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A BIHARMONIC ELLIPTIC PROBLEM WITH DEPENDENCE ON THE GRADIENT AND THE LAPLACIAN

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Abstract. We study the existence of solutions for nonlinear biharmonic equations that depend on the gradient and the Laplacian, under Navier boundary condition. Our main tools are an iterative scheme of the mountain pass "aproximated" solutions, and the truncation method developed by de Figueiredo, Girardi and Matzeu.

1. Introduction

We prove the existence of nontrivial solutions for the equation

$$\begin{gathered} \Delta^2 u + q\Delta u + \alpha(x)u = f(x,u,\nabla u,\Delta u) \quad \text{in } \Omega \\ u(x)=0, \quad \Delta u(x)=0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Δ^2 is the biharmonic operator and $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with smooth boundary $\partial\Omega$.

The above fourth-order semilinear elliptic problem, when f does not depend on derivatives of u, has been studied by many authors; see [4, 5, 20, 22] and references therein. In this case variational techniques are widely applied to obtain existence of solutions.

When $\Omega = \mathbb{R}$ and $q > 0$ the problem (1.1) is is called the Swift-Hohenberg equation, and for $q > 0$ it is called the extended Fisher-Kolmogorov equation. For this class of problems the existence of homoclinic, heteroclinic and periodic solutions have been obtained by several researchers mainly when f does not depend on derivatives; see e. g. [6, 9, 15, 21, 24, 26]. The reader is refereed to [1, 10, 11, 17, 18] for the case $\Omega = (0,1)$ and f depending on the second order derivative but not on the first derivative of u. Recently, the authors in [8] studied a situation where f depends on the first and second order derivatives. For studies with nonlinearities of the form $f(x,u,\Delta u)$ the reader is referred to [13, 19, 23, 25] and references there in.

In our case, due to the presence of the gradient and the Laplacian of u in f, the problem is not variational whihc creates additional difficulties. For instance the critical point theory can not be applied directly. We recall that, to overcome this difficult, Xavier [27] and Yan [28] handled some semilinear elliptic problems of

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the second order involving the gradient by using monotone iterative methods. We apply a technique developed by De Figueiredo, Girard and Matzeu [12] (see also Girard and Matzeu [14]) which "freezes" the gradient variable and use truncation on the nonlinearity f. Thus the new problem becomes variational. The idea of this approach is to consider a class of problems through an iterative scheme where the approximated problem has a nontrivial solution via mountain pass Theorem. Then one obtains estimates in $H^2(\Omega)\cap H^1_0(\Omega)$-norm and C^2-norm. Passing to the limit in a sequence of the approximated solutions we gets a solution of the original problem. In general, a semilinear Navier fourth-order problem is equivalent to the semilinear Dirichlet problem for a system of two coupled second order equations but it is not clear that the truncation method works for the system.

To state our results, let us assume the following conditions:

- (A1) α is a Hölder-continuous function.
- (A2) There are positive constants a, b verifying $0 < a \le \alpha(x) < b$, $\forall x \in \mathbb{R}$.
- (A3) $q \in (-\infty, 2\sqrt{a})$.
- (F0) $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous.
- (F1) $\lim_{t\to 0} f(x,t,\xi_1,\xi_2)/t = 0$ uniformly with respect to $x \in \Omega$, $\xi_1 \in \mathbb{R}^N$ and $\xi_2 \in \mathbb{R}$.
- (F2) There exist $a_1 > 0$, $p \in (1, \frac{N+4}{N-4})$, $(N \ge 5)$, r_1 and r_2 , such that $r := r_1 + r_2 < 1$ and
$$|f(x,t,\xi_1,\xi_2)| \le a_1(1+|t|^p)(1+|\xi_1|^{r_1})(1+|\xi_2|^{r_2}),$$
for all $(x,t,\xi_1,\xi_2) \in \Omega \times \mathbb{R}^{N+2}$.
- (F3) There exist $\theta > 2$ and $t_0 > 0$ such that
$$0 < \theta F(x,t,\xi_1,\xi_2) \le t f(x,t,\xi_1,\xi_2), \quad \forall x \in \Omega, |t| \ge t_0, (\xi_1,\xi_2) \in \mathbb{R}^{N+1},$$
where $F(x,t,\xi_1,\xi_2) = \int_0^t f(x,s,\xi_1,\xi_2)ds$.
- (F4) There exists $a_2, a_3 > 0$ such that
$$F(x,t,\xi_1,\xi_2) \ge a_2|t|^\theta - a_3, \quad \forall x \in \Omega, (t,\xi_1,\xi_2) \in \mathbb{R}^{N+2}.$$

Denote by $y^{i,j}$, $(i = 1,2,3)$ the vectors
$$y^{1,j} = (y_1^j, y_2, y_3), \quad y^{2,j} = (y_1, y_2^j, y_3), \quad y^{3,j} = (y_1, y_2, y_3^j).$$
For $i,k = 1,2,3$ and $j = 1,2$, we define the numbers:
$$L_{\rho_i} = \sup\Big\{\frac{|f(x,y^{i,1}) - f(x,y^{i,2})|}{|y_i^1 - y_i^2|} : (x,y^{i,j}) \in A_i\Big\},$$
where
$$A_i = \{(x,y^{i,j}) \in \Omega \times \mathbb{R}^{N+2},\ |y_i^j| \le \rho_i,\ |y_k| \le \rho_k (i \ne k)\},$$
for some constants $\rho_i > 0$.

- (F5) There exist positive numbers ρ_i $(i = 1,2,3)$ depending on q,θ,a_1,a_2 and a_3, in an explicit way, such that the above positive numbers L_{ρ_i} $(i = 1,2,3)$ satisfy the relation
$$(\tau_1 L_{\rho_1} + \tau_2 L_{\rho_2} + \tau_3 L_{\rho_3})\tau_1 < \gamma,$$
where γ is as in Lemma 2.2 and τ_i $(i = 1,2,3)$ are the optimal constants (that is, the smaller constants) of the inequalities

$$\Big(\int_\Omega |u|^2 dx\Big)^{1/2} \le \tau_1\|u\|, \quad \Big(\int_\Omega |\nabla u|^2 dx\Big)^{1/2} \le \tau_2\|u\|, \quad \Big(\int_\Omega |\Delta u|^2 dx\Big)^{1/2} \le \tau_3\|u\|,$$

where $\|u\|^2 = (u,u)$ and $(u,v) = \int_\Omega (\Delta u \Delta v + \nabla u \nabla v + uv)dx$.

Under such hypotheses, we prove the following result.

Theorem 1.1. *If* (A1)–(A3), (F0)–(F5) *hold, then there exists at least one classical solution of* (1.1).

Example: Suppose β and δ positive and continuous functions. If $f(x,t,\xi_1,\xi_2) = \beta(x)|t|t(1+|\xi_1|)^{1/4}(1+|\xi_2|)^{1/4}+\delta(x)t^3$, then it satisfies all the conditions (F0)–(F5).

Remark 1.2. If $\Omega = \delta\Omega'$, with $\delta > 0$ and Ω' is a bounded domain containing the origin, all functions verifying the growth conditions (F1)–(F4) satisfy the condition (F5) for δ small sufficient. It occurs because the constants τ_i $(i = 2,3)$ and L_{ρ_i} $(i = 1,2,3)$ do not increase as δ approaches zero, and, by Poincaré inequality, we can choose $\tau_1 = (\frac{\delta(\text{diameter of }\Omega')}{w_N})^{1/N}$ with w_N the measure of the unity ball in $\mathbb{R}^N$.

2. Notation and a technical result

Let $X \equiv H^2(\Omega) \cap H_0^1(\Omega)$, which is a Hilbert space with inner product and norm given in the previous section. Since (1.1), in general, is not variational we use a "freezing" technique whose formulation appears initially in [12]. This technique consists of associating to the problem (1.1) a family of problems without dependence of f in the gradient and Laplacian of the solution. That is, for each $w \in X$ fixed we consider the "freezed" problem given by

$$\begin{gathered} \Delta^2 u_w + q\Delta u_w + \alpha(x)u_w = f(x,u_w,\nabla w,\Delta w) \quad \text{in } \Omega \\ u_w(x) = 0, \quad \Delta u_w(x) = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

The nonexistence of a priori estimates, with respect to the norms of the gradient and Laplacian of the solution, is the main difficulty for using variational techniques. Thus, we consider, for each $R > 0$ fixed, the truncated "functions"

$$f_R(x,t,\xi_1,\xi_2) = f(x,t,\xi_1\varphi_R(\xi_1),\xi_2\varphi_R(\xi_2)),$$

and

$$F_R(x,t,\xi_1,\xi_2) = \int_0^t f_R(x,s,\xi_1,\xi_2)ds,$$

where $\varphi_R \in C^1(\mathbb{R})$, $|\varphi_R| \le 1$ and

$$\varphi_R(\xi) = \begin{cases} 1 & \text{if } |\xi| \le R, \\ 0 & \text{if } |\xi| \ge R+1. \end{cases}$$

This argument appears initially in [16]. See also [14].

Remark 2.1. Note that $|\xi\varphi_R(\xi)| \le R+1$, for all $\xi \in \mathbb{R}$.

Thus, for each $w \in X$ and $R > 0$ fixed, we consider "truncated" and "freezed" problem, given by

$$\begin{gathered} \Delta^2 u_w^R + q\Delta u_w^R + \alpha(x)u_w^R = f_R(x,u_w^R,\nabla w,\Delta w) \quad \text{in } \Omega \\ u_w^R(x) = 0, \quad \Delta u_w^R(x) = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.2}$$

The associated functional $I_w^R : X \to \mathbb{R}$ is

$$I_w^R(v) = \frac{1}{2}\int_\Omega [(\Delta v)^2 - q(\nabla v)^2 + \alpha(x)v^2]dx - \int_\Omega F_R(x,v,\nabla w,\Delta w)dx. \tag{2.3}$$

The following technical Lemma gives us a new and equivalent norm in X.

Lemma 2.2. *Suppose α and q satisfy* (A1)–(A3)*. Then there exist positive constants η and γ such that*

$$\gamma\|u\|^2 \leq \int_\Omega (\Delta u^2 - q\nabla u^2 + \alpha(x)u^2)dx \leq \eta\|u\|^2, \quad \forall u \in H^2(\Omega).$$

Proof. The constant η is obtained taking $\eta = \max\{-q, b, 1\}$. To obtain γ, notice that if $q < 0$, it is sufficient to take $\gamma = \min\{-q, a, 1\}$. In the case where $q \geq 0$, and $\Omega \subset \mathbb{R}$, γ will be taken as in [26, Lemma 8]. For $\Omega \subset \mathbb{R}^N, N > 1$, the proof can be adapted from [26, Lemma 8]. $\square$

3. Proof of main theorem

We assume $N \geq 5$; the case $N \in [1, 4]$ is easier. The proof of Theorem 1.1 is achieved with several lemmas. The following result establishes the mountain pass geometry for the functional I_w^R.

Lemma 3.1. *Let $w \in X$ and $R > 0$ be fixed. Then*

- (i) *there exist positive constants $\rho = \rho_R$ and $\alpha = \alpha_R$ such that $I_w^R(v) \geq \alpha$, for all $v \in X$ with $\|v\| = \rho$.*
- ii) *fix v_0 with $\|v_0\| = 1$; there is a $T > 0$ such that $I_w^R(tv_0) \leq 0$, for all $t > T$.*

Proof. By (F1), given any $\varepsilon > 0$ there exists some $\delta > 0$ such that $|v| < \delta$, implies

$$F_R(x, v, \nabla w, \Delta w) \leq \varepsilon \frac{v^2}{2}. \tag{3.1}$$

Now, if $|v| \geq \delta$, by (F2) and by Remark 2.1, there exists some constant $k = k(\delta)$ such that

$$F_R(x, v, \nabla w, \Delta w) \leq k|v|^{p+1}(R + 2)^r. \tag{3.2}$$

Thus, by inequalities (3.1) and (3.2) and by Lemma 2.2 we have

$$I_w^R(v) \geq \frac{\gamma}{2}\|v\|^2 - \frac{\varepsilon}{2}\int_\Omega |v|^2 dx - k(R + 2)^r \int_\Omega |v|^{p+1} dx.$$

So, by the Sobolev embedding Theorem we have

$$I_w^R(v) \geq \frac{1}{2}(\gamma - C\varepsilon)\|v\|^2 - kC(R + 2)^r\|v\|^{p+1},$$

for some positive constant C. Then for a ε small sufficient, we can choose $\rho = \rho_R$ and $\alpha = \alpha_R$, both independent of w, such that the first part of the result holds.

Now, take an arbitrary $v_0 \in X$ with $\|v_0\| = 1$. By (F4) and Lemma 2.2

$$I_w^R(tv_0) \leq \frac{\eta|t|^2}{2}\|v_0\|^2 - a_2|t|^\theta \int_\Omega |v_0|^\theta dx + a_3|\Omega|.$$

Since $\theta > 2$, it is possible to choose $T > 0$ such that $I_w^R(tv_0) \leq 0$, for all $t > T$. $\square$

Lemma 3.2. *For any $w \in X$, $R > 0$, problem* (2.2) *has a nontrivial weak solution.*

Proof. First of all, from Lemma 2.2, (F0), (F1) and (F2), the functional I_w^R is in $C^1(X, \mathbb{R})$; see e.g. [3].

Claim. I_w^R satisfies the Palais-Smale condition; that is, every sequence $(u_n) \subset X$ such that
$$I_w^R(u_n) \to c \quad \text{and} \quad I_w'^R(u_n) \to 0, \quad \text{as } n \to \infty,$$
for some constant c, contains a convergent subsequence.

Verification of the Claim. Note that
$$I_w^R(u_n) - \frac{1}{\theta}\langle I_w'^R(u_n), u_n\rangle \le c + \|u_n\|, \quad \forall n > n_0.$$
since $\theta > 2$, from Lemma 2.2, it is standard to prove that,
$$\|u_n\| < C, \quad C > 0.$$
By the Rellich-Kondrachov Theorem, up to a subsequence, there exists $u \in X$ such that
$$u_n \to u \quad \text{in } L^{p+1}(\Omega) \quad \text{as } n \to \infty.$$
So, as $n \to \infty$, we have
$$f_R(x, u_n, \nabla w, \Delta w) \to f_R(x, u, \nabla w, \Delta w) \quad \text{in } L^{\frac{p+1}{p}}(\Omega).$$
Therefore,
$$\int_\Omega [f_R(x, u_n, \nabla w, \Delta w) - f_R(x, u, \nabla w, \Delta w)](u_n - u)dx \to 0, \quad \text{as } n \to \infty. \tag{3.3}$$
Since $(I_w'^R(u_n) - I_w'^R(u)) \to -I_w'^R(u)$ and $u_n \rightharpoonup u$ weakly in X, we have
$$\langle I_w'^R(u_n) - I_w'^R(u), u_n - u\rangle \to 0, \quad \text{as } n \to \infty. \tag{3.4}$$
Notice that by Lemma 2.2
$$\langle I_w'^R(u_n) - I_w'^R(u), u_n - u\rangle + \int_\Omega [f_R(x, u_n, \nabla w, \Delta w) - f_R(x, u, \nabla w, \Delta w)](u_n - u)dx$$
$$\ge \gamma\|u_n - u\|^2.$$
Using (3.3) and (3.4) in the above inequality, we obtain that $u_n \to u$(strong) in X as $n \to \infty$. Thus, we conclude that the statement is true.

Applying the mountain pass Theorem, due to Ambrosetti-Rabinowitz [3], there exists $u_w^R \neq 0$ weak solution to problem (2.2) $\square$

Lemma 3.3. *Let $R > 0$ be fixed. Then there exist positive constants $d_1 := d_1(R)$, $d_2 := d_2(R)$, independent of w, such that*
$$d_2 \le \|u_w^R\| \le d_1,$$
for all solution u_w^R obtained in Lemma 3.2.

Proof. Notice that
$$I_w^R(u_w^R) \le \max_{t \ge 0} I_w^R(tv_0),$$
with v_0 given as in Lemma 3.1. From (F4) and Lemma 2.2 we obtain
$$I_w^R(tv_0) \le \frac{t^2}{2}\eta - a_2|t|^\theta \int_\Omega |v_0|^\theta dx + a_3|\Omega|.$$
Since $\theta > 2$ and $|v_0|_\theta \neq 0$, the map
$$t \in \mathbb{R} \mapsto \eta\frac{t^2}{2} - a_2|t|^\theta \int_\Omega |v_0|^\theta dx + a_3|\Omega|$$

attains a positive maximum, independent of w and R. So we get a constant C such that

$$I_w^R(u_w^R) \le C. \tag{3.5}$$

Now, define

$$|||u|||^2 = \int_\Omega (\Delta u^2 - q\nabla u^2 + \alpha(x)u^2)dx,$$

which by Lemma 2.2 is an equivalent norm in X. By (3.5), we have

$$\frac{1}{2}|||u_w^R|||^2 \le C + \int_\Omega F_R(x, u_w^R, \nabla w, \Delta w), \quad C > 0. \tag{3.6}$$

Let t_0 be as in condition (F3), and define $D := \{x \in \Omega; |u_w^R(x)| > t_0\}$. Keeping in mind that u_w^R is a solution from (F2) and (F3) and by Remark 2.1, we have

$$\begin{aligned}\int_\Omega F_R(x, u_w^R, \nabla w, \Delta w) &= \int_{\Omega\backslash D} F_R(x, u_w^R, \nabla w, \Delta w) + \int_D F_R(x, u_w^R, \nabla w, \Delta w)\\ &\le a_1(R+2)^r\Big(t_0 + \frac{|t_0|^{p+1}}{p+1}\Big)|\Omega\backslash D| + \frac{1}{\theta}|||u_w^R|||^2.\end{aligned}$$

Returning to equation (3.6) we have

$$\frac{1}{2}|||u_w^R|||^2 \le C + a_1(R+2)^r\Big(t_0 + \frac{|t_0|^{p+1}}{p+1}\Big)|\Omega\backslash D| + \frac{1}{\theta}|||u_w^R|||^2,$$

where $|\Omega\backslash D|$ denotes the Lebesgue measure in $\mathbb{R}^N$ of the set $\Omega\backslash D$. Again by Lemma 2.2, we have

$$\gamma\big(\frac{1}{2} - \frac{1}{\theta}\big)\|u_w^R\|^2 \le \big(\frac{1}{2} - \frac{1}{\theta}\big)|||u_w^R|||^2 \le C + a_1(R+2)^r\big(t_0 + \frac{|t_0|^{p+1}}{p+1}\big)|\Omega\backslash D|.$$

Thus, we can conclude that exists $c_1 > 0$ such that

$$\gamma\big(\frac{1}{2} - \frac{1}{\theta}\big)\|u_w^R\|^2 < c_1(R+2)^r;$$

that is, $\|u_w^R\| \le d_1$, for some $d_1 = d_1(R) > 0$.

Now, we shall prove that there exists $d_2 > 0$ such that $\|u_w^R\| > d_2$. In fact, notice that

$$I_w'^R(u_w^R)(u_w^R) = 0, \tag{3.7}$$

and from (F1) and (F2), given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f_R(x, u_w^R, \nabla w, \Delta w)| \le \varepsilon|u_w^R| + C_\varepsilon|u_w^R|^p(R+2)^r. \tag{3.8}$$

Inserting (3.8) in (3.7) and using Lemma 2.2, we have

$$\gamma\|u_w^R\|^2 \le C_1\varepsilon\|u_w^R\|^2 + C_2C_\varepsilon\|u_w^R\|^{p+1}(R+2)^R,$$

for some constants $C_1, C_2 \ge 0$. Therefore, there exists $d_2 > 0$ such that $\|u_w^R\| \ge d_2$. This completes the proof. $\square$

Lemma 3.4. *Choose $w \in C^{4,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$, and let $R > 0$ be fixed. If $u_w^R \in X$ is a weak solution of problem (2.2), then $u_w^R \in C^{4,\beta}(\overline{\Omega})$, for some $\beta \in (0,1)$, and $\Delta(u_w^R)(x) = 0$ if $x \in \partial\Omega$.*

Proof. Let $u_w^R \in X$ be a weak solution of (2.2). Define $v = \Delta u_w^R$ and

$$g(x) = f_R(x, u_w^R, \nabla w, \Delta w) - q\Delta u_w^R - \alpha(x) u_w^R.$$

By hypotheses (F2), and the Sobolev embedding, notice that $g(x) \in L^2(\Omega)$. So, v is a weak solution of

$$\Delta v = g(x), \quad \text{in } \Omega,$$

in the following sense: For $\phi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega v\Delta\phi dx = \int_\Omega g\phi dx.$$

From Agmon [2, Theorem 7.1'], we have that $v \in H^2_{\rm loc}(\Omega)$. Therefore, $u_w^R \in H^4_{\rm loc}(\Omega)$.

Fix $\phi \in C_c^\infty(\Omega)$, since $u_w^R \in X$ is a weak solution of problem (2.2), we have

$$\int_\Omega \Delta u_w^R \Delta\phi dx - q\int_\Omega \nabla u_w^R \nabla\phi dx + \int_\Omega \alpha(x) u_w^R \phi dx = \int_\Omega f_R(x, u_w^R, \nabla w, \Delta w)\phi dx. \tag{3.9}$$

But $\operatorname{supp}\phi \subset\subset \Omega$, so

$$\int_\Omega (\Delta^2 u_w^R + q\Delta u_w^R + \alpha(x) u_w^R)\phi dx = \int_\Omega f_R(x, u_w^R, \nabla w, \Delta w)\phi dx.$$

From the denseness of $C_c^\infty(\Omega)$ in $X = H^2(\Omega) \cap H_0^1(\Omega)$ we conclude that

$$\int_\Omega (\Delta^2 u_w^R + q\Delta u_w^R + \alpha(x) u_w^R)\phi dx = \int_\Omega f_R(x, u_w^R, \nabla w, \Delta w)\phi dx \quad \forall \phi \in X. \tag{3.10}$$

The Green identities guarantees

$$\int_\Omega (\Delta u_w^R \Delta\phi - \phi\Delta^2 u_w^R) dx = \int_{\partial\Omega} \Big(\Delta u_w^R \frac{\partial\phi}{\partial\nu} - \phi\frac{\partial(\Delta u_w^R)}{\partial\nu}\Big) ds = \int_{\partial\Omega} \Delta u_w^R \frac{\partial\phi}{\partial\nu} ds, \tag{3.11}$$

$$q\int_\Omega \Delta u_w^R \phi dx = q\int_{\partial\Omega} \phi\frac{\partial u_w^R}{\partial\nu} ds - q\int_\Omega \nabla u_w^R \nabla\phi dx. \tag{3.12}$$

So, combining (3.9), (3.10), (3.11) and (3.12), we have

$$\int_{\partial\Omega} \Delta u_w^R \frac{\partial\phi}{\partial\nu} ds = 0. \tag{3.13}$$

From (3.10), we obtain that

$$\Delta^2 u_w^R + q\Delta u_w^R + \alpha(x) u_w^R = f_R(x, u_w^R, \nabla w, \Delta w) \quad \text{a.e. in } \Omega. \tag{3.14}$$

By Green's identity,

$$\int_\Omega \Delta^2 u_w^R \Delta u_w^R dx = -\int_\Omega (\nabla\Delta u_w^R)^2 dx + \int_{\partial\Omega} \frac{\partial\Delta u_w^R}{\partial\nu} \Delta u_w^R ds.$$

By (3.13), we obtain

$$\int_\Omega \Delta^2 u_w^R \Delta u_w^R dx = -\int_\Omega (\nabla\Delta u_w^R)^2 dx. \tag{3.15}$$

By Green's identity,

$$\int_\Omega \Delta u_w^R u_w^R dx = -\int_\Omega (\nabla u_w^R)^2 dx + \int_{\partial\Omega} \frac{\partial u_w^R}{\partial\nu} u_w^R ds\,.$$

Since $u_w^R \in H_0^1(\Omega)$, we get

$$\int_\Omega \Delta u_w^R u_w^R dx = -\int_\Omega (\nabla u_w^R)^2 dx. \tag{3.16}$$

Multiplying Δu_w^R equation (3.14), and integrating by parts and using (3.15) and (3.16), we obtain

$$\begin{aligned} &-\int_\Omega (\nabla \Delta u_w^R)^2 dx + q\int_\Omega (\Delta u_w^R)^2 dx - \int_\Omega \alpha(x)(\nabla u_w^R)^2 dx \\ &= \int_\Omega f_R(x, u_w^R, \nabla w, \Delta w)\Delta u_w^R dx \end{aligned} \tag{3.17}$$

By (F2) and Sobolev embbedding we can assume that

$$\Big(\int_\Omega (f_R(x, u_w^R, \nabla w, \Delta w))^2 dx\Big)^{1/2} < \infty\,.$$

Thus, from (3.17) we have $\|u_w^R\|_{W^{3,2}(\Omega)} < \infty$.

Let be $\varphi \in C_c^1(\mathbb{R}^N)$. Integrating by parts, we have

$$\int_\Omega \Delta u_w^R \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega \varphi \frac{\partial \Delta u_w^R}{\partial x_i} + \int_{\partial\Omega} \Delta u_w^R \varphi \nu^i ds.$$

Then,

$$\Big|\int_\Omega \Delta u_w^R \frac{\partial \varphi}{\partial x_i} dx\Big| \le \Big|\int_\Omega \varphi \frac{\partial \Delta u_w^R}{\partial x_i} dx\Big| \le \|u_w^R\|_{W^{3,2}(\Omega)} |\varphi|_{L^2(\Omega)}.$$

Now, by [7, Prop IX.18] we obtain

$$\Delta u_w^R \in H_0^1(\Omega). \tag{3.18}$$

Now, let us consider the following notation

$$\begin{gathered} v = \Delta u_w^R + q u_w^R, \\ g(x) = f_R(x, u_w^R, \nabla w, \Delta w) - \alpha(x) u_w^R. \end{gathered}$$

Notice that v and u_w^R are solutions in the weak sense of the respective differential equations with Dirichlet boundary condition, namely,

$$\begin{gathered} \Delta v = g(x), \quad \text{in } \Omega \\ v(x) = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.19}$$

and

$$\begin{gathered} \Delta u_w^R + q u_w^R = v(x), \quad \text{in } \Omega \\ u_w^R(x) = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{3.20}$$

By the Sobolev embedding, we have $u_w^R \in L^q(\Omega)$, with $q = 2N/(N-4)$. By (F2), we have that $g \in L^s(\Omega)$ with $s = \frac{q}{p}$, where p is given in (F2).

We want to show that $u_w^R \in W^{4,r}(\Omega)$, for some r such that $4r > N$. If $4s > N$, it is sufficient to take $r = s$. In fact, applying Agmon [2, Theorem 8.2], we have $u_w^R \in W^{4,r}(\Omega)$. Now, suppose that $4s < N$. By the Sobolev embedding,

$$u_w^R \in L^{q_1}(\Omega), \quad \text{where } q_1 = \frac{Ns}{N-4s}.$$

By (F2), we have $g \in L^{s_1}$ with $s_1 = q_1/p$.

From [2, Theorem 8.2], we have $v \in W^{2,s_1}(\Omega)$ and $u_w^R \in W^{4,s_1}(\Omega)$. Since $1 < p < \frac{N+4}{N-4}$, there exists a $\epsilon > 0$ such that

$$s = (1+\epsilon)\frac{2N}{N+4}.$$

Thus,

$$\frac{s_1}{s} = \frac{q_1}{q} = \frac{sN}{N-4s}\frac{N-4}{2N} = (1+\epsilon)\frac{N(N-4)}{(N+4)(N-4s)}.$$

But notice that (it is sufficient we substitute $s = (1+\epsilon)2N/(N+4)$),

$$\frac{N(N-4)}{(N+4)(N-4s)} > 1.$$

Therefore, $s_1/s > 1+\epsilon$. This argument is known as a *bootstrap.*

If $4s_1 < N$, applying again the bootstrap argument, we obtain $u_w^R \in W^{4,s_2}$, where

$$s_2 = \frac{Ns_1}{p(N-4s_1)}.$$

Therefore,

$$\frac{s_2}{s_1} = \frac{Ns_1(N-4s)}{Ns(N-4s_1)} > (1+\epsilon)\frac{N-4s}{N-4s_1} > (1+\epsilon).$$

We can repeat this last argument a finite times to obtain that $u_w^R \in W^{4,r}(\Omega)$, for some r such that $4r \geq N$.

For the case $4r = N$, since $g \in L^r(\Omega)$, we have that $g \in L^k(\Omega)$ for some $k < r$ such that $(1+\epsilon)k > r$. Applying again the bootstrap argument, we conclude that $u_w^R \in W^{4,r}(\Omega)$, with $4r > N$.

Therefore, we can apply the Sobolev-Morrey Theorem to show that $u_w^R \in C^{\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$. By (F0) and $(A1)$, we have that

$$g(x) = f_R(x, u_w^R, \nabla w, \Delta w) - \alpha(x)u_w^R \in C^{\beta}(\overline{\Omega}), \quad \text{for some } \beta \in (0,1).$$

By applying the Schauder estimates in (3.19), we obtain that $v \in C^{2,\beta}(\overline{\Omega})$. By applying the Schauder estimates again, in (3.20), we obtain

$$u \in C^{4,\beta}(\overline{\Omega}). \tag{3.21}$$

To conclude, notice that by (3.18) and (3.21), we have $\Delta u_w^R(x) = 0$, if $x \in \partial\Omega$. □

Lemma 3.5. *There exist positive constants μ_0, μ_1 and μ_2, independent of $R > 0$ and of $w \in X$, such that*

$$\|u_w^R\|_{C^0} \leq \mu_0(R+2)^r,$$
$$\|\nabla(u_w^R)\|_{C^0} \leq \mu_1(R+2)^r,$$
$$\|\Delta(u_w^R)\|_{C^0} \leq \mu_2(R+2)^r\,.$$

Also, there exists $\overline{R} > 0$ such that $\mu_i(\overline{R}+2)^r \leq \overline{R}$, for $i = 0,1,2$.

Proof. This result follows combining Lemma 3.3 and the results of the Sobolev embedding by arguing as in the proof Lemma 3.4.

To obtain $\overline{R} > 0$ such that $\mu_i(\overline{R}+2)^r \leq \overline{R}$, it is sufficient to observe that $r < 1$, and therefore

$$\frac{\mu_i}{\overline{R}^{1-r}}\Big(\frac{\overline{R}+2}{\overline{R}}\Big)^r \leq 1$$

for $\overline{R}$ sufficiently large. □

Now, let us "construct" a nontrivial solution for problem (1.1). Consider the following problem: Let $u_0 \in X \cap C^{4,\lambda}(\overline{\Omega})$, $\lambda \in (0,1)$, and u_n $(n = 1, 2, \dots)$ be a weak solution of the problem (P_n), that is, problem (2.2), with $w = u_{n-1}$, which was found by the mountain pass Theorem in Lemma 3.2 and $R = \overline{R}$ obtained in Lemma 3.5.

Note that from Lemma 3.4 we have $u_n \in C^4(\overline{\Omega})$ and from Lemmas 3.3 and 3.5, we infer that $\|u_n\| \geq d_2$ and

$$\|u_n\|_{C^0}, \|\nabla u_n\|_{C^0}, \|\Delta u_n\|_{C^0} \leq \overline{R},$$

respectively. Thus,

$$\begin{aligned} f_{\overline{R}}(x, u_n, \nabla u_{n-1}, \Delta u_{n-1}) &= f(x, u_n, \nabla u_{n-1}\varphi_{\overline{R}}(\nabla u_{n-1}), \Delta u_{n-1}\varphi_{\overline{R}}(\Delta u_{n-1})) \\ &= f(x, u_n, \nabla u_{n-1}, \Delta u_{n-1}). \end{aligned}$$

So, u_n is a weak solution of problem (P_n).

Remark 3.6. When the diameter of Ω approaches zero, by an easy calculation in the proof of Lemma 3.5, it is possible to choose μ_i $(i = 0, 1, 2)$ sufficiently small.

Lemma 3.7. *In hypothesis* (F5)*, let us take*

$$\begin{gathered} \rho_1 = \inf\{k_1 : \|u_n\|_{C^0} \leq k_1, \ \forall\, n \in \mathbb{N}\} > 0, \\ \rho_2 = \inf\{k_2 : \|\nabla u_n\|_{C^0} \leq k_2, \ \forall\, n \in \mathbb{N}\} > 0, \\ \rho_3 = \inf\{k_3 : \|\Delta u_n\|_{C^0} \leq k_3, \ \forall\, n \in \mathbb{N}\} > 0. \end{gathered}$$

Then $\{u_n\}$ converges strongly in X.

Remark 3.8. We recall that the constant d_1 (Lemma 3.3) is obtained using only the conditions (F1)–(F4), and the constants ρ_1, ρ_2, ρ_3 are exhibited combining the constant d_1 with the Sobolev embedding constants. Thus, as is pointed out in [14], the condition (F5) can be read as a constraint on the growth coefficients of f with respect to dimension N.

Proof of Lemma 3.7. In this proof we will use a similar argument that used in [12] and [14]. Let u_n and u_{n+1} be a weak solutions of problems (P_n) and (P_{n+1}), respectively. Then, multiplying (P_{n+1}) resp. (P_n) by $(u_{n+1} - u_n)$ and integrating by parts, and applying Lemma 2.2 we obtain

$$\begin{aligned} &\gamma\|u_{n+1} - u_n\|^2 \\ &\leq \int_\Omega [f(x, u_{n+1}, \nabla u_n, \Delta u_n) - f(x, u_n, \nabla u_n, \Delta u_n)](u_{n+1} - u_n)dx \\ &\quad + \int_\Omega [f(x, u_n, \nabla u_n, \Delta u_n) - f(x, u_n, \nabla u_{n-1}, \Delta u_n)](u_{n+1} - u_n)dx \\ &\quad + \int_\Omega [f(x, u_n, \nabla u_{n-1}, \Delta u_n) - f(x, u_n, \nabla u_{n-1}, \Delta u_{n-1})](u_{n+1} - u_n)dx. \end{aligned}$$

Thus, by (F5) and the Hölder inequality we obtain

$$\begin{aligned} \gamma\|u_{n+1} - u_n\|^2 \leq \tau_1^2 L_{\rho_1}\|u_{n+1} - u_n\|^2 &+ \tau_1\tau_2 L_{\rho_2}\|u_n - u_{n-1}\|\|u_{n+1} - u_n\| \\ &+ \tau_1\tau_3 L_{\rho_3}\|u_n - u_{n-1}\|\|u_{n+1} - u_n\|. \end{aligned}$$

Therefore,

$$\|u_{n+1} - u_n\| \leq \frac{(\tau_1\tau_2 L_{\rho_2} + \tau_1\tau_3 L_{\rho_3})}{\gamma - \tau_1^2 L_{\rho_1}}\|u_n - u_{n-1}\|. \tag{3.22}$$

Hence it follows that the sequence u_n converges strongly to function u, in X. $\square$

Proof of Theorem 1.1. First of all, as before, we obtain that $\|u_n\| \geq d_2 > 0$. Also, we see that,

$$\|u_n\|_{C^0}, \quad \|\nabla u_n\|_{C^0}, \quad \|\Delta u_n\|_{C^0}$$

are uniformly bounded. Now, from (P_n), notice that $v_n = \Delta u_n$ verifies the equation

$$\Delta v_n = h(x), \quad x \in \overline{\Omega},$$

where

$$h(x) = f(x, u_n, \nabla u_{n-1}, \Delta u_{n-1}) - q\Delta u_n - \alpha(x)u_n.$$

Since $\|h\|_{C^\beta} \leq C$, for some positive constant C, by the Schauder Theorem follows that there exists a constant $C > 0$ such that $\|v_n\|_{C^{2,\beta}} \leq C$; therefore,

$$\|u_n\|_{C^{4,\beta}} \leq C.$$

From Arzela-Ascoli Theorem, passing to a subsequence, if necessary, we conclude that

$$\frac{\partial^j}{\partial x_i^j} u_n \to \frac{\partial^j}{\partial x_i^j} u, \quad \text{as } n \to \infty,$$

uniformly in $\overline{\Omega}$ for $j = 0, 1, \dots, 4$ and $i = 1, \dots, N$. Actually, from Lemma 3.7, all the subsequences of $\frac{\partial^j}{\partial x_i^j} u_n$ have the same limit, so the whole sequence

$$\frac{\partial^j}{\partial x_i^j} u_n \to \frac{\partial^j}{\partial x_i^j} u, \quad \text{as } n \to \infty, \text{ for } j = 0, 1, \dots, 4.$$

Therefore, passing to the limit in (P_n), we obtain that u is a classical solution of (1.1). Hence, the proof of Theorem 1.1 is complete.

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