every bounded set $K \subset \mathbb{X}$, the set $\{f(t, x) : t \geq 0, x \in K\}$ is bounded and $\lim_{t\to\infty}(f(t, x) - f(t + \omega, x)) = 0$ uniformly in $x \in K$.*

Definition 2.8 ([18])**.** A continuous function $f : [0, \infty[\times\mathbb{X} \to \mathbb{X}$ is said to be asymptotically uniformly continuous on bounded sets if for every $\epsilon > 0$ and every bounded set $K \subset \mathbb{X}$, there exist $L_{\epsilon,K} > 0$ and $\delta_{\epsilon,K} > 0$ such that $||f(t, x)-f(t, y)\| < \epsilon$ for all $t \geq L_{\epsilon,K}$ and all $x, y \in K$ with $\|x - y\| < \delta_{\epsilon,K}$.

Theorem 2.9 ([18])**.** *Let $f : [0, \infty[\times\mathbb{X} \to \mathbb{X}$ be a function which uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Let $u : [0, \infty[$ be S-asymptotically ω-periodic function. Then the Nemytskii operator $\phi(\cdot) := f(\cdot, u(\cdot))$ is S-asymptotically ω-periodic function.*

Fractional powers of the operator A**.** Let $\varrho(A)$ denote the resolvent set of A. We assume without loss of generality that

$$0 \in \varrho(A). \tag{2.1}$$

This allows us, on the one hand, to say that there exist constants $M > 1$ and $\delta > 0$ such that

$$\|T(t)x\| \leq Me^{-\delta t}\|x\|, \quad \forall t \geq 0,\ x \in \mathbb{X}, \tag{2.2}$$

and on the other hand, to define the fractional power A^α for $0 < \alpha < 1$, as a closed linear operator on its domain $D(A^\alpha)$ with inverse $A^{-\alpha}$ given by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^t t^{\alpha-1} T(t) dt$$

where Γ denotes the Gamma function

$$\Gamma(\alpha) = \int_0^t t^{\alpha-1} e^{-\alpha t} dt.$$

We have the following basic properties for A^α.

Theorem 2.10 ([26, pp. 69-75])**.** *For $0 < \alpha < 1$, the following properties hold.*

(i) $\mathbb{X}_\alpha = D(A^\alpha)$ *is a Banach space with the norm* $|x|_\alpha = \|A^\alpha x\|$ *for* $x \in D(A^\alpha)$*;*
(ii) $A^{-\alpha}$ *is the closed linear operator with* $Im(A^{-\alpha}) = D(A^\alpha)$ *and we have* $A^\alpha = (A^{-\alpha})^{-1}$*;*
(iii) $A^{-\alpha} \in \mathbb{B}(\mathbb{X}, \mathbb{X})$*;*
(iv) $T(t) : \mathbb{X} \to \mathbb{X}_\alpha$ *for every* $t > 0$*;*
(v) $A^\alpha T(t)x = T(t)A^\alpha x$ *for each* $x \in D(A^\alpha)$ *and* $t \geq 0$*;*
(vi) $0 < \alpha \leq \beta$ *implies* $D(A^\beta) \hookrightarrow D(A^\alpha)$*;*
(vii) *There exists* $M_\alpha > 1$ *such that*

$$\|A^\alpha T(t)x\| \leq M_\alpha \frac{e^{-\delta t}}{t^\alpha} \|x\| \quad \text{for } x \in \mathbb{X},\ t > 0.$$

where $\delta > 0$ *is given by* (2.2)

Remark 2.11. Observe as in [20, 22] that from Theorem 2.10 (iv) and (v), the restriction $T_\alpha(t)$ of $T(t)$ to $\mathbb{X}_\alpha$ is exactly the part of $T(t)$ in $\mathbb{X}_\alpha$.

Let $x \in \mathbb{X}_\alpha$.

$$|T(t)x|_\alpha = \|A^\alpha T(t)x\| = \|T(t)A^\alpha x\| \leq |T(t)| \, \|A^\alpha x\| = |T(t)| \, |x|_\alpha,$$

and as t decreases to 0,

$$|T(t)x - x|_\alpha = \|A^\alpha T(t)x - A^\alpha x\| = \|T(t)A^\alpha x - A^\alpha x\| \to 0,$$

for all $x \in \mathbb{X}_\alpha$; it follows that $(T(t))_{t\geq 0}$ is a family of strongly continuous semigroup on $\mathbb{X}_\alpha$ and $|T_\alpha(t)| \leq |T(t)|$ for all $t \geq 0$.

Proposition 2.12 ([11, 28])**.** *($(T(t)_{t\geq 0})$ is a strongly continuous semigroup on $\mathcal{C}_\alpha$; that is,*

(i) *for all $t \geq 0$ $T(t)$ is a bounded linear operator on $\mathcal{C}_\alpha$;*
(ii) $T(0) = I$*;*
(iii) $T(t+s) = T(t)T(s)$ *for all* $t, s \geq 0$*;*
(iv) *for all $\varphi \in \mathcal{C}_\alpha$, $T(t)\varphi$ is a continuous function of $t \geq 0$ with values in $\mathcal{C}_\alpha$.*

3. Applications to partial differential equations with finite delay

Definition 3.1. Let $\varphi \in \mathcal{C}_\alpha$. A function $u : [-r, +\infty[\to \mathbb{X}_\alpha$ is said to be a mild solution of (1.2) if the following conditions hold:

(i) $u : [-r, +\infty[\to \mathbb{X}_\alpha$ is continuous;
(ii) $u(t) = T(t)\varphi(0) + \int_0^t T(t-s)[L(u_s) + f(s)]ds$ for $t \geq 0$;
(iii) $u_0 = \varphi$.

For the rest of this article, we define

$$\Omega = \{u : [-r, +\infty[\to \mathbb{X}_\alpha \text{ such that } u|_{[-r,0]} \in \mathcal{C}_\alpha \text{ and } u|_{\mathbb{R}^+} \in SAP_\omega(\mathbb{X}_\alpha)\}.$$

Note that if $u \in \Omega$ then u is bounded on $[-r, +\infty[$. We set

$$\|u\|_\Omega = \sup_{s\in[-r,+\infty[} |u(s)|_\alpha. \tag{3.1}$$

It is clear that $\|u\|_\infty \le \|u\|_\Omega$.

Lemma 3.2. *Under assumption* (2.1)*, the function l defined by*

$$l(t) = T(t)\varphi(0)$$

belongs to $SAP_\omega(\mathbb{X}_\alpha)$.

Proof. Since $\varphi(0) \in \mathbb{X}_\alpha$, we have on the one hand that $(T(t))_{t\ge0}$ is a family of strongly continuous semigroup on $\mathbb{X}_\alpha$ (see Remark 2.11), and on the other hand that $|l(t)|_\alpha \le M|\varphi(0)|_\alpha$ because (2.2) holds. Consequently $l \in BC(\mathbb{R}^+, \mathbb{X}_\alpha)$.

Now using (2.2) and Remark 2.11, we obtain for $t \ge 0$,

$$\begin{aligned} |l(t+\omega) - l(t)|_\alpha &= |T(t+\omega)\varphi(0) - T(t)\varphi(0)|_\alpha \\ &\le |T(t+\omega)\varphi(0)|_\alpha + |T(t)\varphi(0)|_\alpha \\ &\le |T(t+\omega)||\varphi(0)|_\alpha + |T(t)||\varphi(0)|_\alpha \\ &\le Me^{-\delta(t+\omega)}|\varphi(0)|_\alpha + Me^{-\delta t}|\varphi(0)|_\alpha. \end{aligned}$$

As $\delta > 0$, we deduce that

$$\lim_{t\to\infty} |l(t+\omega) - l(t)|_\alpha = 0.$$

Thus $l \in SAP_\omega(\mathbb{X}_\alpha)$. □

Lemma 3.3. *If $u \in \Omega$, then*

$$|u_t|_{\mathcal{C}_\alpha} \le \|u\|_\Omega, \tag{3.2}$$

$$|L(u_t)|_\alpha \le |L|_{\mathbb{B}(\mathcal{C}_\alpha,\mathbb{X}_\alpha)}\|u\|_\Omega \tag{3.3}$$

$$\lim_{t\to+\infty} |u_{t+\omega} - u_t|_{\mathcal{C}_\alpha} = 0. \tag{3.4}$$

Proof. For any $\theta \in [-r, 0]$ and $t \ge 0$, we have

$$|u_t(\theta)|_\alpha = |u(t+\theta)|_\alpha.$$

Since u_t is continuous on $[-r, 0]$ which is compact, we know that there exists $\theta^* \in [-r, 0]$ such that

$$|u_t|_{\mathcal{C}_\alpha} = \sup_{-r\le\theta\le0} |u(t+\theta)|_\alpha = |u(t+\theta^*)|_\alpha.$$

Since $u \in \Omega$, we deduce that (3.2) holds. As $L \in \mathbb{B}(\mathcal{C}_\alpha, \mathbb{X}_\alpha)$, we can write

$$|L(u_t)|_{\mathbb{X}_\alpha} \le |L|_{\mathbb{B}(\mathcal{C}_\alpha,\mathbb{X}_\alpha)}|u_t|_{\mathcal{C}_\alpha}.$$

Therefore, using (3.2), we obtain (3.3).

To complete the proof of the lemma, it suffices to prove (3.4). As u_t is continuous on $[-r, 0]$ which is compact, there exists $\theta^* \in [-r, 0]$ such that

$$\begin{aligned} |u_{t+\omega} - u_t|_{\mathcal{C}_\alpha} &= \sup_{-r\le\theta\le0} |u(t+\theta+\omega) - u(t+\theta)|_\alpha \\ &= |u(t+\theta^*+\omega) - u(t+\theta^*)|_\alpha. \end{aligned}$$

Set $s = t + \theta$. Then, as t tends to $+\infty$ we have s tends to $+\infty$. Consequently

$$\lim_{t\to\infty} |u(t+\theta^* + \omega) - u(t+\theta^*)|_\alpha = \lim_{s\to\infty} |u(s+\omega) - u(s)|_\alpha = 0$$

since $u \in \Omega$. Hence, $\lim_{t\to\infty} |u_{t+\omega} - u_t|_{\mathcal{C}_\alpha} = 0$. □

Lemma 3.4. *Assume that* (2.1) *holds. Let $f \in SAP_\omega(\mathbb{X}_\alpha)$ and $\phi \in \Omega$. Then the function $\Phi : t \mapsto L(\phi_t) + f(t)$ belongs to $SAP_\omega(\mathbb{X}_\alpha)$.*

Proof. It is clear that $\Phi \in C(\mathbb{R}^+, \mathbb{X}_\alpha)$. Using Lemma 3.3, we obtain

$$|\Phi(t)|_\alpha \le |L(\phi_t)|_\alpha + |f(t)|_\alpha \le |L|_{\mathbb{B}(\mathcal{C}_\alpha,\mathbb{X}_\alpha)} \|\phi\|_\Omega + \|f\|_\infty.$$

This implies that $\Phi \in BC(\mathbb{R}^+, \mathbb{X}_\alpha)$. Hence

$$\|\Phi\|_\infty \le |L|_{\mathbb{B}(\mathcal{C}_\alpha,\mathbb{X}_\alpha)} \|\phi\|_\Omega + \|f\|_\infty. \tag{3.5}$$

On the other hand, for all $t \ge 0$,

$$\begin{aligned} |\Phi_{t+\omega} - \Phi_t|_\alpha &\le |L(\phi_{t+\omega} - \phi_t)|_\alpha + |f(t+\omega) - f(t)|_\alpha \\ &\le |L|_{\mathbb{B}(\mathcal{C}_\alpha,\mathbb{X}_\alpha)} |\phi_{t+\omega} - \phi_t|_{\mathcal{C}_\alpha} + |f(t+\omega) - f(t)|_\alpha, \end{aligned}$$

Since $\phi \in \Omega$, using Lemma 3.3-(3.4) and the fact that $f \in SAP_\omega(\mathbb{X}_\alpha)$, we deduce that

$$\lim_{t\to\infty} |\Phi_{t+\omega} - \Phi_t|_\alpha = 0. \tag{3.6}$$

This completes the proof. □

Proposition 3.5. *Assume that* (2.1) *holds. Let $f \in SAP_\omega(\mathbb{X}_\alpha)$. For each $\phi \in \Omega$, define the nonlinear operator $\wedge_0$ by*

$$(\wedge_0 \phi)(t) = \begin{cases} \varphi(t) & \text{if } t \in [-r, 0], \\ T(t)\varphi(0) + \int_0^t T(t-s)[L(\phi_s) + f(s)]ds & \text{if } t \ge 0. \end{cases}$$

Then $\wedge_0$ maps Ω into itself.

Proof. It is clear that $(\wedge_0\phi)$ is defined on $[-r, +\infty[$ and because $\varphi \in \mathcal{C}_\alpha$, we have $(\wedge_0\phi)|_{[-r,0]} \in \mathcal{C}_\alpha$. Thus it suffices to show that the function

$$v : t \to \int_0^t T(t-s)[L(\phi_s) + f(s)]ds \in SAP_\omega(\mathbb{X}_\alpha)$$

to complete the proof, since by Lemma 3.2, $T(t)\varphi(0) \in SAP_\omega(\mathbb{X}_\alpha)$.

For $t \ge 0$, let $\Phi(t) = L(\phi_t) + f(t)$. Then

$$\begin{aligned} v(t+\omega) - v(t) &= \int_0^{t+\omega} T(t+\omega-s)\Phi(s)\,ds - \int_0^t T(t-s)\Phi(s)\,ds \\ &= \int_0^\omega T(t+\omega-s)\Phi(s)\,ds + \int_\omega^{t+\omega} T(t+\omega-s)\Phi(s)\,ds \\ &\quad - \int_0^t T(t-s)\Phi(s)\,ds. \end{aligned}$$

Then

$$|v(t+\omega) - v(t)|_\alpha \le |I_1(t)|_\alpha + |I_2(t)|_\alpha,$$

where

$$I_1(t) = \int_0^\omega T(t+\omega-s)\Phi(s)\,ds,$$

$$\begin{gathered}
I_2(t)=\int_{\omega}^{t+\omega} T(t+\omega-s)\Phi(s)\,ds-\int_0^t T(t-s)\Phi(s)\,ds,\\
|I_1(t)|_\alpha=\big|\int_0^{\omega} T(t+\omega-s)\Phi(s)\,ds\big|_\alpha\leq\int_0^{\omega}|T(t+\omega-s)\Phi(s)|_\alpha ds
\end{gathered}$$

Since

$$\begin{aligned}
\int_0^{\omega}|T(t+\omega-s)\Phi(s)|_\alpha ds&=\int_0^{\omega}||A^\alpha T(t+\omega-s)\Phi(s)||ds\\
&=\int_0^{\omega}||T(t+\omega-s)A^\alpha\Phi(s)||ds\\
&\leq\int_0^{\omega}Me^{-\delta(t+\omega-s)}\|A^\alpha\Phi(s)\|\,ds,
\end{aligned}$$

using (3.5) we deduce that

$$\begin{aligned}
|I_1(t)|_\alpha&\leq Me^{-\delta(t+\omega)}\int_0^{\omega}e^{\delta s}|\Phi(s)|_\alpha ds\\
&\leq Me^{-\delta(t+\omega)}\|\Phi\|_\infty\int_0^{\omega}e^{\delta s}ds\\
&\leq\frac{1}{\delta}Me^{-\delta(t+\omega)}\|\Phi\|_\infty(e^{\delta w}-1)\\
&\leq\frac{1}{\delta}M\|\Phi\|_\infty e^{-\delta t}
\end{aligned}$$

Consequently, $\lim_{t\to\infty}|I_1(t)|_\alpha=0$ In view of (3.6), we can find T_ϵ sufficiently large such that

$$|\Phi(t+\omega)-\Phi(t)|_\alpha<\frac{\delta}{M}\epsilon,\quad\text{for } t>T_\epsilon.$$

After a change of variable, we obtain

$$I_2(t)=\int_0^t T(t-s)(\Phi(s+\omega)-\Phi(s))\,ds.$$

Thus we obtain

$$|I_2(t)|_\alpha\leq\big|\int_0^{T_\epsilon}T(t-s)(\Phi(s+\omega)-\Phi(s))\,ds\big|_\alpha+\big|\int_{T_\epsilon}^{t}T(t-s)(\Phi(s+\omega)-\Phi(s))\,ds\big|_\alpha.$$

Observing that

$$\begin{aligned}
\big|\int_0^{T_\epsilon}T(t-s)(\Phi(s+\omega)-\Phi(s))\,ds\big|_\alpha&\leq\int_0^{T_\epsilon}\big|T(t-s)\big(\Phi(s+\omega)-\Phi(s)\big)\big|_\alpha ds\\
&\leq\int_0^{T_\epsilon}Me^{-\delta(t-s)}|\Phi(s+\omega)-\Phi(s)|_\alpha ds\\
&\leq 2\int_0^{T_\epsilon}Me^{-\delta(t-s)}\|\Phi\|_\infty ds\\
&\leq 2M\|\Phi||_\infty e^{-\delta t}\int_0^{T_\epsilon}e^{\delta s}ds\\
&\leq 2M\|\Phi||_\infty e^{-\delta t}\big(\frac{e^{\delta T_\epsilon}}{\delta}-\frac{1}{\delta}\big),
\end{aligned}$$

we deduce that

$$\lim_{t\to\infty} \Big| \int_0^{T_\epsilon} T(t-s)(\Phi(s+\omega)-\Phi(s))\,ds\Big|_\alpha = 0$$

since $\lim_{t\to\infty}[2M\|\Phi\|_\infty e^{-\delta t}(\frac{e^{\delta T_\epsilon}}{\delta}-\frac{1}{\delta})]=0$. Also we have

$$\begin{aligned}
\Big| \int_{T_\epsilon}^{t} T(t-s)(\Phi(s+\omega)-\Phi(s))\,ds\Big|_\alpha &\le \int_{T_\epsilon}^{t} \big|T(t-s)(\Phi(s+\omega)-\Phi(s))\big|_\alpha ds\\
&\le \int_{T_\epsilon}^{t} |T(t-s)||(\Phi(s+\omega)-\Phi(s))|_\alpha ds\\
&\le \int_{T_\epsilon}^{t} Me^{-\delta(t-s)}\frac{\delta}{M}\epsilon \le \epsilon.
\end{aligned}$$

Therefore

$$\lim_{t\to\infty} \int_{T_\epsilon}^{t} T(t-s)(\Phi(s+\omega)-\Phi(s))\,ds = 0.$$

Finally, we obtain $\lim_{t\to\infty} I_2(t)=0$ and we have $t\to\int_0^t T(t-s)[L(\phi_s)+f(s)]ds \in SAP_\omega(\mathbb{X}_\alpha)$. In summary, we have proved that

- $(\wedge_0\phi)$ is defined $[-r,+\infty[$,
- $(\wedge_0\phi)|_{[}-r,0]\in \mathcal{C}_\alpha$,
- $(\wedge_0\phi)|_{\mathbb{R}^+}\in SAP_\omega(\mathbb{X}_\alpha)$;

that is, $(\wedge_0\phi)\in\Omega$. □

Theorem 3.6. *Suppose that* (2.1) *holds and $f\in SAP_\omega(\mathbb{X}_\alpha)$. Let v be the restriction of the mild solution of* (1.2) *on $\mathbb{R}^+$. Then $v\in SAP_\omega(\mathbb{X}_\alpha)$.*

Proof. According to the definition of mild solution of (1.2) given by Definition 3.1, we have for any $t\ge 0$,

$$v(t)=T(t)\varphi(0)+\int_0^t T(t-s)[L(u_s)+f(s)]ds.$$

Hence it suffices to apply Proposition (3.5), with $u=\phi$, to obtain that v belongs to $SAP_\omega(\mathbb{X}_\alpha)$. □

We make the following assumption.

(H1) The function $g: R^+\times\mathbb{X}_\alpha\to\mathbb{X}_\alpha$, $t\to g(t,u)$ is continuous and there exists a constant $K_f\ge 0$ such that

$$|g(t,u)-g(t,v)|_\alpha \le K_g|u-v|_\alpha \quad \text{for all } t\in\mathbb{R}^+\ (u,v)\in\mathbb{X}^2.$$

(H2) $M\big(|L|_{\mathbb{B}(\mathcal{C}_\alpha,\mathbb{X}_\alpha)}+K_g\big)/\delta<1$.

Definition 3.7. *Let $\varphi\in\mathcal{C}_\alpha$. A function $u:[-r,+\infty[\to\mathbb{X}_\alpha$ is said to be a mild solution of* (1.1) *if the following conditions hold:*

(i) *$u:[-r,+\infty[\to\mathbb{X}_\alpha$ is continuous;*
(ii) $u(t)=T(t)\varphi(0)+\int_0^t T(t-s)[L(u_s)+g(s,u(s))]ds$ *for $t\ge 0$;*
(iii) $u_0=\varphi$.

Proposition 3.8. *Suppose that* (2.1) *holds. Assume also that the function g is uniformly S-asymptotically ω-periodic on bounded sets and* (H1) *hold. For each $\phi \in \Omega$, define the nonlinear operator $\wedge_1$ by*

$$(\wedge_1\phi)(t) = \begin{cases} \varphi(t) & \text{if } t \in [-r, 0], \\ T(t)\varphi(0) + \int_0^t T(t-s)[L(\phi_s) + g(s, \phi(s))]ds & \text{if } t \geq 0. \end{cases}$$

Then $\wedge_1$ maps Ω into itself.

Proof. We have $\phi|_{\mathbb{R}^+} \in SAP_\omega(\mathbb{X}_\alpha)$ since $\phi \in \Omega$. Since g satisfying (H1), it follows from Theorem 2.9 that the function $h : t \mapsto g(t, \phi(t))$ belongs to $SAP_\omega(\mathbb{X}_\alpha)$. Hence, it suffices to proceed exactly as for the proof of the Proposition 3.5 replacing $f(\cdot)$ by $h(\cdot)$ to obtain that $\wedge_1$ maps Ω into itself. □

Theorem 3.9. *Suppose that* (2.1) *and* (H2) *hold. Also assume that the function g is uniformly S-asymptotically ω-periodic on bounded sets and* (H1) *hold. Then for all $\varphi \in \mathcal{C}_\alpha$, Equation* (1.1) *has a unique mild solution in Ω.*

Proof. Consider the operator $Q : \Omega \to \Omega$ defined by:

$$(Qu)(t) = \begin{cases} \varphi(t) & \text{if } t \in [-r, 0], \\ T(t)\varphi(0) + \int_0^t T(t-s)[L(u_s) + g(s, u(s))]ds & \text{if } t \geq 0. \end{cases}$$

Observe that in view of Proposition 3.8, Q is well defined. Consider $u, v \in \Omega$. For all $t \in [-r, +\infty[$, we have

$$\begin{aligned} &|(Qu)(t) - (Qv)(t)|_\alpha \\ &= \Big| \int_0^t T(t-s)[(L(u_s) - L(v_s)) + (g(s, u(s)) - g(s, v(s)))]ds \Big|_\alpha \\ &\leq \int_0^t \big| T(t-s)[(L(u_s) - L(v_s)) + (g(s, u(s)) - g(s, v(s)))] \big|_\alpha ds. \end{aligned}$$

Therefore, using (2.2) and (3.2), we obtain

$$\begin{aligned} &|(Qu)(t) - (Qv)(t)|_\alpha \\ &\leq \int_0^t Me^{-\delta(t-s)}[|L(u_s) - L(v_s)|_\alpha + |g(s, u(s)) - g(s, v(s))|_\alpha]ds \\ &\leq Me^{-\delta t}|L|_{\mathbb{B}(\mathcal{C}_\alpha, \mathbb{X}_\alpha)} \int_0^t e^{\delta s}|u_s - v_s|_{\mathcal{C}_\alpha} ds \\ &\quad + Me^{-\delta t}K_g \int_0^t e^{\delta s}|u(s) - v(s)|_\alpha ds \\ &\leq Me^{-\delta t}|L|_{\mathbb{B}(\mathcal{C}_\alpha, \mathbb{X}_\alpha)} \|u - v\|_\Omega \int_0^t e^{\delta s} ds \\ &\quad + Me^{-\delta t}K_g \|u - v\|_\infty \int_0^t e^{\delta s} ds. \end{aligned}$$

Since $\|u - v\|_\infty \leq \|u - v\|_\Omega$, we deduce that for all $t \geq -r$,

$$\begin{aligned} |(Qu)(t) - (Qv)(t)|_\alpha &\leq Me^{-\delta t}|L|_{\mathbb{B}(\mathcal{C}_\alpha, \mathbb{X}_\alpha)} \|u - v\|_\Omega \int_0^t e^{\delta s} ds \\ &\quad + Me^{-\delta t}K_g \|u - v\|_\infty \int_0^t e^{\delta s} ds \end{aligned}$$

$$\begin{aligned}
&\leq \frac{Me^{-\delta t}}{\delta}\big(|L|_{\mathbb{B}(\mathcal{C}_\alpha,\mathbb{X}_\alpha)} + K_g\big)\|u-v\|_\Omega(e^{\delta t}-1)\\
&\leq \frac{M}{\delta}\big(|L|_{\mathbb{B}(\mathcal{C}_\alpha,\mathbb{X}_\alpha)} + K_g\big)\|u-v\|_\Omega.
\end{aligned}$$

Hence

$$\|(Qu)(t)-(Qv)(t)\|_\Omega \leq \frac{M}{\delta}\big(|L|_{\mathbb{B}(\mathcal{C}_\alpha,\mathbb{X}_\alpha)} + K_g\big)\|u-v\|_\Omega.$$

Hence assumption (H2) allows us to conclude in view of the contraction mapping principle that Q has a unique point fixed in $u \in \Omega$. The proof is now complete. □

4. Application

Consider the functional partial differential equation

$$\begin{gathered}
\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + \int_{-r}^0 q(\theta)y(t+\theta,x)d\theta + g(t,u(t,x)) \quad t\in\mathbb{R}^+, x\in[0,\pi]\\
u(t,0)=u(t,\pi)=0 \quad t\in\mathbb{R}^+\\
u(\theta,x)=\phi(\theta,x), \quad \text{for } \theta\in[-r,0] \text{ and } x\in[0,\pi]
\end{gathered} \tag{4.1}$$

where $q:[-r,0]\to\mathbb{R}$ is continuous. To study this system in the abstractt form (1.1), we choose $\mathbb{X}=L^2([0,\pi])$ and the operator $A: D(A)\subset\mathbb{X}\to\mathbb{X}$ is given by $Au=-u''$ with domain

$$D(A)=\{u\in\mathbb{X}: u'\in\mathbb{X},\, u''\in\mathbb{X},\, u(0)=u(\pi)=0\}.$$

Then $-A$ generates an analytic semigroup $T(\cdot)$ such that $\|T(t)\|\leq e^{-t}$, $t\geq 0$ ([15]). Moreover, the eigenvalues of A are $n^2\pi^2$ and the corresponding normalized eigenvectors are $e_n(x)=\sqrt{2}\sin(n\pi x)$, $n=1,2,\cdots$. Hence, we have

(a) $Au=\sum_{n=1}^\infty n^2\pi^2\langle u,e_n\rangle e_n$ if $u\in D(A)$;
(b) $A^{-1/2}u=\sum_{n=1}^\infty \frac{1}{n}\langle u,e_n\rangle e_n$ if $u\in\mathbb{X}$;
(c) The operator $A^{1/2}$ is given by

$$A^{1/2}u=\sum_{n=1}^\infty n\langle u,e_n\rangle e_n$$

for each $u\in D(A^{1/2})=\{u\in\mathbb{X}: \sum_{n=1}^\infty \frac{1}{n}\langle u,e_n\rangle e_n\in\mathbb{X}\}$.

Let $\mathbb{X}_{1/2}=\big(D(A^{1/2}),|\cdot|_{1/2}\big)$ where $|x|_{1/2}=\|A^{1/2}x\|_2$ for each $x\in D(A^{1/2})$. Let $\mathcal{C}_\alpha$ be the Banach space $C([-r\,,0],\mathbb{X}_{1/2})$ equipped with norm $|\cdot|_\infty$. We define $g:\mathbb{R}^+\times\mathbb{X}_{1/2}\to\mathbb{X}_{1/2}$ and $\varphi:[-r,0]\times[0,\pi]\to\mathbb{X}_{1/2}$ by $g(t,u(t))(x)=g(t,u(t,x))$ and $\phi(\theta)(x)=\phi(\theta,x)$ respectively. We define the operator L by

$$L(\phi)(x)=\int_{-r}^0 q(\theta)\phi(\theta)(x)d\theta \quad \text{for } x\in[0,\,\pi],\ \phi\in\mathcal{C}_{1/2}.$$

we have $A^{1/2}\phi(\theta)(x)\in L^2([-r,\,0])$ since $\phi\in\mathcal{C}_{1/2}$. It follows that

$$\begin{aligned}
\big|A^{1/2}L(\phi)(x)\big|^2 &\leq \int_{-r}^0 q(\theta)^2d\theta\int_{-r}^0\big|A^{1/2}\phi(\theta)(x)\big|^2d\theta\\
&\leq r\big(\sup_{-r\leq\theta\leq 0} q(\theta)\big)^2\int_{-r}^0|A^{1/2}\phi(\theta)(x)|^2d\theta
\end{aligned}$$

since q is continuous on $[-r, 0]$ which is a compact set of $\mathbb{R}$. Therefore we deduce that

$$\begin{aligned}\int_0^\pi |A^{1/2}L(\phi)(x)|^2\,dx &\le r\big(\sup_{-r\le\theta\le 0} q(\theta)\big)^2\int_0^\pi\int_{-r}^0 |A^{1/2}\phi(\theta)(x)|^2 d\theta\,dx\\ &= r\big(\sup_{-r\le\theta\le 0} q(\theta)\big)^2\int_{-r}^0\int_0^\pi |A^{1/2}\phi(\theta)(x)|^2 dx\,d\theta.\end{aligned}$$

Hence, we obtain

$$|L(\phi)|_{1/2} \le r^2\big(\sup_{-r\le\theta\le 0} q(\theta)\big)^2|\phi|^2_{\mathcal{C}_{1/2}}.$$

This means that L is a bounded linear operator from $\mathcal{C}_{1/2}$ to $\mathbb{X}_{1/2}$. Therefore, (4.1) takes the abstract form (1.1).

Assume $\int_{-r}^0 |q(\theta)|d\theta < 1$ and that the function $g : R^+\times\mathbb{X}_\alpha \to \mathbb{X}_\alpha$, $t \to g(t,u)$ is continuous and there exists a constant $K_f \ge 0$ such that

$$|g(t,u) - g(t,v)|_\alpha \le K_g|u-v|_\alpha \quad \text{for all } t\in\mathbb{R}^+,\ (u,v)\in\mathbb{X}^2.$$

Note that such a function exists. Take for instance Let $f(t,x) = e^{-t}x$ then $|f(t,x)-f(t,y)|_{1/2} \le |x-y|_{1/2}$.

Theorem 4.1. *Assume that g is uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Then System* (4.1) *has a unique solution defined on $[-r,\infty[$ such that its restriction on $\mathbb{R}^+$ belongs to $SAP_\omega(\mathbb{X}_\alpha)$ provided $(r^2\left(\sup_{-r\le\theta\le 0}q(\theta)\right)^2 + K_g) < 1$.*

Proof. It suffices to apply Theorem 3.9, observing that (H2) is satisfied since $r^2\big(\sup_{-r\le\theta\le 0}q(\theta)\big)^2 + K_g < 1$ and $M = \delta = 1$. $\square$

References

[1] M. Adimy, A. Elazzouzi, K. Ezzinbi; *Reduction principle and dynamic behaviors for a class of partial functional differential equations*, Nonlinear Analysis. Theory, Methods and Applications, 71(56), 2009, 1709-1727.

[2] M. Adimy, K. Ezzinbi; *Existence and linearized stability for partial neutral functional differential equations*, Differential Equations Dynam. Systems, Vol.7 (1999) 371-417.

[3] M. Adimy, K. Ezzinbi, M.Laklach; *Spectral decomposition for partial neutral functional differential equations*, Canadian. Appl. Math. Q. 9, Vol. 1 (2001), 1-34.

[4] J. Blot, P. Cieutat, G. M. N'Guérékata; *S-asymptotically ω-periodic functions and applications to evolution equations* (preprint).

[5] S. Bochner; *Continuous mappings of almost automorphic and almost periodic functions,* Proc. Nat. Acad. Sci. USA 52(1964), 907-910.

[6] C. Cuevas, M. Pierri, A. Sepulveda; *Weighted S-asymptotically ω-periodic solutions of a class of fractional differential equations*, Advances in Difference Equations, Vol. 2011, Art. ID 584874, 13 p.

[7] T. Diagana, G. M. N'Guérékata; *Almost automorphic solutions to some classes of partial evolution equations*, Applied Math. Letters 20(2007), 462-466.

[8] T. Diagana; *Existence of Pseudo-Almost Automorphic Mild Solutions to some Nonautonomous Partial Evolution Equations*, Advances in Difference Equations 2011 Article ID 895079, 23 pages doi: 10.1155/2011/895079

[9] T. Diagana, E. Hernández, M. Rabello; *Pseudo almost periodic solutions to some non-autonomous neutral functional differential equations with unbounded delay*, Mathematical and Computer Modelling 45 (2007) 1241-1252.

[10] W. Dimbour, G. M. N'Guérékata; *S-asymptotically ω-periodicsolutions to some classes of partial evolution equations*(preprint).

[11] A. Elazzouzi; *$C^{(n)}$-almost periodic and $C^{(n)}$-almost automorphic solutions for a class of partial functional differential equations with finite delay.*
[12] K. J. Engel, R. Nagel; *One Parameter Semigroups of Linear Evolution Equations*, Grad. text in Math., Vol. 194, Springer(2001).
[13] K. Ezzinbi, N. Boukli-Hacene; *Weighted pseudo almost periodic solutions for some partial functional differential equationq*, Vol. 71 (2009) 3612-3621.
[14] K. Ezzinbi, G. M. N'Guérékata; *Almost automorphic solutions for some partial functional differential equations*, Journal of Mathematical Analysis and Applications, Vol. 328 (2007), 344-358.
[15] A. Lunardi; *Analytic semigroup and optimal regularity in parabolic problems*, in PNLDE Vol.16, Birkhäuser Verlag Basel, 1995.
[16] J. K. Hale, V. Lunel; *Introduction to functional Differential Equations, in: Applied Mathematical Sciences*, Vol. 99, Springer-verlag, New-York/Berlin (1993).
[17] C. C. Travis, G. F. Webb; *Existence and stability for partial functional differential equations*, Transactions of the American Mathematical Society 200 (1974), 395-418.
[18] H. R. Henríquez, M. Pierre, P. Táboas; *On S-asymptotically ω-periodic function on Banach spaces and applications*, J. Math. Anal. Appl 343(2008), 1119-1130.
[19] H. R. Henríquez, M. Pierre, P. Táboas; *Existence of S-asymptotically ω-periodic solutions for abstract neutral equations*, Bull. Austr. Math. Soc 78(2008), 365-382.
[20] Hsiang Liu, Jung-Chan Chang; *Existence for a class of partial differential equations with nonlocal conditions*, Nonlinear Analysis, TMA, (in press).
[21] C. Lizama, G. M. N'Guérékata; *Bounded mild solutions for semilinear integro differential equations in Banach spaces*, Integral Equations and Operators Theory, 68(2010), 207-227.
[22] G. Mophou, G. M. N'Guérékata ; *Mild Solutions for Semilinear Fractional Differential Equations*, Electronic Journal of Differential Equations, Vol.2009(2009), No. 21, pp.1-9.
[23] G. M. N'Guérékata; *Almost Automorphy and almost periodic function in Abstract Spaces* Kluwer Academic / Plenum Publisher, New York-Berlin-Moscow, 2001.
[24] G. M. N'Guérékata; *Topics in almost automorphy*, Springer-Verlag, New York 2005.
[25] S. Nicola, M. Pierri; *A note on S-asymptotically ω-periodic functions*, Nonlinear Analysis, R.W.A., 10 (2009), 2937-2938.
[26] A. Pazy; *Semigroups of linear Operators and Application to Partial Differential Equations*, in:Applied Mathematical sciences, vol. 44, Springer-Verlag, New-York, 2001.
[27] G. Da Prato, E. Sinestrari; *Differential operators with nondense domains*, Ann. Sci. Norm. Super. Pisa Cl. Sci., Vol. 2 (1987) 285-344.
[28] C. C. Travis, G. F. Webb; *Partial differential equations with deviating arguments in the time variable*, Journal of Mathematical Analysis and Applications, 56 (1976) 397-409.
[29] J. Wu; *Theory and Applications of Partial Functional Differential Equations*, Springer (1996).

William Dimbour
Laboratoire C.E.R.E.G.M.I.A., Université des Antilles et de la Guyane, Campus Fouillole 97159 Pointe-à-Pitre Guadeloupe (FWI)
E-mail address: William.Dimbour@univ-ag.fr

Gisèle Mophou
Laboratoire C.E.R.E.G.M.I.A., Université des Antilles et de la Guyane, Campus Fouillole 97159 Pointe-à-Pitre Guadeloupe (FWI)
E-mail address: gisele.Mophou@univ-ag.fr

Gaston M. N'Guérékata
Department of Mathematics, Morgan State University, 1700 East Cold Spring Lane, Baltimore, MD 21251, USA
E-mail address: Gaston.N'Guerekata@morgan.edu, nguerekata@aol.com