A PARABOLIC-HYPERBOLIC SYSTEM MODELLING A MOVING CELL

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Abstract. In this article, we study the existence and uniqueness of local solutions for a moving boundary problem governed by a coupled parabolic-hyperbolic system. The results can be applied to cell movement, extending a result obtained by Choi, Groulx, and Lui in 2005.

1. Introduction

In this article, we consider the system of n hyperbolic equations coupled with a single parabolic equation

$$\mathbf{w}_t = -\sigma_x \mathbf{w}_x - \mathbf{F}(\mathbf{w}, \sigma),$$
$$\sigma_t = g(\mathbf{w}, \sigma)\sigma_{xx} - \sigma_x^2 + h(\mathbf{w}, \sigma), \tag{1.1}$$

where $\mathbf{F}$, g, and h are given C^1 functions in their respective variables and $x \in [r(t), f(t)]$. The boundary conditions are

$$\mathbf{w} = \mathbf{w}_f(t), \quad \sigma = 0, \quad \text{at } x = f(t),$$
$$\sigma = 0, \quad \text{at } x = r(t) \tag{1.2}$$

with $\mathbf{w}_f$ being a given C^1 function. Motion of the boundaries are defined by

$$\frac{df}{dt} = V|_{f(t)-r(t)} + \sigma_x|_{x=f(t)},$$
$$\frac{dr}{dt} = 1 + \sigma_x|_{x=r(t)}. \tag{1.3}$$

Here $V : (0, \infty) \to (0, \infty)$ is a given C^1 function with $V(\ell) > 0$ when $\ell > 0$. We observe that $\ell(t) = f(t) - r(t)$ represents the instantaneous domain size. The moving boundary problem consists of equations (1.1), (1.2), (1.3), and initial conditions

$$r(0) = 0, \quad f(0) = \ell_0 > 0, \quad \mathbf{w}(x, 0) = \mathbf{w}_0(x), \quad \sigma(x, 0) = \psi(x) \tag{1.4}$$

with $\mathbf{w}_0 \in C^1[0, \ell_0]$ and $\psi \in C^{2+\beta}[0, \ell_0]$ for some $0 < \beta < 1$. In order for (1.1)b) to be parabolic, we impose

$$g(\mathbf{w}_0, \psi) > 0 \quad \text{for all } x \in [0, \ell_0]. \tag{1.5}$$

We also require the initial conditions be compatible with the boundary conditions and the moving conditions to first order (see p.319, [4] for the parabolic case).

The zeroth order compatibility is equivalent to

$$\psi(0) = \psi(\ell_0) = 0, \quad \mathbf{w}_0(\ell_0) = \mathbf{w}_f(0). \tag{1.6}$$

For the first order compatibility one differentiates $\sigma(f(t), t) = 0$ which yields $\sigma_x \frac{df}{dt} + \sigma_t = 0$. Therefore, at $(x, t) = (\ell_0, 0)$, we have

$$\psi_x(V + \psi_x) + g(\mathbf{w}_0, \psi)\psi_{xx} - \psi_x^2 + h(\mathbf{w}_0, \psi) = 0 \tag{1.7}$$

with ψ, ψ_x, ψ_{xx}, $\mathbf{w}_0$ and V all evaluated at ℓ_0. Similarly the first order compatibility for σ at $(x, t) = (0, 0)$ leads to

$$\psi_x(1 + \psi_x) + g(\mathbf{w}_0, \psi)\psi_{xx} - \psi_x^2 + h(\mathbf{w}_0, \psi) = 0 \tag{1.8}$$

with all the terms evaluated at $x = 0$.

One needs only the first order compatibility for $\mathbf{w}_0$ at $(x, t) = (\ell_0, 0)$. From $\mathbf{w}(f(t), t) = \mathbf{w}_f(t)$, we obtain

$$\mathbf{w}_{0,x}(V + \psi_x) - \psi_x \mathbf{w}_{0,x} - \mathbf{F}(\mathbf{w}_0, \psi) = \frac{d\mathbf{w}_f}{dt}(0) \tag{1.9}$$

with ψ, ψ_x, $\mathbf{w}_0$, $\mathbf{w}_{0,x}$ and V all evaluated at ℓ_0.

Under such conditions we will prove the following theorem by mapping the changing domain $Q_\epsilon \equiv \{(x, t) : 0 < t < \epsilon, \ r(t) < x < f(t)\}$ to a rectangle with a unit length.

Theorem 1.1. *Consider the moving boundary problem* (1.1), (1.2) *and* (1.3) *with initial conditions* (1.4) *with* $\mathbf{w}_0 \in C^1[0, \ell_0]$ *and* $\psi \in C^{2+\beta}[0, \ell_0]$ *for some* $0 < \beta < 1$. *Furthermore assume* (1.5) *holds and the boundary and the initial conditions are compatible to first order; i.e.,* (1.6) *to* (1.9) *hold. Then there exists an* $\epsilon > 0$ *such that the moving boundary problem has unique solutions* $\mathbf{w} \in C^{1,1}$ *and* $\sigma \in C^{2+\beta,(2+\beta)/2}$ *in the domain* $\overline{Q_\epsilon}$.

We now explain the number of boundary conditions needed for $\mathbf{w}$. Let the characteristics associated with hyperbolic equations (1.1)a) through a point (x_0, t_0) be denoted by $x = \tilde{x}(x_0, t_0, t)$. Then $\partial \tilde{x}/\partial t = \sigma_x(\tilde{x}, t)$. From the moving conditions (1.3), the front end is moving at a positive speed $V(\ell(t))$ faster than the characteristic. In other words the characteristics from the front $x = f(t)$ are going into the domain $[r(t), f(t)]$. Hence a boundary condition is needed for $\mathbf{w}$ at the front. On the other hand the characteristic at $x = r(t)$ is going outside the domain $[r(t), f(t)]$. No boundary condition can be imposed at the rear end.

2. MOTIVATION

The study of the system (1.1) is motivated by the one-dimensional model for the movement of a nematode sperm cell proposed by Mogilner and Verzi [5]. Based on the principles of mechanics, they proposed

$$\begin{aligned}
\frac{\partial b}{\partial t} &= -\frac{\partial}{\partial x}(bv) - \gamma_b b, \\
\frac{\partial p}{\partial t} &= -\frac{\partial}{\partial x}(pv) + \gamma_b b - \gamma_p p, \\
\frac{\partial c}{\partial t} &= -\frac{\partial}{\partial x}(cv),
\end{aligned} \tag{2.1}$$

where b, p denote the length densities of the bundled filaments and free filaments, respectively, c is the density of the cytoskeletal nodes (responsible for cell adhesion), v is the cytoskeletal velocity, γ_b is the rate of unbundling of the bundled filaments, and γ_p is the rate of disassembly of the free filaments. System (2.1) is derived from mass balance and is assumed to hold for $x \in [r(t), f(t)]$, which denotes the spatial interval defined by the rear and front ends of the cell at time t.

A balance between frictional and elastic filament forces leads Mogilner and Verzi to assume

$$v(x, t) = \frac{1}{\xi} \frac{\partial \sigma}{\partial x}, \tag{2.2}$$

where ξ is the effective drag coefficient between the cell and the substratum, and σ is the total filament stress with a constitutive law defined by

$$\sigma = Kb(\frac{1}{c} - \rho) + \kappa \frac{p}{c}, \tag{2.3}$$

where K and κ are the effective spring constants for the bundled and free filaments, respectively, and ρ is the rest length of the bundled filament while the free filament is assumed to have natural length 0. This formula for stress is based on Hooke's law with the average distance between two cytoskeletal nodes being $1/c$.

Motion of the boundaries are defined by

$$\frac{df}{dt} = V_p|_{f(t)-r(t)} + v|_{f(t)}, \tag{2.4}$$
$$\frac{dr}{dt} = V_d + v|_{r(t)}$$

with initial conditions $f(0) = \ell_0$ and $r(0) = 0$, respectively. Here V_d is a given positive constant representing the rate of disassembly at the rear, and V_p is a given function, depending on the instantaneous cell length, which represents the rate of filament polymerization at the front.

In [1], Choi, Groulx, and Lui proved the local existence of solutions assuming that $\gamma_p = 0$ and $K = \kappa$ so that $(b + p)/c$ is a conserved quantity as time evolves.

It can be shown that system (1.1) is a generalization of (2.1). In fact, let $w = \frac{b}{c}$. Using (2.1)a and (2.1)c, we then obtain the following equation for w:

$$\begin{aligned} w_t &= \frac{b_t c - bc_t}{c^2} \\ &= \frac{(-b_x v - bv_x - \gamma_b b)c - b(-c_x v - cv_x)}{c^2} \\ &= -v\left[\frac{b_x c - bc_x}{c^2}\right] - \gamma_b \frac{b}{c} \\ &= -vw_x - \gamma_b w. \end{aligned}$$

Let $u = \frac{p}{b}$. Using (2.1)b and (2.1)c, we arrive at

$$u_t = -vu_x - (\gamma_p - \gamma_b)u + \gamma_b.$$

With $K = 1$ and $\rho = 1$, (2.3) can be recast in terms of w and u as

$$\sigma = w(1 + \kappa u) - b. \tag{2.5}$$

On taking the time derivative and substituting w_t and u_t from the above calculations,

$$\sigma_t = bv_x - v\sigma_x - \gamma_b w - \kappa \gamma_p uw + \kappa w \gamma_b + \gamma_b b.$$

Replacing v by $\frac{\sigma_x}{\xi}$ and rearranging and using (2.5), system (2.1) is equivalent to

$$w_t = -\left(\frac{\sigma_x}{\xi}\right)w_x - \gamma_b w,$$

$$u_t = -\left(\frac{\sigma_x}{\xi}\right)u_x - (\gamma_p - \gamma_b)u + \gamma_b, \tag{2.6}$$

$$\sigma_t = b\left(\frac{\sigma_x}{\xi}\right)_x - \frac{\sigma_x^2}{\xi} - \gamma_b\sigma + \kappa\gamma_b w(1 + u) - \kappa\gamma_p uw.$$

With $b = w(1 + \kappa u) - \sigma$, (2.6) can be cast totally in term of w, u and σ.

It is now clear that the system consists of a second-order equation and a pair of first-order hyperbolic equations that can be rewritten as system (1.1).

3. Preliminary lemma

Throughout this article, we will use $u \in C^{1,0}$ to denote u and u_x are continuous functions in a (x,t) domain, while $u \in C^{1,1}$ means u, u_x and u_t are continuous. The following lemma will be needed in Section 5 to prove the main theorem.

Lemma 3.1. *Let $\mathcal{R}_\delta = \{(x,t) : 0 < x < 1, 0 < t < \delta\}$. Let $a \in C^{1,0}(\overline{\mathcal{R}_\delta})$, $\mathbf{f} : \mathbf{R}^n \times \overline{\mathcal{R}_\delta}$ be a continuous function of $(\mathbf{u}, x, t)$ with $\mathbf{f_u}$ and $\mathbf{f}_x$ being continuous, $\mathbf{g} \in C^1[0,\delta]$, and $\mathbf{u}_0 \in C^1[0,1]$. Consider the system*

$$\mathbf{u}_t + a(x,t)\mathbf{u}_x = \mathbf{f}(\mathbf{u},x,t),$$

$$\mathbf{u}(1,t) = \mathbf{g}(t) \quad \text{for } 0 \le t \le \delta, \tag{3.1}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0(x) \quad \text{for } 0 \le x \le 1.$$

Suppose $a(0,t) < 0$ and $a(1,t) < 0$ for $0 \le t \le \delta$, and the compatibility conditions $\mathbf{g}(0) = \mathbf{u}_0(1)$ and $\mathbf{g}_t(0) + a(1,0)\mathbf{u}_0'(1) = \mathbf{f}(\mathbf{u}_0(1),1,0)$ holds. Then there exists a unique solution $\mathbf{u} \in C^{1,1}(\overline{\mathcal{R}_{\delta_1}})$ for some positive $\delta_1 \le \delta$ to the above system. Furthermore,

(a) there exists a constant $M_1 > 0$, depending on the L^∞-norm of $a, \mathbf{g}, \mathbf{u}_0$ and $\mathbf{f}$ in the compact set $[-\|\mathbf{u}_0\|_\infty - \|\mathbf{g}\|_\infty - 1, \|\mathbf{u}_0\|_\infty + \|\mathbf{g}\|_\infty + 1] \times \overline{\mathcal{R}_\delta}$, such that $\|\mathbf{u}\|_\infty \le M_1$,

(b) there exists a constant $M_2 > 0$, depending on the C^1 norm of $\mathbf{g}$ and $\mathbf{u}_0$, the $C^{1,0}$-norm of coefficient a, and the L^∞-norm of $\mathbf{f}, \mathbf{f_u}, \mathbf{f}_x$ in the compact set $[-\|\mathbf{u}_0\|_\infty - \|\mathbf{g}\|_\infty - 1, \|\mathbf{u}_0\|_\infty + \|\mathbf{g}\|_\infty + 1] \times \overline{\mathcal{R}_\delta}$, such that $\|\mathbf{u}\|_{C^{1,1}} \le M_2$. The time interval of existence $[0,\delta_1]$ also depends on the same norms.

Proof. Let $x = \tilde{x}(t,\tau)$ be the characteristic curve coming out from $x = 1$ at time τ into the domain $\mathcal{R}_\delta$. In other words, $\tilde{x}$ satisfies

$$\frac{\partial \tilde{x}}{\partial t} = a(\tilde{x}(t,\tau),t),$$

$$\tilde{x}(\tau,\tau) = 1. \tag{3.2}$$

Now define $\mathbf{v}(t,\tau) = \mathbf{u}(\tilde{x}(t,\tau),t)$. Then using (3.2)a and the governing equation on $\mathbf{u}$, we obtain

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{f}(\mathbf{v},\tilde{x}(t,\tau),t),$$

$$\mathbf{v}(\tau,\tau) = \mathbf{g}(\tau). \tag{3.3}$$

By ODE theory there exists a unique solution $\mathbf{v}$ for a small time interval $[\tau, \tau + \delta_1]$ with δ_1 being uniform with respect to initial conditions $\mathbf{g}(\tau)$ for all $\tau \in [0,\delta]$. One

can perform similar calculations for characteristics starting from initial point $(x, 0)$ for $x \in [0, 1]$ by shrinking the time interval of existence $[0, \delta_1]$ if necessary. These lead to a L^∞-norm bound on $\mathbf{v}$, which leads to statement (a) in the theorem. Continuity of solutions across the characteristic Γ_0 coming out from $(x, t) = (1, 0)$ is an easy consequence of the above analysis and the compatibility condition $\mathbf{g}(0) = \mathbf{u}_0(1)$.

Now differentiating the initial condition (3.2)b with respect to τ yields $\tilde{x}_t(\tau, \tau) + \tilde{x}_\tau(\tau, \tau) = 0$, which simplifies to $\tilde{x}_\tau(\tau, \tau) = -a(1, \tau) > 0$. Hence from (3.2),

$$\frac{\partial}{\partial t}\Big(\frac{\partial \tilde{x}}{\partial \tau}\Big) = a_x(\tilde{x}(t, \tau), t)\frac{\partial \tilde{x}}{\partial \tau},$$
$$\frac{\partial \tilde{x}}{\partial \tau}(\tau, \tau) = -a(1, \tau).$$

Therefore, if $\|a_x\|_\infty \leq M$, then

$$|a(1, \tau)|e^{-M(t-\tau)} \leq \frac{\partial \tilde{x}}{\partial \tau} \leq |a(1, \tau)|e^{M(t-\tau)}. \tag{3.4}$$

Similarly we differentiate the initial condition (3.3)b obtaining $\mathbf{v}_t(\tau, \tau) + \mathbf{v}_\tau(\tau, \tau) = \mathbf{g}_t(\tau)$, which simplifies to $\mathbf{v}_\tau(\tau, \tau) = \mathbf{g}_t(\tau) - \mathbf{f}(\mathbf{g}(\tau), 1, \tau)$. Therefore (3.3) yields the following governing equation for $\frac{\partial \mathbf{v}}{\partial \tau}$,

$$\frac{\partial}{\partial t}\Big(\frac{\partial \mathbf{v}}{\partial \tau}\Big) = \mathbf{f_u}(\mathbf{v}, \tilde{x}(t, \tau), t)\frac{\partial \mathbf{v}}{\partial \tau} + \mathbf{f}_x(\mathbf{v}, \tilde{x}(t, \tau), t)\frac{\partial \tilde{x}}{\partial \tau},$$
$$\frac{\partial \mathbf{v}}{\partial \tau}(\tau, \tau) = \mathbf{g}_t(\tau) - \mathbf{f}(\mathbf{g}(\tau), 1, \tau). \tag{3.5}$$

Thus we get the linear system $\frac{\partial}{\partial t}\big(\frac{\partial \mathbf{v}}{\partial \tau}\big) = A(t)\frac{\partial \mathbf{v}}{\partial \tau} + \mathbf{b}(t)$ with matrix A, vector $\mathbf{b}$ and initial condition all with L^∞-norm bounds. Using $< \cdot, \cdot >$ to denote scalar product in $\mathbf{R}^n$, which is related to ℓ_2 norm. Then there exist positive constants c_1 and c_2 such that

$$\frac{\partial}{\partial t}\big(\|\frac{\partial \mathbf{v}}{\partial \tau}\|_{\ell^2}^2\big) = 2\big\langle \frac{\partial \mathbf{v}}{\partial \tau}, \frac{\partial}{\partial t}\big(\frac{\partial \mathbf{v}}{\partial \tau}\big)\big\rangle$$
$$= 2\big\langle \frac{\partial \mathbf{v}}{\partial \tau}, A(t)\frac{\partial \mathbf{v}}{\partial \tau} + b(t)\big\rangle$$
$$\leq c_1\|\frac{\partial \mathbf{v}}{\partial \tau}\|_{\ell^2}^2 + c_2,$$

which leads to boundedness of $\|\frac{\partial \mathbf{v}}{\partial \tau}\|_\infty$.

From the definition of $\mathbf{v}$ we have that $\frac{\partial \mathbf{v}}{\partial \tau} = \frac{\partial \mathbf{u}}{\partial x}\frac{\partial \tilde{x}}{\partial \tau}$. With $\partial \tilde{x}/\partial \tau$ in (3.4) having a positive lower bound, it is immediate that there exists a positive constant m such that $\|\frac{\partial \mathbf{u}}{\partial x}\|_\infty \leq m$. Next with $\frac{\partial \mathbf{v}}{\partial t} = a\frac{\partial \mathbf{u}}{\partial x} + \frac{\partial \mathbf{u}}{\partial t}$, we can also bound $\frac{\partial \mathbf{u}}{\partial t}$. Therefore, there exists a positive constant M_2 such that $\|\mathbf{u}\|_{C^{1,1}} \leq M_2$ for those solutions whose characteristics originate from $x = 1$.

A similar analysis can be performed with solutions whose characteristics originate from $t = 0$. To complete the proof of statement (b), we need $\mathbf{u}_t$ and $\mathbf{u}_x$ to be continuous across Γ_0.

Assuming the solution is smooth in these two regions for the time being, we have

$$(\mathbf{u}_x)_t + a(\mathbf{u}_x)_x + a_x\mathbf{u}_x = \mathbf{f_u}\mathbf{u}_x + \mathbf{f}_x.$$

Let $[\cdot]$ to denote the jump across Γ_0. Due to continuity of $\mathbf{u}$ across Γ_0, on subtraction of the above equations in the two regions we obtain

$$([\mathbf{u}_x])_t + a([\mathbf{u}_x])_x + (a_x - \mathbf{f_u})[\mathbf{u}_x] = 0,$$
$$[\mathbf{u}_x](0,0) = 0,$$

(3.6)

where the zero initial condition is a consequence of the second compatibility condition. Hence $[\mathbf{u}_x] = 0$ at all subsequent time by integrating along the characteristic Γ_0.

Since the coefficients a and $a_x - \mathbf{f_u}$ in (3.6) do not depend on higher smoothness of solution $\mathbf{u}$, one can obtain (3.6) by using approximations of a and $\mathbf{f}$ by smoother functions and take the limit. The proof of the Lemma is now complete. $\qquad\square$

4. Fixing the domain

To facilitate our discussion, we let $Q_\epsilon = \{(x,t) \mid r(t) < x < f(t), 0 < t < \epsilon\}$. It is convenient to work on a fixed domain so we first straighten out the moving boundaries. Let $\ell(t) = f(t) - r(t)$ and $x = r(t) + \bar{x}\ell(t)$. The region Q_ϵ is mapped onto the region $\mathcal{R}_\epsilon = \{(\bar{x},t) \mid 0 < \bar{x} < 1, 0 < t < \epsilon\}$. Define $\tilde{\mathbf{w}}(\bar{x},t) = \mathbf{w}(r(t) + \bar{x}\ell(t), t)$ and $\tilde{\sigma}(\bar{x},t) = \sigma(r(t) + \bar{x}\ell(t), t)$. Then $\tilde{\mathbf{w}}_{\bar{x}} = \mathbf{w}_x \ell$ and

$$\tilde{\mathbf{w}}_t = (r' + \bar{x}\ell')\mathbf{w}_x + \mathbf{w}_t$$
$$= \frac{(r' + \bar{x}\ell')}{\ell}\tilde{\mathbf{w}}_{\bar{x}} - \frac{\sigma_x}{\ell}\tilde{\mathbf{w}}_{\bar{x}} - \mathbf{F}(\tilde{\mathbf{w}})$$
$$= -\left(\frac{\tilde{\sigma}_{\bar{x}}}{\ell^2} - \frac{(r' + \bar{x}\ell')}{\ell}\right)\tilde{\mathbf{w}}_{\bar{x}} - \mathbf{F}(\tilde{\mathbf{w}})$$

with boundary condition $\tilde{\mathbf{w}}(1,t) = \mathbf{w}_f(t)$. Similarly we can obtain the governing equation for $\tilde{\sigma}$. System (1.1) is then transformed into the system

$$\tilde{\mathbf{w}}_t = -\left(\frac{\tilde{\sigma}_{\bar{x}}}{\ell^2} - \frac{(r' + \bar{x}\ell')}{\ell}\right)\tilde{\mathbf{w}}_{\bar{x}} - \mathbf{F}(\tilde{\mathbf{w}}),$$
$$\tilde{\sigma}_t = \frac{g(\tilde{\mathbf{w}},\tilde{\sigma})}{\ell^2}\tilde{\sigma}_{\bar{x}\bar{x}} - \left(\frac{\tilde{\sigma}_{\bar{x}}}{\ell^2} - \frac{(r' + \bar{x}\ell')}{\ell}\right)\tilde{\sigma}_{\bar{x}} + h(\tilde{\mathbf{w}},\tilde{\sigma}),$$

(4.1)

which holds in $\mathcal{R}_\epsilon$. The boundary conditions (1.2) become

$$\tilde{\sigma}(0,t) = 0,$$
$$\tilde{\sigma}(1,t) = 0, \quad \tilde{\mathbf{w}}(1,t) = \mathbf{w}_f(t),$$

(4.2)

and the equations for the moving boundaries (1.3) become

$$\frac{df}{dt} = V(\ell(t)) + \frac{\tilde{\sigma}_{\bar{x}}(1,t)}{\ell}, \quad f(0) = \ell_0,$$
$$\frac{dr}{dt} = 1 + \frac{\tilde{\sigma}_{\bar{x}}(0,t)}{\ell}, \quad r(0) = 0.$$

(4.3)

We observe that the first order compatibility conditions (1.6) to (1.9) between the initial and the boundary conditions in the domain Q_ϵ give rise to the corresponding compatibility conditions of $\tilde{\sigma}$ at $(\bar{x},t) = (0,0), (1,0)$ and $\tilde{\mathbf{w}}$ at $(\bar{x},t) = (1,0)$ in the domain $\mathcal{R}_\epsilon$. Hence one can establish Theorem 1.1 by considering (4.1)-(4.3) with corresponding initial conditions which are compatible to the boundary conditions to first order.

The idea of our existence proof is to make a guess for $\tilde{\sigma}$. Next using such a guess and (4.3) we find the moving boundaries f, r and $\ell = f - r$. Via Lemma 3.1, we

have enough a priori bounds for the solution $\tilde{\mathbf{w}}$ of the hyperbolic equations (4.1)a). The final step is to solve (4.1)b) for a new $\tilde{\sigma}$. If we can find a fixed point of such an iterative procedure, this will be a solution we are looking for.

5. Proof of Theorem 1.1

Define $g_0 = g(\tilde{\mathbf{w}}, \tilde{\sigma})|_{t=0}$ and $h_0 = h(\tilde{\mathbf{w}}, \tilde{\sigma})|_{t=0}$. Let z be the solution of the following initial-boundary value problem

$$z_t = \frac{g_0(\bar{x}l_0)}{\ell_0^2} z_{\bar{x}\bar{x}} - \Big(\psi'(\bar{x}\ell_0) - \frac{[r'(0) + \bar{x}\ell'(0)]}{\ell_0}\Big)\psi'(\bar{x}\ell_0) + h_0(\bar{x}l_0),$$
$$z(0,t) = z(1,t) = 0, \tag{5.1}$$
$$z(\bar{x},0) = \psi(\bar{x}l_0),$$

which is a linear second order parabolic equation. It is derived from (4.1)b) with all the terms, except for the ones involving the time derivative and the second spatial derivative, evaluated using the initial conditions. By hypothesis the coefficients and the non-homogeneous terms in the above equation are time-independent and in $C^1([0,1])$. Since the initial conditions satisfy the first order compatibility conditions at $(0,0)$ and $(1,0)$, respectively, a unique solution z exists and is in $C^{2+\beta,(2+\beta)/2}(\overline{\mathcal{R}_\epsilon})$ for some $0 < \beta < 1$ as defined in the hypothesis. It is also clear that if solution $\tilde{\sigma}$ to (4.1)b) exists, then $\tilde{\sigma}(\bar{x},0) = z(\bar{x},0)$ and $\tilde{\sigma}_t(\bar{x},0) = z_t(\bar{x},0)$. Let $0 < \epsilon \le 1$ and let

$$\mathcal{S}_\epsilon = \big\{\sigma \in C^{2,1}(\overline{\mathcal{R}_\epsilon}) : \|\sigma - z\|_{C^{2,1}(\overline{\mathcal{R}_\epsilon})} \le 1,\ \sigma(\bar{x},0) = z(\bar{x},0),$$
$$\sigma_t(\bar{x},0) = z_t(\bar{x},0),\ \sigma(0,\cdot) = \sigma(1,\cdot) = 0\big\}.$$

The goal is to define, for sufficiently small ϵ, a compact continuous map $T : \mathcal{S}_\epsilon \to \mathcal{S}_\epsilon$ and then apply the Schauder fixed-point Theorem.

Recall $[0, \delta_1]$ is the interval of existence in Lemma 3.1. Let $\sigma \in \mathcal{S}_{\epsilon_1}$ where $\epsilon_1 \le \delta_1$ will be determined later and let ℓ be the solution to the equation $\ell' = L(\ell, t)$ with initial condition $\ell(0) = \ell_0$, where

$$L(\ell, t) = V(\ell) - 1 + \frac{\sigma_{\bar{x}}(1,t) - \sigma_{\bar{x}}(0,t)}{\ell}.$$

Since L is C^1 in ℓ, $C^{1/2}$ in t, there exists an $\epsilon_1 > 0$, uniform with respect to $\sigma \in \mathcal{S}_1$, such that the solution ℓ exists, belongs to $C^{1+1/2}([0, \epsilon_1])$ and satisfies $2\ell_0 \ge \ell \ge \ell_0/2$. Now solve (4.3) separately for f and r. It is clear that $f, r \in C^{1+1/2}[0, \epsilon_1]$ and $f - r = \ell$.

Using f, r, and σ, the next step is to solve the hyperbolic equations (4.1)a) for $\tilde{\mathbf{w}}$. By Lemma 3.1, the solution $\mathbf{w}$ exists and has uniform $C^{1,1}$ bound which is independent of the choice of $\sigma \in \mathcal{S}_{\epsilon_1}$.

Now let $\hat{\sigma}$ be the solution to the linear initial-boundary value problem

$$\hat{\sigma}_t = \frac{g(\mathbf{w}(\bar{x},t), \sigma(\bar{x},t))}{\ell^2(t)} \hat{\sigma}_{\bar{x}\bar{x}} + G(\bar{x},t),$$
$$\hat{\sigma}(0,t) = \hat{\sigma}(1,t) = 0, \tag{5.2}$$
$$\hat{\sigma}(\bar{x},0) = \psi(\bar{x}l_0)$$

in $\overline{\mathcal{R}_{\epsilon_1}}$, where

$$G(\bar{x},t) = -\Big(\frac{\sigma_{\bar{x}}}{\ell^2} - \frac{(r' + \bar{x}\ell')}{\ell}\Big)\sigma_{\bar{x}} + h(\mathbf{w}, \sigma). \tag{5.3}$$

Having uniform time derivative bounds on $\mathbf{w}$ and σ, they stay close to $\mathbf{w}_0$ and ψ by reducing ϵ_1 if necessary. Hence we have the parabolicity $g(\mathbf{w}, \sigma) > 0$ because of (1.5).

With $\sigma \in \mathcal{S}_{\epsilon_1}$ and our established estimates on h and ℓ, we have $\|G\|_{C^{1,1/2}(\overline{\mathcal{R}}_{\epsilon_1})}$ being uniformly bounded independently of the choice of $\sigma \in \mathcal{S}_{\epsilon_1}$. Define the operator $T : \mathcal{S}_{\epsilon_1} \to \mathcal{S}_{\epsilon_1}$ by $T\sigma = \hat{\sigma}$.

To show that $\hat{\sigma} \in \mathcal{S}_{\epsilon_1}$, let $\tilde{g}(\bar{x}, t) = g(\mathbf{w}(\bar{x}, t), \sigma(\bar{x}, t))$ and $u = \hat{\sigma} - z$. Observe that (5.1) is the same as

$$z_t = \frac{\tilde{g}(\bar{x}, 0)}{\ell_0^2} z_{\bar{x}\bar{x}} + G(\bar{x}, 0) \,.$$

Then it can readily be checked that u satisfies the equation

$$u_t = \frac{\tilde{g}(\bar{x}, t)}{\ell^2} u_{\bar{x}\bar{x}} + H(\bar{x}, t), \tag{5.4}$$

where

$$H(\bar{x}, t) = \Big(\frac{\tilde{g}(\bar{x}, t)}{\ell^2} - \frac{\tilde{g}(\bar{x}, 0)}{\ell_0^2} \Big) z_{\bar{x}\bar{x}} + G(\bar{x}, t) - G(\bar{x}, 0) \,. \tag{5.5}$$

The established estimates allow us to conclude that there is a uniform bound on $\|H\|_{C^{\beta, \beta/2}(\overline{\mathcal{R}}_{\epsilon_1})}$, which is independent of the choice of $\sigma \in \mathcal{S}_{\epsilon_1}$. Observe that u has zero initial and boundary conditions and $H(\cdot, 0) = 0$. Hence by [4, ch.4, Thm. 5.4],

$$\|u\|_{C^{2+\beta, (2+\beta)/2}(\overline{\mathcal{R}}_{\epsilon_1})} \leq M_1 \|H\|_{C^{\beta, \beta/2}(\overline{\mathcal{R}}_{\epsilon_1})} \,, \tag{5.6}$$

where M_1 is independent of the choice of $\sigma \in \mathcal{S}_{\epsilon_1}$ and remains bounded as $\epsilon \downarrow 0$. Since $u(\cdot, 0) = u_t(\cdot, 0) = 0$, by choosing ϵ_1 smaller if necessary, (5.6) allows us to conclude $\|u\|_{C^{2,1}(\overline{\mathcal{R}}_{\epsilon_1})} \leq 1$ so that $\hat{\sigma} \in \mathcal{S}_{\epsilon_1}$. Inequality (5.6) also implies that $\|\hat{\sigma}\|_{C^{2+\beta,(2+\beta)/2}(\overline{\mathcal{R}}_{\epsilon_1})}$ is bounded independently of the choice of σ in $\mathcal{S}_{\epsilon_1}$. Thus T is a compact operator.

As $\sigma \in \mathcal{S}_{\epsilon_1}$ varies continuously in $C^{2,1}(\overline{\mathcal{R}}_{\epsilon_1})$ norm, it is readily checked that r, f, ℓ varies continuously in $C^{1+1/2}[0, \epsilon_1]$ norm, which leads to a corresponding variation of $\mathbf{w}(\bar{x}, t)$ in $C^{1,1}(\overline{\mathcal{R}}_{\epsilon_1})$ norm. Standard parabolic estimate then requires $\hat{\sigma}$ to vary continuously in $C^{2,1}(\overline{\mathcal{R}}_{\epsilon_1})$ norm. Hence T is continuous on $\mathcal{S}_{\epsilon_1}$. Schauder fixed point Theorem implies that T has a fixed point and the proof of the existence of solution is complete.

Proof of uniqueness. Now we turn our attention to the uniqueness of smooth solutions. Let $(\tilde{\sigma}_i, \tilde{\mathbf{w}}_i, f_i, r_i), i = 1, 2$, be two solutions of the moving boundary problem with the same initial conditions. Let $g_i = g(\tilde{\mathbf{w}}_i, \tilde{\sigma}_i)$ and $h_i = h(\tilde{\mathbf{w}}_i, \tilde{\sigma}_i)$ for $i = 1, 2$. Define $\hat{\sigma} = \tilde{\sigma}_1 - \tilde{\sigma}_2$, $\hat{g} = g_1 - g_2$, $\hat{h} = h_1 - h_2$, $\hat{\mathbf{w}} = \tilde{\mathbf{w}}_1 - \tilde{\mathbf{w}}_2$, $\hat{\ell} = \ell_1 - \ell_2$, and $\hat{r} = r_1 - r_2$. Then from (4.1)b), $\hat{\sigma}$ satisfies

$$\hat{\sigma}_t = \frac{g_1}{\ell_1^2} \hat{\sigma}_{\bar{x}\bar{x}} + \frac{(r_1' - \bar{x}\ell_1')}{\ell_1} \hat{\sigma}_{\bar{x}} - \frac{(\tilde{\sigma}_{1\bar{x}} + \tilde{\sigma}_{2\bar{x}})}{\ell_1^2} \hat{\sigma}_{\bar{x}} + G_\sigma(\bar{x}, t) \tag{5.7}$$

where the nonhomogeneous term is

$$G_\sigma = \Big(\frac{g_1}{\ell_1^2} - \frac{g_2}{\ell_2^2} \Big) \tilde{\sigma}_{2\bar{x}\bar{x}} + \Big(\frac{(r_1' - \bar{x}\ell_1')}{\ell_1} - \frac{(r_2' - \bar{x}\ell_2')}{\ell_2} \Big) \tilde{\sigma}_{2\bar{x}} - \Big(\frac{1}{\ell_1^2} - \frac{1}{\ell_2^2} \Big) \tilde{\sigma}_{2\bar{x}}^2 + (h_1 - h_2) \,.$$

Since $G_\sigma(\bar{x}, 0) = 0$ and $\hat{\sigma}$ has zero boundary and initial conditions in $\mathcal{R}_\tau$, by [4, ch.4, Thm. 9.2], for any $q > 1$, there exists a constant $K_{q,\tau} > 0$, which remains bounded as $\tau \to 0$, such that

$$\|\hat{\sigma}\|_{W_q^{2,1}(\mathcal{R}_\tau)} \leq K_{q,\tau} \left(\|\hat{g}\|_{C(\overline{\mathcal{R}_\tau})} + \|\hat{r}\|_{C^1([0,\tau])} + \|\hat{\ell}\|_{C^1([0,\tau])} + \|\hat{h}\|_{C(\overline{\mathcal{R}_\tau})} \right),$$

where the right hand side is obtained by estimating L^∞-norm of G_σ.

By increasing the constant $K_{q,\tau}$ if necessary, we can recast the above estimate as

$$\|\hat{\sigma}\|_{W_q^{2,1}(\mathcal{R}_\tau)} \leq K_{q,\tau} \left(\|\hat{\sigma}\|_{C(\overline{\mathcal{R}_\tau})} + \|\hat{r}\|_{C^1([0,\tau])} + \|\hat{\ell}\|_{C^1([0,\tau])} + \|\hat{\mathbf{w}}\|_{C(\overline{\mathcal{R}_\tau})} \right). \quad (5.8)$$

A similar calculation for $\hat{\mathbf{w}}$ using (4.1)a) yields

$$\hat{\mathbf{w}}_t + \Big[\frac{\tilde{\sigma}_{1\bar{x}}}{\ell_1^2} - \frac{(r_1' - \bar{x}\ell_1')}{\ell_1} \Big] \hat{\mathbf{w}}_{\bar{x}} = G_w(\bar{x}, t),$$

where

$$G_w(\bar{x}, t) = -\Big[\frac{\tilde{\sigma}_{1\bar{x}}}{\ell_1^2} - \frac{\tilde{\sigma}_{2\bar{x}}}{\ell_2^2} - \frac{(r_1' - \bar{x}\ell_1')}{\ell_1} + \frac{(r_2' - \bar{x}\ell_2')}{\ell_2} \Big] \tilde{\mathbf{w}}_{2\bar{x}} - (\mathbf{F}(\tilde{\mathbf{w}}_1) - \mathbf{F}(\tilde{\mathbf{w}}_2)).$$

With $G_w(\bar{x}, 0) = 0$ and $\hat{\mathbf{w}}$ vanishing at $t = 0$ and on the right boundary of $\mathcal{R}_\tau$, the compatibility conditions at $(x, t) = (1, 0)$ are satisfied. By integrating along the characteristics, there exists a constant $K_1 > 0$ such that

$$\|\hat{\mathbf{w}}\|_{C(\overline{\mathcal{R}_\tau})} \leq K_1 \|G_w\|_{C(\overline{\mathcal{R}_\tau})} \leq K_1 (\|\hat{\sigma}\|_{C^{1,0}(\overline{\mathcal{R}_\tau})} + \|\hat{r}\|_{C^1([0,\tau])} + \|\hat{\ell}\|_{C^1([0,\tau])}). \quad (5.9)$$

Next we estimate $\hat{\ell}$ and $\hat{r}$. By subtracting (4.3)b) from (4.3)a), we obtain a governing equation for ℓ. Thus $\hat{\ell}$ satisfies an equation of the form

$$\hat{\ell}' = m(t)\hat{\ell} + n(t) \quad (5.10)$$

for some functions m and n with initial condition $\hat{\ell}(0) = 0$. We note that $\|m\|_{C([0,\tau])}$ is bounded and $\|n\|_{C([0,\tau])} \leq K_3 \|\hat{\sigma}\|_{C^{1,0}(\overline{\mathcal{R}_\tau})}$ for some constant $K_3 > 0$. From (5.10),

$$\|\hat{\ell}\|_{C^1([0,\tau])} \leq K_4 \|n\|_{C(0,\tau])} \leq K_5 \|\hat{\sigma}\|_{C^{1,0}(\overline{\mathcal{R}_\tau})} \quad (5.11)$$

for some constant $K_5 > 0$. A similar calculation gives

$$\|\hat{r}\|_{C^1([0,\tau])} \leq K_6(\|\hat{\sigma}\|_{C^{1,0}(\overline{\mathcal{R}_\tau})} + \|\hat{\ell}\|_{C([0,\tau])}) \leq K_7 \|\hat{\sigma}\|_{C^{1,0}(\overline{\mathcal{R}_\tau})} \quad (5.12)$$

for some positive constants K_6, K_7.

Substituting (5.9), (5.11), (5.12) in (5.8), we have $\|\hat{\sigma}\|_{W_q^{2,1}(\mathcal{R}_\tau)} \leq K_8 \|\hat{\sigma}\|_{C^{1,0}(\overline{\mathcal{R}_\tau})}$ for some $K_8 > 0$. Note that the constants K_1 to K_8 remain bounded as $\tau \downarrow 0$. Lemma 3.3 in [4, ch.4], with $\ell = 1$, $r = 0$, $s = 1$, and $q = 6$ implies that $\|\hat{\sigma}\|_{C^{1+\lambda, \frac{1+\lambda}{2}}(\overline{\mathcal{R}_\tau})} \leq K_9 \|\hat{\sigma}\|_{W_6^{2,1}(\mathcal{R}_\tau)}$ where $\lambda = \frac{1}{2}$ and K_9 is independent of τ. This means that $\hat{\sigma}_{\bar{x}}$ is Hölder continuous in t with exponent $1/4$. Since $\hat{\sigma}(\cdot, 0) = 0$, combining the above inequalities, we have

$$\|\hat{\sigma}\|_{C^{1+\frac{1}{2}, \frac{3}{4}}(\overline{\mathcal{R}_\tau})} \leq K_{10} \tau^{1/4} \|\hat{\sigma}\|_{C^{1+\frac{1}{2}, \frac{3}{4}}(\overline{\mathcal{R}_\tau})} \quad (5.13)$$

for some constant $K_{10} > 0$. By choosing τ small enough that $K_{10} \tau^{1/4} < 1$, we have $\hat{\sigma} = 0$; i.e., $\tilde{\sigma}_1 = \tilde{\sigma}_2$. That $\tilde{\mathbf{w}}_1 = \tilde{\mathbf{w}}_2$, $\tilde{\ell}_1 = \tilde{\ell}_2$, and $\tilde{r}_1 = \tilde{r}_2$ follow immediately from (5.9), (5.11), (5.12). The uniqueness part of the proof is complete.

6. Conclusion and open questions

In [1], the existence and uniqueness of local solutions to a moving boundary problem (2.1) modelling cell motility is established when $\gamma_p = 0$ and $K = \kappa$ in (2.3). Such assumptions are discarded in this paper so that some conservation relation becomes unavailable.

If one puts (2.2) and (2.3) into (2.1), at first glance the governing equations look like a strongly coupled parabolic system. It is, however, a system of first order hyperbolic equations coupled with a parabolic equation and requires a careful reformulation to make such an issue clear. Through a clever choice of new independent variables in section 2, the transformed equations (2.6) are weakly coupled, allowing a simpler analysis to establish a priori bounds. The corresponding generalized problem presented in section 1 is then transformed into the problem (4.1)-(4.3) with a fixed domain. A fixed point iterative scheme leads to the existence of solutions to this moving boundary problem. Uniqueness then follows by applying a priori estimates on the difference of two solutions. We now cite some open problems associated with this model:

(a) Having proved the local existence and uniqueness of the solution, a natural step is the study of global existence of solutions. Besides $\gamma_p = 0$ and $K = \kappa$, some special initial conditions are needed in [1] to prove the global existence of solution. Such simplifications allow the reduction of the model to a scalar parabolic equation with some non-local moving boundary conditions. This reduction allows certain techniques which are not possible for a system of equations. There is some progress in the global existence for a single simple hyperbolic equation coupled with a parabolic equation (Choi and Miller, in preparation). For the system (1.1) with appropriate restrictions on $\mathbf{F}$, it will be interesting to see if a modification of such ideas will work or some totally different tricks are necessary in the study of its global existence of solution.

(b) The constitutive law (2.3) proposed by Mogilner and Verzi in [5] is based on the assumption that the stress can be modelled as the sum of two linear spring forces. The actual stress-strain relationship inside a cell may be more complicated. For example one may just require that stress increases with extension beyond its natural length. Under such more general conditions, the local and global existence of solutions can be studied.

(c) A two-dimensional model has been proposed by Choi and Lui in [3]. The model was shown to admit a travelling domain solution, in the sense that both the shape of the domain and the steady travelling speed are parts of the solution. Both the local and the global existence of solution to such a 2D model has not been established.

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