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WEIGHTED PSEUDO ALMOST AUTOMORPHIC SEQUENCES AND THEIR APPLICATIONS

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Abstract. In this article we define the concept of weighted pseudo almost automorphic sequence, and establish some basic properties of these sequences. Further, as an application, we show the existence, uniqueness and global attractivity of weighted pseudo almost automorphic sequence solutions of a neural network model.

1. Introduction

Pseudo almost automorphic functions are natural generalization of almost automorphic functions introduced by Xiao et.al. [20]. Recently, Blot et.al [8] have proposed an extension of pseudo almost automorphic functions called weighted pseudo almost automorphic functions. The existence and uniqueness of the pseudo almost automorphic and almost periodic solutions of differential equations have been investigated by many authors recently [2, 3, 4, 5, 8, 13, 19].

The theory of almost periodic and almost automorphic sequence parallels those of the corresponding functions. They have been studied by several researchers in the past [7, 14, 16]. Unlike differential equations, solution to difference equations result in sequences. The almost periodicity/almost automorphy of the solution to difference equations have been investigated in some of the recent literatures. For instance, Cao [10] and Huang et. al. [15] showed the existence of almost periodic sequence solutions of a discrete model of neural networks. Araya et. al. [7] proved the existence and uniqueness of almost automorphic sequence solutions of some difference equations.

Difference equations play an important role in fields like numerical methods for differential equations, finite element methods, control theory etc. They arise as variational equations along the orbits of discrete dynamical systems. The theory of almost periodic sequences and their related extensions are thus important for discrete dynamical systems in general. In this paper we present yet another generalization of almost automorphic sequences, the notion of weighted pseudo almost automorphy of a sequence. Further, we explore some of the properties of these sequences and derive the conditions under which the solution of some difference

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equations would be weighted pseudo almost automorphic. Further in the next section we consider a discrete model of cellular neural network and show the existence of a unique weighted pseudo almost automorphic sequence solution.

The organization of the rest of the paper is as follows: In section 2 the definition of weighted pseudo almost automorphic sequence is presented. Some important results like closedness property and composition theorem have been established. In section 3 we investigate the conditions for the existence and uniqueness of weighted pseudo almost automorphic sequence solution of a discrete model of a neural network. We further show that this solution is globally attractive.

2. Preliminaries

Let X be a real or complex Banach space endowed with the norm $\|\cdot\|_X$.

Denote by $\mathbb{U}$ the collection of all positive functions $\rho : \mathbb{Z} \to \mathbb{R}$. For each $\rho \in \mathbb{U}$ define

$$m(r,\rho) = \sum_{-r}^{r} \rho(k).$$

Denote by $\mathbb{U}_\infty$ the set of all $\rho \in \mathbb{U}$ such that $\lim_{r\to\infty} m(r,\rho) = \infty$. Denote by $\mathbb{U}_b$ the set of all bounded $\rho \in \mathbb{U}_\infty$ such that $\inf_{k\in\mathbb{Z}} \rho(k) > 0$.

Definition 2.1. A function $f : \mathbb{Z} \to X$ is said to be almost automorphic sequence if for every sequence of integer $\{k_l\}_{l\in\mathbb{N}}$ there exists a subsequence $\{k_n\}_{n\in\mathbb{N}}$ such that

$$f(k + k_n) \to g(k) \quad \text{and} \quad g(k - k_n) \to f(k)$$

for each $k \in \mathbb{Z}$. This is also equivalent to

$$\lim_{n\to\infty} \lim_{m\to\infty} f(k + k_n - k_m) = f(k)$$

for each $k \in \mathbb{Z}$.

Denote by $AAS(X)$ the set of all almost automorphic sequences from $\mathbb{Z}$ to X. Then $(AAS(X), \|\cdot\|_{AAS(X)})$ is a Banach space with the supremum norm,

$$\|u\|_{AAS(X)} = \sup_{k\in\mathbb{Z}} \|u(k)\|_X.$$

Definition 2.2. A function $f : \mathbb{Z} \times X \to X$ is said to be almost automorphic sequence in k for each $x \in X$ if for every sequence of integers $\{k_l\}_{l\in\mathbb{N}}$ there exists a subsequence $\{k_n\}_{n\in\mathbb{N}}$ such that

$$f(k + k_n, x) \to g(k, x) \quad \text{and} \quad g(k - k_n, x) \to f(k, x)$$

for each $k \in \mathbb{Z}$ and $x \in X$. The set of all such functions are denoted by $AAS(\mathbb{Z} \times X, X)$.

A bounded function $h : \mathbb{Z} \times X \to X$ is said to be in $PAA_0S(\mathbb{Z} \times X, \rho)$ for some $\rho \in \mathbb{U}_\infty$ if

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \sum_{-r}^{r} \|h(k,x)\|_X \rho(k) = 0$$

for each $x \in X$.

Definition 2.3. A function $f : \mathbb{Z} \times X \to X$ is said to be weighted pseudo almost automorphic sequence if it can be decomposed as a sum of two functions

$$f = f_1 + f_2,$$

where f_1 is almost automorphic sequence and $f_2 \in PAA_0S(\mathbb{Z} \times X, \rho)$.

The following theorem is from Blot et. al. [8].

Theorem 2.4. *The decomposition of a weighted pseudo almost automorphic function is unique for any $\rho \in U_b$.*

We assume that for any weighted pseudo almost automorphic functions $f(t)$ over $\mathbb{R}$, the sequence $\{x_n\}$ define by $x(n) = f(n)$ for $n \in \mathbb{Z}$ is weighted pseudo almost automorphic.

As an example, consider the function

$$f(k) = \text{signum}(\cos 2\pi k\theta) + e^{-|k|}$$

It is well known that the function signum$(\cos(2\pi k\theta))$, $k \in \mathbb{Z}$ is almost automorphic sequence for θ irrational [9]. Now consider the weight function ρ_1 defined by

$$\rho_1(k) = 1 + k^2 \quad k \in \mathbb{Z} \tag{2.1}$$

It is easy to verify that

$$m(r, \rho_1) = \sum_{-r}^{r} \rho_1(k) = \sum_{-r}^{r} (1 + k^2) = \frac{r(r+1)(2r+1)}{3}.$$

Thus $\lim_{r\to\infty} m(r, \rho_1) = \infty$. Thus $\rho_1 \in \mathbb{U}_\infty$. Further

$$\begin{aligned} \lim_{r\to\infty} \frac{1}{m(r,\rho_1)} \sum_{-r}^{r} e^{-|k|}\rho_1(k) &= \lim_{r\to\infty} \frac{1}{m(r,\rho_1)} \sum_{-r}^{r} e^{-|k|}(1 + k^2) \\ &= \lim_{r\to\infty} \frac{2}{m(r,\rho_1)} \Big(-1 + \sum_{0}^{r} e^{-k}(1 + k^2) \Big) = 0 \end{aligned} \tag{2.2}$$

since

$$\sum_{0}^{r} e^{-k}k^2 \le \int_0^r k^2 e^{-k} dk < \infty.$$

Hence $e^{-|k|} \in PAA_0S(\mathbb{Z}, \rho_1)$ and so $f(k) \in WPAAS(\mathbb{Z})$.

Further consider another weight function ρ_2 defined by

$$\rho_2(k) = \begin{cases} e^{\beta k} & k < 0, \\ 1 & k \ge 0 \end{cases}$$

for some $\beta > 0$. It is easy to verify that

$$m(r, \rho_2) = \sum_{-r}^{r} \rho_2(k) = \sum_{-r}^{0} \rho_2(k) + \sum_{1}^{r} \rho_2(k) = r + \sum_{-r}^{0} e^{\beta k}.$$

Thus $\lim_{r\to\infty} m(r, \rho_2) = \lim_{r\to\infty}(r + \frac{1}{1-e^{-\beta}}) \to \infty$ which implies that $\rho_2 \in \mathbb{U}_\infty$. Further

$$\lim_{r\to\infty} \frac{1}{m(r,\rho_2)} \sum_{-r}^{r} e^{-|k|}\rho_2(k) = \lim_{r\to\infty} \frac{1}{m(r,\rho_2)} \Big(\sum_{-r}^{0} e^{(\beta+1)k} + \sum_{1}^{r} e^{-k} \Big) = 0$$

for $\beta > 0$. Because $e^{-|k|} \in PAA_0S(\mathbb{Z}, \rho_2)$ and so $f(k) \in WPAAS(\mathbb{Z})$.

Now we state some important properties of weighted pseudo almost automorphic sequences.

Theorem 2.5. *Let u, v be weighted pseudo almost automorphic sequences with same weight function ρ, then the following assertions are true:*

(i) *The sum $u + v$ is weighted pseudo almost automorphic sequence;*
(ii) *For every scalar $\lambda, \lambda u$ is weighted pseudo almost automorphic sequence;*
(iii) *The translation $u_k(l) = u(k + l)$ is weighted pseudo almost automorphic sequence for fixed l if the sequences $\frac{\rho(k-l)}{\rho(k)}$ and $\frac{\rho(k)}{\rho(k-l)}$ are bounded.*

The proof of the above theorem is easy to verify. The proof of assertion (iii) is similar to the proof in [6].

Theorem 2.6. *Let X and Y be two Banach spaces. Also let $u : \mathbb{Z} \to X$ be a weighted pseudo almost automorphic sequence. If $T : X \to Y$ is a bounded linear continuous function, then the composition $Tou : \mathbb{Z} \to Y$ is weighted pseudo almost automorphic sequence.*

Proof. Let $u = u_1 + u_2$, where u_1 is almost automorphic and $u_2 \in PAA_0S$. Now for $\rho \in \mathbb{U}_\infty$, consider

$$\begin{aligned}
\lim_{r\to\infty} \frac{1}{m(r,\rho)} \sum_{-r}^{r} \|Tou_2(k)\|\rho(k) &\le \lim_{r\to\infty} \frac{1}{m(r,\rho)} \sum_{-r}^{r} \|T\|\|u_2(k)\|\rho(k) \\
&\le \lim_{r\to\infty} \frac{C}{m(r,\rho)} \sum_{-r}^{r} \|u_2(k)\|\rho(k) \to 0.
\end{aligned}$$

To show that Tou_1 is almost automorphic sequence. As u_1 is almost automorphic sequence, thus for any sequence k_l there exists a subsequence k_n such that

$$u_1(k + k_n) \to \bar{u}_1(k) \quad \text{and} \quad \bar{u}_1(k - k_n) \to u_1(k)$$

for each $k \in \mathbb{Z}$. Now for the sequence k_n we have

$$\begin{aligned}
\lim_{n\to\infty} T(u_1(k + k_n)) = T(\lim_{n\to\infty} u_1(k + k_n)) = T(\bar{u}_1(k)), \\
\lim_{n\to\infty} T(\bar{u}_1(k - k_n)) = T(\lim_{n\to\infty} \bar{u}_1(k - k_n)) = T(u_1(k))
\end{aligned}$$

because T is continuous. Thus Tou_1 is almost automorphic sequence. Hence we conclude that Tou is weighted pseudo almost automorphic sequence. □

Theorem 2.7. *Let $u : \mathbb{Z} \to \mathbb{C}$ be weighted pseudo almost automorphic sequence and $f : \mathbb{Z} \to X$ be almost automorphic sequence. Then the sequence $uf : \mathbb{Z} \to X$ defined by $(uf)(k) = u(k)f(k)$, $k \in \mathbb{Z}$ is also weighted pseudo almost automorphic.*

Proof. As u is weighted pseudo almost automorphic, thus u can be written as $u = u_1 + u_2$, where u_1 is almost automorphic and $u_2 \in PAA_0S$. Hence we get $u(k)f(k) = u_1(k)f(k) + u_2(k)f(k)$. Now for $\rho \in \mathbb{U}_\infty$, consider

$$\begin{aligned}
\lim_{r\to\infty} \frac{1}{m(r,\rho)} \sum_{-r}^{r} \|u_2(k)f(k)\|\rho(k) &\le \lim_{r\to\infty} \frac{1}{m(r,\rho)} \sum_{-r}^{r} \|u_2(k)\|\|f(k)\|\rho(k) \\
&\le \lim_{r\to\infty} \frac{\|f\|}{m(r,\rho)} \sum_{-r}^{r} \|u_2(k)\|\rho(k) \to 0,
\end{aligned}$$

as we know that $\|f\| = \sup_{k\in\mathbb{Z}} \|f(k)\| < \infty$ from [7]. For any sequence k_l there exists a subsequence k_n such that

$$u_1(k+k_n) \to \bar{u}_1(k), \quad \bar{u}_1(k-k_n) \to u_1(k),$$
$$f(k+k_n) \to \bar{f}(k), \quad \bar{f}(k-k_n) \to f(k)$$

for each $k \in \mathbb{Z}$. Now consider

$$\begin{aligned}
&\|u_1(k+k)n)f(k+k_n) - \bar{u}_1(k)\bar{f}(k)\| \\
&\le \|f(k+k_n) - \bar{f}(k)\|\|u(k+k_n)\| + \|u(k+k_n) - \bar{u}_1(k)\|\|\bar{f}(k)\| \to 0,
\end{aligned}$$

as $n \to \infty$. Also we have

$$\begin{aligned}
&\|\bar{u}_1(k-k)n)\bar{f}(k-k_n) - u_1(k)f(k)\| \\
&\le \|\bar{f}(k-k_n) - f(k)\|\|\bar{u}(k-k_n)\| + \|\bar{u}(k-k_n) - u_1(k)\|\|f(k)\| \to 0,
\end{aligned}$$

as $n \to \infty$. Thus we conclude that $u_1(k)f(k)$ is almost automorphic sequence and hence $u(k)f(k)$ is weighted pseudo almost automorphic sequence. $\square$

Theorem 2.8. *Let $f : \mathbb{Z} \times X \to X$ be almost automorphic sequence which satisfies a Lipschitz condition in x uniformly in k, that is*

$$\|f(k,x) - f(k,y)\| \le L\|x-y\|,$$

for each $x, y \in X$. Assume $\phi : \mathbb{Z} \to X$ is weighted pseudo almost automorphic sequence, then the function $f(k,\phi(k))$ is weighted pseudo almost automorphic sequence.

Proof. We know that ϕ is weighted pseudo almost automorphic sequence, thus $\phi = \phi_1 + \phi_2$ where ϕ_1 is almost automorphic sequence and $\phi_2 \in PAA_0S$. Writing

$$f(k,\phi(k)) = f(k,\phi_1(k)+\phi_2(k)) - f(k,\phi_1(k)) + f(k,\phi_1(k)).$$

As f is Lipschitz, we have

$$\|f(k,\phi_1(k)+\phi_2(k)) - f(k,\phi_1(k))\| \le L\|\phi_2(k)\|.$$

Thus for $\rho \in \mathbb{U}_\infty$ we obtain

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{m(r,\rho)} \sum_{-r}^{r} \|f(k,\phi_1(k)+\phi_2(k)) - f(k,\phi_1(k))\|\rho(k) \\
&\le \lim_{r\to\infty} \frac{L}{m(r,\rho)} \sum_{-r}^{r} \|\phi_2(k)\|\rho(k) \to 0.
\end{aligned}$$

We know that f and ϕ_1 are almost automorphic sequences. So for each sequence k_l there exist a subsequence k_n such that

$$f(k+k_n,x) \to g(k,x) \quad \text{and} \quad g(k-k_n,x) \to f(k,x)$$

for all $k \in \mathbb{Z}$, $x \in X$. Also we have

$$\phi_1(k+k_n) \to \psi(k) \quad \text{and} \quad \psi(k-k_n) \to \phi_1(k)$$

for all $k \in \mathbb{Z}$, $x \in X$.

Now consider

$$\begin{aligned}
&f(k+k_n,\phi_1(k+k_n)) - g(k,\psi(k)) \\
&= f(k+k_n,\phi_1(k+k_n)) - f(k+k_n,\psi(k)) + f(k+k_n,\psi(k)) - g(k,\psi(k)),
\end{aligned}$$

we have

$$\begin{aligned}
&\|f(k+k_n,\phi_1(k+k_n))-g(k,\psi(k))\| \\
&\le \|f(k+k_n,\phi_1(k+k_n))-f(k+k_n,\psi(k))\|+\|f(k+k_n,\psi(k))-g(k,\psi(k))\| \\
&\le L\|\phi(k+k_n)-\psi(k)\|+\|f(k+k_n,\psi(k))-g(k,\psi(k))\|\to 0,
\end{aligned}$$

as $n\to\infty$. Thus we conclude that

$$f(k+k_n,\phi_1(k+k_n))\to g(k,\psi(k)).$$

Next assume

$$\begin{aligned}
&g(k-k_n,\psi(k-k_n))-f(k,\phi_1(k)) \\
&= g(k-k_n,\psi(k-k_n))-g(k-k_n,\phi_1(k))+g(k-k_n,\phi_1(k))-f(k,\phi_1(k)),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\|g(k-k_n,\psi(k-k_n))-f(k,\phi_1(k))\| \\
&\le \|g(k-k_n,\psi(k-k_n))-g(k-k_n,\phi_1(k))\|+\|g(k-k_n,\phi_1(k))-f(k,\phi_1(k))\| \\
&\le L\|\psi(k-k_n)-\phi_1(k)\|+\|g(k-k_n,\phi_1(k))-f(k,\phi_1(k))\|\to 0,
\end{aligned}$$

as $n\to\infty$. Hence

$$g(k-k_n,\psi(k-k_n))\to f(k,\phi_1(k)).$$

It is easy to verify that g satisfies the same Lipschitz condition as f. By the above analysis one can conclude that $f(k,\phi_1(k))$ is almost automorphic sequence. Hence $f(k,\phi)$ is weighted pseudo almost automorphic sequence.

Define the operator $\Delta u(k)=u(k+1)-u(k)$. Now it is easy to verify that for any weighted pseudo almost automorphic sequence $\{u(k)\}_{k\in\mathbb{Z}}$, $\Delta u(k)$ is weighted pseudo almost automorphic sequence. $\square$

The next result is important to study weighted pseudo almost automorphic sequence solutions of difference equations. For the theorems listed below we assume that the function ρ satisfy condition (iii) of theorem 2.5.

Theorem 2.9. *Assume $v:\mathbb{Z}\to\mathbb{C}$ is a summable function, that is, $\sum_{k\in\mathbb{Z}}|v(k)|<\infty$. Then for any weighted pseudo almost automorphic sequence $u:\mathbb{Z}\to X$ the following function*

$$w(k)=\sum_{l\in\mathbb{Z}}v(l)u(k-l),\quad k\in\mathbb{Z}$$

is weighted pseudo almost automorphic sequence.

Proof. Consider $w_2(k)=\sum_{l\in\mathbb{Z}}v(l)u_2(k-l)$, where $u_2\in PAA_0S$ is the second component of u. We have

$$\|w_2(k)\|\le\sum_{l\in\mathbb{Z}}|v(l)|\|u_2(k-l)\|\le\sum_{l\in\mathbb{Z}}|v(l)|\|u_2\|\le M\|u_2\|,$$

for some positive constant M. Thus one gets

$$\lim_{r\to\infty}\frac{1}{m(r,\rho)}\sum_{-r}^{r}\|w_2(k)\|\rho(k)\le\lim_{r\to\infty}\frac{M}{m(r,\rho)}\sum_{-r}^{r}\|u_2(k)\|\rho(k)\to 0.$$

Let $w_1(k)=\sum_{l\in\mathbb{Z}}v(l)u_1(k-l)$, where u_1 is the almost automorphic component of u. The almost automorphy of w_1 follows from [7, theorem 2.13]. $\square$

The theorems stated below are easy generalization of [7, theorems 3.1 and 4.1]. Consider the difference equation

$$u(n+1) = \lambda u(n) + f(n), \tag{2.3}$$

where λ is any real or complex number. Now denote the region $\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$.

Theorem 2.10. *Let X be a Banach space. If $\lambda \in \mathbb{C}\backslash\mathbb{D}$ and $f : \mathbb{Z} \to X$ is weighted pseudo almost automorphic sequence, then the weighted pseudo almost automorphic sequence solution of equation* (2.3) *is given by*

(i) $u(n) = \sum_{k=-\infty}^{n} \lambda^{n-k} f(k-1)$ *if* $|\lambda| < 1$;
(ii) $u(n) = -\sum_{k=n}^{\infty} \lambda^{n-k-1} f(k)$ *if* $|\lambda| > 1$.

We have the following theorem for the difference equation

$$u(n+1) = \lambda u(n) + f(n, u(n)). \tag{2.4}$$

Theorem 2.11. *Let X be a Banach space. If $\lambda \in \mathbb{C}\backslash\mathbb{D}$ and $f : \mathbb{Z} \times X \to X$ be a weighted pseudo almost automorphic sequence that satisfies the following Lipschitz condition*

$$\|f(k,x) - f(k,y)\| \le L\|x - y\|$$

for each $x, y \in X$ and $k \in \mathbb{Z}$. Then the weighted pseudo almost automorphic sequence solution of difference equation (2.4) *is given by*

(i) $u(n) = \sum_{k=-\infty}^{n} \lambda^{n-k} f(k-1, u(k-1))$ *if* $|\lambda| < 1 - L$;
(ii) $u(n) = -\sum_{k=n}^{\infty} \lambda^{n-k-1} f(k, u(k))$ *if* $|\lambda| > 1 + L$.

3. Weighted pseudo almost automorphic sequence in neural networks

In this section we consider a discrete model of cellular neural network and show the existence of a unique weighted pseudo almost automorphic sequence solution. A cellular neural network is a nonlinear dynamic circuit consisting of many processing units called cells arranged in two or three dimensional array. This is very useful in the areas of signal processing, image processing, pattern classification and associative memories. Hence, the application of cellular networks is of great interest to many researchers. In [10, 11, 12, 17], the authors have dealt with the global exponential stability and the existence of a periodic solution of a cellular neural network with delays using the general method of Lyapunov functional. The discrete analogue of continuous time cellular network models are very important for theoretical analysis as well as for implementation. Thus, it is essential to formulate a discrete time analogue of continuous time network. A most acceptable method is to discretize the continuous time network. For detailed analysis on the discretization method the reader may consult Mohamad and Gopalsamy [17], Stewart [18].

Consider the following model of cellular neural network consisting of m interconnected cells

$$\frac{dx_i(t)}{dt} = -a_i([t])x_i(t) + \sum_{j=1}^{m} b_{ij}([t]) f_j(x_j([t])) + I_i([t]),$$

where $i = 1, 2, \dots, m$ and $[\cdot]$ denote the greatest integer function. The differential equations of above kind are called equations with piecewise constant argument. The functions $x_i(t)$ denotes the potential of the cell i at time t. The terms $a_i(t)$ denotes the rate which the cell i resets its potential to the resting state when isolated from other cells and inputs. $b_{ij}(t)$ denotes the strengths of connectivity between the jth cell and the ith cell. The functions f_i and I_i denote the nonlinear output function and external input source introduced from outside the network to ith component, respectively.

The discrete analogue of the above model is given by

$$x_i(n+1) = x_i(n)e^{-a_i(n)} + \frac{1-e^{-a_i(n)}}{a_i(n)}\Big\{\sum_{j=1}^{m} b_{ij}(n)f_j(x_j(n)) + I_i(n)\Big\}, \tag{3.1}$$

where $i = 1, 2, \dots, m$, $n \in \mathbb{Z}$. The existence of weighted pseudo almost automorphic sequence solutions of equation (3.1) has been established in this section. For more details on the model we refer to Huang et. al. [15] in which the authors have proved the existence of an almost periodic sequence solution of equations (3.1). The present author has shown the existence of k pseudo almost periodic sequence solutions of a neural network model [1].

The assumptions described below are necessary to show the existence of weighted almost automorphic solutions of equation (3.1).

Assumptions.

(A1) $a_i(n) > 0$ is almost automorphic sequence and $b_{ij}(n)$, $I_i(n)$ are weighted pseudo almost automorphic sequence for $i, j = 1, 2, \dots, m$.

(A2) There exist positive constants M_j and L_i such that $|f_j(x)| \le M_j$ and $|f_i(x) - f_i(y)| \le L_i|x-y|$ for each $x, y \in \mathbb{R}$ and $j = 1, 2, \dots, m$; $i = 1, 2, \dots, m$.

For the discrete equation (3.1), define

$$C_i(n) = e^{-a_i(n)}, \quad D_{ij}(n) = b_{ij}(n)\frac{1-e^{-a_i(n)}}{a_i(n)}, \quad F_i(n) = I_i(n)\frac{1-e^{-a_i(n)}}{a_i(n)}.$$

Using the above notation, (3.1) can be written as,

$$x_i(n+1) = C_i(n)x_i(n) + \sum_{j=1}^{m} D_{ij}(n)f_j(x_j(n)) + F_i(n), \tag{3.2}$$

for $i = 1, 2, \dots, m$. Denote:

$$C_i^* = \sup_{n\in\mathbb{Z}} |C_i(n)|, \quad I_i^* = \sup_{n\in\mathbb{Z}} |I_i(n)|,$$
$$D_{ij}^* = \sup_{n\in\mathbb{Z}} |D_{ij}(n)|, \quad F_i^* = \sup_{n\in\mathbb{Z}} |F_i(n)|,$$
$$b_{ij}^* = \sup_{n\in\mathbb{Z}} |b_{ij}(n)|, \quad a_i^* = \inf_{n\in\mathbb{Z}} a_i(n), \quad P_i = \sum_{j=1}^{m} D_{ij}^* M_j + F_i^*.$$

Definition 3.1. A solution $x(\nu) = (x_1(\nu), \dots, x_m(\nu))^T$ of (3.2) is said to be globally attractive if for any other solution $y(\nu) = (y_1(\nu), \dots, y_m(\nu))^T$ of (3.2), we have

$$\lim_{\nu\to\infty} |x_i(\nu) - y_i(\nu)| = 0.$$

Lemma 3.2. *Suppose assumption* (A1) *holds, then* $C_i \in AAS$ *and* $D_{ij}, F_i \in WPAAS$ *for* $i, j = 1, 2, \dots, m$.

Proof. From the assumption (A1) we know that $a_i(n)$ is almost automorphic sequence. Thus for any sequence k_l there exists a subsequence k_m such that

$$a_i(n+k_m) \to a_{i1}(n) \text{ and } a_{i1}(n-k_m) \to a_i(n).$$

Denoting $C_{i1}(n) = e^{-a_{i1}(n)}$, we have

$$|C_i(n+k_m) - C_{i1}(n)| = |e^{-a_i(n+k_m)} - e^{-a_{i1}(n)}| \le |a_i(n+k_m) - a_{i1}(n)| \to 0,$$

as $m \to \infty$. Also

$$|C_{i1}(n-k_m) - C_i(n)| = |e^{-a_{i1}(n-k_m)} - e^{-a_i(n)}| \le |a_{i1}(n-k_m) - a_i(n)| \to 0,$$

as $m \to \infty$. Thus one can conclude that $C_i(n)$ are almost automorphic. Now since b_{ij} and I_i are weighted pseudo almost automorphic, we can decompose them into two parts

$$b_{ij} = b_{ij1} + b_{ij2} \quad \text{and} \quad I_i = I_{i1} + I_{i2}$$

such that $b_{ij1}, I_{i1} \in AAS$ and $b_{ij2}, I_{i2} \in PAA_0S$. We have

$$\begin{gathered}
b_{ij1}(n+k_m) \to \bar{b}_{ij1}(n), \quad \bar{b}_{ij1}(n-k_m) \to b_{ij1}(n), \\
I_{i1}(n+k_m) \to \bar{I}_{i1}(n), \quad \bar{I}_{i1}(n-k_m) \to I_{i1}(n).
\end{gathered}$$

Also for $\rho \in \mathbb{U}_\infty$ we obtain

$$\begin{gathered}
\lim_{r\to\infty} \frac{1}{m(\rho,r)} \sum_{n=-r}^{r} |b_{ij2}(n)|\rho(n) = 0, \\
\lim_{r\to\infty} \frac{1}{m(\rho,r)} \sum_{n=-r}^{r} |I_{i2}(n)|\rho(n) = 0.
\end{gathered}$$

Denote $D_{ij1}(n) = b_{ij1}(n)\frac{1-e^{-a_i(n)}}{a_i(n)}$ and $\bar{D}_{ij1}(n) = \bar{b}_{ij1}(n)\frac{1-e^{-a_{i1}(n)}}{a_{i1}(n)}$ we have

$$\begin{aligned}
&|D_{ij1}(n+k_m) - \bar{D}_{ij1}(n)| \\
&= \Big|b_{ij1}(n+k_m)\frac{1-e^{-a_i(n+k_m)}}{a_i(n+k_m)} - \bar{b}_{ij1}(n)\frac{1-e^{-a_{i1}(n)}}{a_{i1}(n)}\Big| \\
&\le |b_{ij1}(n+k_m) - \bar{b}_{ij1}(n)| \times \Big|\frac{1-e^{-a_i(n+k_m)}}{a_i(n+k_m)}\Big| \\
&\quad + |\bar{b}_{ij1}(n)| \times \Big|\frac{1-e^{-a_i(n+k_m)}}{a_i(n+k_m)} - \frac{1-e^{-a_{i1}(n)}}{a_{i1}(n)}\Big| \\
&\to \infty, \quad \text{as } m \to \infty.
\end{aligned}$$

Also

$$\begin{aligned}
&|\bar{D}_{ij1}(n-k_m) - D_{ij1}(n)| \\
&= \Big|\bar{b}_{ij1}(n-k_m)\frac{1-e^{-a_{i1}(n-k_m)}}{a_{i1}(n-k_m)} - b_{ij1}(n)\frac{1-e^{-a_i(n)}}{a_i(n)}\Big| \\
&\le |\bar{b}_{ij1}(n-k_m) - b_{ij1}(n)| \times \Big|\frac{1-e^{-a_{i1}(n-k_m)}}{a_{i1}(n-k_m)}\Big| \\
&\quad + |b_{ij1}(n)| \times \Big|\frac{1-e^{-a_{i1}(n-k_m)}}{a_{i1}(n-k_m)} - \frac{1-e^{-a_i(n)}}{a_i(n)}\Big| \\
&\to \infty, \quad \text{as } m \to \infty.
\end{aligned}$$

Considering $D_{ij2}(n) = b_{ij2}(n)\frac{1-e^{-a_i(n)}}{a_i(n)}$ one obtains

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{m(r,\rho)} \sum_{n=-r}^{r} |D_{ij2}(n)|\rho(n) \\
&\leq \lim_{r\to\infty} \frac{1}{m(r,\rho)} \sum_{n=-r}^{r} |b_{ij2}(n)| \times \Big|\frac{1-e^{-a_i(n)}}{a_i(n)}\Big|\rho(n) \\
&\leq \lim_{r\to\infty} \frac{1}{a_i^* m(r,\rho)} \sum_{n=-r}^{r} |b_{ij2}(n)|\rho(n).
\end{aligned}$$

Since b_{ij2} are weighted pseudo almost automorphic sequences, we have

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \sum_{n=-r}^{r} |D_{ij2}(n)|\rho(n) = 0.$$

Thus D_{ij} are weighted pseudo almost automorphic sequences. By the similar analysis one can easily show that F_i are also weighted pseudo almost automorphic sequences. □

Lemma 3.3. *Under assumptions* (A1), (A2), *every solution of* (3.2) *is bounded.*

Proof. One can easily observe that the relation

$$C_i(n)x_i(n) - R_i \leq x_i(n+1) \leq C_i(n)x_i(n) + R_i,$$

where $R_i = \sum_{j=1}^m D_{ij}^* + F_i^*$ holds. Consider the difference equations

$$\bar{x}_i(n+1) = C_i(n)\bar{x}_i(n) + R_i,$$

where $\bar{x}_i(0) = x_i(0)$. Using induction we have

$$\begin{aligned}
\bar{x}_i(n) &= \prod_{k=1}^{n} C_i(k)\bar{x}_i(0) + R_i\Big(\sum_{l=1}^{n-1}\prod_{k=1}^{l} C_i(k) + 1\Big) \\
&\leq e^{-na_i^*}\bar{x}_i(0) + R_i(\sum_{l=1}^{n-1} e^{-la_i^*} + 1) \\
&\leq |\bar{x}_i(0)| + \frac{R_i}{1-e^{-a_i^*}}.
\end{aligned}$$

One can easily observe that $x_i(n) \leq \bar{x}_i(n)$. Now using the difference equation

$$\tilde{x}_i(n+1) = C_i(n)\tilde{x}_i(n) - R_i$$

and doing similar calculation we obtain

$$\tilde{x}_i(n) \geq -|\bar{x}_i(0)| - \frac{R_i}{1-e^{-a_i^*}}.$$

Combining the two inequalities above, we have the estimate

$$-|x_i(0)| - \frac{R_i}{1-e^{-a_i^*}} \leq x_i(n) \leq |x_i(0)| + \frac{R_i}{1-e^{-a_i^*}}.$$

Thus x_i are bounded. □

Now consider the difference equations

$$x_i(n+1) = C_i(n)x_i(n) + F_i(n). \tag{3.3}$$

Lemma 3.4. *Under assumption* (A1), *there exists a weighted pseudo almost automorphic sequence solution of* (3.3).

Proof. Using an induction argument, one obtain

$$\begin{aligned} x_i(n+1) &= \prod_{k=0}^{n} C_i(k)x_i(0) + \sum_{l=0}^{n} \prod_{k=n-l+1}^{n} C_i(k)F_i(n-l) \\ &= e^{-\sum_{k=0}^{n} a_i(k)}x_i(0) + \sum_{l=0}^{n} I_i(n-l)\frac{1-e^{-a_i(n-l)}}{a_i(n-l)}e^{-\sum_{k=n-l+1}^{n} a_i(k)}. \end{aligned}$$

Consider the sequence

$$\hat{x}_i(n) = \sum_{l=0}^{\infty} I_i(n-l)\frac{1-e^{-a_i(n-l)}}{a_i(n-l)}e^{-\sum_{k=n-l+1}^{n} a_i(k)}.$$

Since

$$|\hat{x}_i(n)| \le \sum_{l=0}^{\infty} I_i^* \frac{1-e^{-a_i^*}}{a_i^*}e^{-(l-1)a_i^*} = \sum_{l=0}^{\infty} e^{a_i^*}\frac{1-e^{-a_i^*}}{a_i^*}e^{-la_i^*} \le \frac{I_i^* e^{a_i^*}}{a_i^*},$$

the sequence $\hat{x}_i(n)$ is well defined. It is easy to verify that

$$\hat{x}_i(n+1) = C_i(n)\hat{x}_i(n) + F_i(n).$$

Hence the sequence $\hat{x}_i = \{\hat{x}_i(n)\}$ is bounded. Now define

$$\hat{x}_{i1}(n) = \sum_{l=0}^{\infty} I_{i1}(n-l)\frac{1-e^{-a_i(n-l)}}{a_i(n-l)}e^{-\sum_{k=n-l+1}^{n} a_i(k)}$$

and

$$\hat{y}_{i1}(n) = \sum_{l=0}^{\infty} \bar{I}_{i1}(n-l)\frac{1-e^{-a_{i1}(n-l)}}{a_{i1}(n-l)}e^{-\sum_{k=n-l+1}^{n} a_{i1}(k)}.$$

Also let

$$\hat{x}_{i2}(n) = \sum_{l=0}^{\infty} I_{i2}(n-l)\frac{1-e^{-a_i(n-l)}}{a_i(n-l)}e^{-\sum_{k=n-l+1}^{n} a_i(k)},$$

where I_{i1} and I_{i2} are two component of I_i. For any sequence k_l there exists a sequence k_m such that

$$\begin{aligned} &|\hat{x}_{i1}(n+k_m) - \hat{y}_{i1}(n)| \\ &= \Big| \sum_{l=0}^{\infty} I_{i1}(n+k_m-l)\frac{1-e^{-a_i(n+k_m-l)}}{a_i(n+k_m-l)}e^{-\sum_{k=n-l+1}^{n} a_i(k+k_m)} \\ &\quad - \sum_{l=0}^{\infty} \bar{I}_{i1}(n-l)\frac{1-e^{-a_{i1}(n-l)}}{a_{i1}(n-l)}e^{-\sum_{k=n-l+1}^{n} a_{i1}(k)}\Big| \\ &\le \sum_{l=0}^{\infty} \Big(|I_{i1}(n+k_m-l) - \bar{I}_{i1}(n-l)| \frac{1-e^{-a_i(n+k_m-l)}}{|a_i(n+k_m-l)|}e^{-\sum_{k=n-l+1}^{n} |a_i(k+k_m)|} \\ &\quad + \Big|\bar{I}_{i1}(n-l)\Big| \times \Big| \frac{1-e^{-a_i(n+k_m-l)}}{a_i(n+k_m-l)}e^{-\sum_{k=n-l+1}^{n} a_i(k+k_m)} \\ &\quad - \frac{1-e^{-a_{i1}(n-l)}}{a_{i1}(n-l)}e^{-\sum_{k=n-l+1}^{n} a_{i1}(k)}\Big|\Big) \end{aligned}$$

$$
\begin{aligned}
&\le \epsilon\frac{e^{a_i^*}}{a_i^*}+\epsilon\frac{I_i^* e^{a_i^*}}{a_i^{*2}}\\
&\le \frac{e^{a_i^*}}{a_i^*}\epsilon+\frac{I_i^* e^{a_i^*}}{a_i^{*2}}\epsilon.
\end{aligned}
$$

The above calculations imply that

$$\hat{x}_{i1}(n+k_m)\to\hat{y}_{i1}(n)\quad m\to\infty.$$

Similarly one can show that

$$\hat{y}_{i1}(n-k_m)\to\hat{x}_{i1}(n)\quad m\to\infty.$$

Hence x_{i1} is almost automorphic sequence.

For $\rho\in\mathbb{U}_\infty$, we have

$$
\begin{aligned}
\sum_{n=-r}^{r}|\hat{x}_{i2}(n)|\rho(n) &\le \sum_{n=-r}^{r}\sum_{l=0}^{\infty}|I_{i2}(n-l)|\frac{1-e^{-a_i(n-l)}}{|a_i(n-l)|}e^{-\sum_{k=n-l+1}^{n}|a_i(k)|}\rho(n)\\
&\le \sum_{n=-r}^{r}\sum_{l=0}^{\infty}|I_{i2}(n-l)|\frac{e^{-(l-1)a_i^*}}{a_i^*}\rho(n)\\
&\le \frac{e^{a_i^*}}{a_i^*}\sum_{n=-r}^{r}|I_{i2}(n)|\rho(n).
\end{aligned}
$$

Now we can easily observe that

$$\lim_{r\to\infty}\frac{1}{m(r,\rho)}\sum_{n=-r}^{r}|\hat{x}_{i2}(n)|\rho(n)\le\frac{e^{a_i^*}}{a_i^*}\lim_{r\to\infty}\frac{1}{m(r,\rho)}|I_{i2}(n)|\rho(n)=0,$$

because $I_{i2}\in PAA_0S$. We can easily see that $\hat{x}_i=\hat{x}_{i1}+\hat{x}_{i2}$. Hence $\hat{x}_i$ are weighted pseudo almost automorphic sequences (3.3). $\square$

Theorem 3.5. *Assume* (A1), (A2) *hold. There exists a unique weighted pseudo almost automorphic sequence solution of* (3.2) *which is globally attractive, if*

$$\max_{1\le i\le m}\{C_i^*+\sum_{j=1}^{m}D_{ij}^*L_j\}<1.$$

Proof. Denote a metric $d: WPAAS\times WPAAS\to\mathbb{R}^+$, by

$$d(x,y)=\sup_{n\in\mathbb{Z}}\max_{1\le i\le m}|x_i(n)-y_i(n)|.$$

Now define a mapping $F: WPAAS\to WPAAS$ by $Fx=y$, where

$$Fx=(F_1x,F_2x,\dots,F_mx)^T$$

such that $F_ix=y_i$ and $y_i=\{y_i(n)\}$. Define

$$y_i(n+1)=C_i(n)x_i(n)+\sum_{j=1}^{m}D_{ij}(n)f_j(x_j(n))+F_i(n),$$

where $\hat{x}_i$ is weighted pseudo almost automorphic sequence solution of (3.3). Using lemma 3.2 and assumption (A2), one can observe that F maps weighted pseudo almost automorphic sequences into weighted pseudo almost automorphic sequences. Now denote

$$\max_{1\le i\le m}\{C_i^* + \sum_{j=1}^{m} D_{ij}^* L_j\} = r < 1.$$

For $x, y \in WPAAS$, we have

$$\begin{aligned}
\|Fx - Fy\| &= \sup_{n\in\mathbb{Z}} \max_{1\le i\le m} \sum_{j=1}^{m} \Big|[(D_{ij}(n)(f_j(x_j(n)) - f_j(y_j(n)))]\Big| \\
&\le \sup_{n\in\mathbb{Z}} \max_{1\le i\le m} \sum_{j=1}^{m} D_{ij}^* L_j |x_j(n) - y_j(n)| \\
&\le r\|x - y\|.
\end{aligned}$$

Hence F is a contraction. It follows that equation (3.2) has a unique weighted pseudo almost automorphic sequence x. Let y be any sequence satisfying equation (3.2). Consider $Q(n) = x(n) - y(n)$, then we obtain

$$Q_i(n+1) = C_i(n)Q_i(n) + \sum_{j=1}^{m} D_{ij}(n)(f_j(x_j(n)) - f_j(y_j(n))).$$

Taking modulus of both sides one has

$$|Q_i(n+1)| \le C_i^*|Q_i(n)| + \sum_{j=1}^{m} D_{ij}^* L_j |Q_j(n)|.$$

Define $Q(n) = \max_{1\le i\le m} |Q_i(n)|$, we have

$$|Q(n+1)| \le C_i^*|Q(n)| + \sum_{j=1}^{m} D_{ij}^* L_j Q(n) \le rQ(n).$$

By induction we have $Q(n) \le r^n Q(0)$. Hence

$$|x_i(n) - y_i(n)| \to 0 \quad \text{as } n \to \infty.$$

Thus x is a unique weighted pseudo almost automorphic sequence solution of (3.2) which is globally attractive. □

Remark: For $\rho = 1$ the solution is pseudo almost automorphic and is denoted by $PAAS$. The results presented in this paper can also be applied to more general models like delayed model of neural networks.

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