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# DISTRIBUTION-VALUED WEAK SOLUTIONS TO A PARABOLIC PROBLEM ARISING IN FINANCIAL MATHEMATICS

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Abstract. We study distribution-valued solutions to a parabolic problem that arises from a model of the Black-Scholes equation in option pricing. We give a minor generalization of known existence and uniqueness results for solutions in bounded domains $\Omega \subset \mathbb{R}^{n+1}$ to give existence of solutions for certain classes of distributions $f \in \mathcal{D}'(\Omega)$. We also study growth conditions for smooth solutions of certain parabolic equations on $\mathbb{R}^n \times (0, T)$ that have initial values in the space of distributions.

## 1. Introduction and Motivation

Recently, there has been an increased interest in the study of parabolic differential equations that arise in financial mathematics. A particular instance of this is the Black-Scholes model of option pricing via a reversed-time parabolic differential equation [5]. In 1973 Black and Scholes developed a theory of market dynamic assumptions, now known as the Black-Scholes model, to which the Itô calculus can be applied. Merton [18] further added to this theory completing a system for measuring, pricing and hedging basic options. The pricing formula for basic options is known as the Black-Scholes formula, and is numerically found by solving a parabolic partial differential equation using Itô's formula. In this frame, general parabolic equations in multidimensional domains arise in problems for barrier options for several assets [21].

Much of the current research in mathematical finance deals with removing the simplifying assumptions of the Black-Scholes model. In this model, an important quantity is the volatility that is a measure of the fluctuation (i.e. risk) in the asset prices; it corresponds to the diffusion coefficient in the Black-Scholes equation. While in the standard Black-Scholes model the volatility is assumed constant, recent variations of this model allow for the volatility to take the form of a stochastic variable [10]. In this approach the underlying security $S$ follows, as in the classical Black-Scholes model, a stochastic process

$$dS_t = \mu S_t dt + \sigma_t S_t dZ_t$$

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where $Z$ is a standard Brownian motion. Unlike the classical model, however, the variance $v(t) = (\sigma(t))^2$ also follows a stochastic process given by

$$dv_t = \kappa(\theta - v(t))dt + \gamma\sqrt{v_t}dW_t$$

where $W$ is another standard Brownian motion. The correlation coefficient between $W$ and $Z$ is denoted by $\rho$:

$$E(dZ_t, dW_t) = \rho dt.$$

This leads to the generalized Black-Scholes equation

$$\begin{aligned}&\frac{1}{2}vS^2(D_{SS}U) + \rho\gamma vS(D_vD_sU) + \frac{1}{2}v\gamma^2(D_{vv}U) + rSD_SU\\&+ [\kappa(\theta - v) - \lambda v]D_vU - rU + D_tu = 0.\end{aligned}$$

Introducing the change of variables given by $y = \ln S$, $x = \frac{v}{\gamma}$, $\tau = T - t$, we see that $u(x, y) = U(S, v)$ satisfies

$$D_\tau u = \frac{1}{2}\gamma x[\Delta u + 2\rho D_{xy}u] + \frac{1}{\gamma}[\kappa(\theta - \gamma x) - \lambda\gamma x]D_xu + (r - \frac{\gamma x}{2})D_yu - ru$$

in the cylindrical domain $\Omega \times (0, T)$ with $\Omega \subset \mathbb{R}^2$. Using the Feynman-Kac relation, more general models with stochastic volatility have been considered (see [4]) leading to systems such as

$$\begin{gathered}D_\tau u = \frac{1}{2}\operatorname{trace}(M(x,\tau)D^2u) + q(x,\tau)\cdot Du\\u(x, 0) = u_0(x)\end{gathered}$$

for some diffusion matrix $M$ and payoff function $u_0$.

These considerations motivate the study of the general parabolic equation

$$\begin{aligned}Lv &= f(v, x, t) \quad \text{in } \Omega\\v(x, t) &= v_0(x, t) \quad \text{on } \mathcal{P}\Omega\end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^{n+1}$ is a smooth domain, $f : \mathbb{R}^{n+2} \mapsto \mathbb{R}$ is continuous and continuously differentiable with respect to $v$, $v_0 \in C(\mathcal{P}\Omega)$, and $\mathcal{P}\Omega$ is the parabolic boundary of $\Omega$. Here, $L$ is a second order elliptic operator of the form

$$Lv = \sum_{i,j=1}^{n} a_{ij}(x, t)D_{ij}v + \sum_{i=1}^{n} b_i(x, t)D_iv + c(x, t)v - \eta D_tv \tag{1.2}$$

where $\eta \in (0, 1)$ and $a_{ij}$, $b_i$, $c$ satisfy the following 4 conditions:

$$a_{ij}, b_i, c \in C(\overline{\Omega}) \tag{1.3}$$

$$\lambda\|\xi\|^2 \le \sum_{ij} a_{ij}(x, t)\xi_i\xi_j \le \Lambda\|\xi\|^2, \quad (0 < \lambda \le \Lambda) \tag{1.4}$$

$$\|a_{ij}\|_\infty, \|b_i\|_\infty, \|c\|_\infty < \infty \tag{1.5}$$

$$c \le 0. \tag{1.6}$$

Existence and uniqueness results for (1.1) when $\Omega$ is a bounded domain and the coefficients belong to the Hölder space $C^{\delta,\delta/2}(\overline{\Omega})$ have been well-established (c.f. [15] and [13]). Extensions of these results to domains of the form $\Omega \times (0, T)$ where $\Omega \subset \mathbb{R}^n$ is in general an unbounded domain are also given, as in [2] and [3].

Our concern in this work, however, is in the interpretation and solution of (1.1) in the sense of distributions. This is inspired primarily by the study in [15], Chapter 3, which obtains weak solutions $v$ of the divergence-form operator

$$\sum_{i,j=1}^{n} D_i(a_{ij}D_j v) - \eta D_t v = f$$

where the matrix $a_{ij}$ is constant and $f$ belongs to the Sobolev space $W^{1,\infty}(\Omega)$, where $\Omega \subset \mathbb{R}^{n+1}$ is a bounded domain. The solutions $v$ are weak in the sense that the derivatives of $v$ can only be defined in the context of distributions, as we discuss in more detail below. Our goal is to generalize these results to the well-known classical space $\mathcal{D}(\Omega)$ of test functions and its strong-dual space, $\mathcal{D}'(\Omega)$. In particular, we let $f \in \mathcal{D}'(\Omega)$ be of the form $f = D_\alpha g$ for some $g \in C(\overline{\Omega})$, and ask what conditions are sufficient on $f$ and the coefficients $a_{ij}$, $b_i$, and $c$ so that $Lv = f$ makes sense for some other $v \in \mathcal{D}'(\Omega)$.

Another facet of this question, however, is to consider characterizations of classical solutions to parabolic differential equations that define distributions at their boundary. This problem has been extensively studied in the case that $L$ is associated with an operator semigroup, beginning with the work of [11] and [16] to realize various spaces of distributions as initial values to solutions of the heat equation. The problem is to consider the action of a solution $v(x,t)$ to the heat equation on $\mathbb{R}^n \times (0,T)$ on a test function $\phi$ in the following sense:

$$(v,\phi) = \lim_{t\to 0^+} \int_{\mathbb{R}^n} v(x,t)\phi(x)dx. \tag{1.7}$$

The authors in [17] and [9] characterize those solutions $v$ for which (1.7) defines a hyperfunction in terms of a suitable growth condition on the solution $v(x,t)$, while [6] extends these results to describe solutions with initial values in the spaces of Fourier hyperfunctions and infra-exponentially tempered distributions. [7] gives a characterization of the growth of smooth solutions to the Hermite heat equation $L = \triangle - |x|^2 - D_t$ with initial values in the space of tempered distributions. In all of these cases, the ability to express a solution $v$ of the equation $Lv = 0$ as integration against an operator kernel (the heat kernel for the Heat semigroup and the Mehler kernel [20] for the Hermite heat semigroup) plays an important role in establishing sufficient and necessary growth conditions. While this is not possible for a general parabolic operator of the form (1.2), in this paper we propose a sufficient growth condition for a solution of $Lv = 0$ on $\mathbb{R}^n \times (0,T)$ to define a particular type of distribution, and we show the necessity of this condition in a few special cases.

The terminology we use in this paper is standard. We will denote $X = (x,t)$ as an element of $\mathbb{R}^{n+1}$ where $x \in \mathbb{R}^n$. Derivatives will be denoted by $D_i$ with $1 \le i \le n$ or $D_t$ for single derivatives, and by $D_\alpha$ with $\alpha \in \mathbb{N}^n$ for higher-order derivatives. If $\alpha \in \mathbb{N}^n$ then $|\alpha|$ denotes the sum

$$|\alpha| = \alpha_1 + \dots + \alpha_n.$$

Constants will generally be denoted by $C$, $K$, $M$, etc. with indices representing their dependence on certain parameters of the equation.

We give also a brief introduction to the theory of weak solutions and distributions as they pertain to our results. For $n \ge 1$, take $\Omega \subset \mathbb{R}^{n+1}$ to be open. Let $u, v \in L^1_{\text{loc}}(\Omega)$ and $\alpha \in \mathbb{N}^n$. We say that $v$ is the weak partial derivative of $u$ of order $|\alpha|$, denoted simply by $D_\alpha u = v$, provided that

$$\int_\Omega u(D_\alpha \phi)dx = (-1)^{|\alpha|}\int_\Omega v\phi dx$$

for all test functions $\phi \in C_0^\infty(\Omega)$. Observe that $v$ is unique only up to a set of zero measure. This leads to the following definition of the Sobolev space $W^{k,p}(\Omega)$:

Let $p \in [1,\infty)$, $k \in \mathbb{N}$, and $\Omega \subset \mathbb{R}^{n+1}$ be open. We define the Sobolev space $W^{k,p}(\Omega)$ as those $u \in L^1_{\rm loc}(\Omega)$ for which the weak derivatives $D_\alpha u$ are defined and belong to $L^p(\Omega)$ for each $0 \le |\alpha| \le k$. Observe that $W^{k,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{k,p} = \sum_{0\le|\alpha|\le k} \|D_\alpha u\|_{L^p(\Omega)}.$$

Furthermore, we denote by $W_0^{k,p}(\Omega)$ the closure of the test-function space $C_0^\infty(\Omega)$ under the Sobolev norm $\|\cdot\|_{k,p}$.

The classical space $\mathcal{D}(\Omega)$ of test functions with support in the domain $\Omega \subset \mathbb{R}^{n+1}$ originates from the constructions of [19]. To begin, let $K \subset \Omega$ be a regular, compact set. We denote by $\mathcal{D}_k(K)$ the space of functions $\phi \in C_0^\infty(K)$ for which

$$\|\phi\|_{k,K} = \|(1+|x|)^k\hat{\phi}(x)\|_\infty < \infty.$$

In fact, the norm $\|\cdot\|_{k,K}$ makes $\mathcal{D}_k(K)$ into a Banach space of smooth functions with support contained in $K$. Observe that the sequence $\mathcal{D}_k(K)$ for $k \in \mathbb{N}$ is a sequence of Banach spaces with the property that

$$\mathcal{D}_{k+1}(K) \subset \mathcal{D}_k(K)$$

for each $k$, where the inclusion is continuous. It follows that we may take the projective limit of these spaces to define the space

$$\mathcal{D}(K) = \operatorname*{proj}_{k\to\infty} \mathcal{D}_k(K)$$

of test functions $\phi$ which satisfy $\|\phi\|_{k,K} < \infty$ for every $k \in \mathbb{N}$.

Now, let $K_i$ be an increasing sequence of compact subsets of $\Omega$ whose union is all of $\Omega$. We refer to such a sequence as a compact exhaustion of $\Omega$. Then we have the continuous inclusions

$$\mathcal{D}(K_i) \subset \mathcal{D}(K_{i+1})$$

for each $i$. Thus, we may take an inductive limit to define

$$\mathcal{D}(\Omega) = \operatorname*{ind}_{i\to\infty} \mathcal{D}(K_i).$$

This is a space of continuous functions $\phi$ for which there exists a compact set $K \subset \Omega$ with $\|\phi||_{k,K} < \infty$ for all $k \in \mathbb{N}$. The topology on this space can equivalently be described as follows: a sequence $\phi_i$ in $\mathcal{D}(\Omega)$ converges to 0 if and only if there is a compact set $K \subset \Omega$ such that $\{\phi_i\}_{i=1}^\infty \subset \mathcal{D}(K)$ and $\|\phi_i\|_{k,K} \to 0$ for each $k$.

We consolidate these statements in the following definition:

**Definition 1.1.** Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with a countable, compact exhaustion $K_i$. We define $\mathcal{D}(\Omega)$ as the locally convex topological vector space

$$\mathcal{D}(\Omega) = \operatorname*{ind}_{i\to\infty} \operatorname*{proj}_{k\to\infty} \mathcal{D}_k(K_i).$$

The space $\mathcal{D}(\Omega)$ is separable, complete, and bornologic. We recall that a locally convex topological vector space $X$ is bornologic if and only if the continuous linear operators from $X$ to another locally convex topological vector space $Y$ are exactly the bounded linear operators from $X$ to $Y$. We denote by $\mathcal{D}'(\Omega)$ the topological dual of this space with the strong-operator topology, also referred to as a space of distributions. The space $\mathcal{D}'(\Omega)$ includes such objects as $u = \sum_\alpha D_\alpha g$, where $g \in C(\Omega)$. In particular, the action of $u$ on a test function $\phi$ is interpreted in the weak sense:

$$u(\phi) = \sum_\alpha (-1)^{|\alpha|} \int_\Omega g D_\alpha(\phi) dx.$$

The layout of this paper is as follows: In Section 2 we give existence and uniqueness results to certain divergence-form parabolic differential equations in sufficiently small domains $\Omega \subset \mathbb{R}^{n+1}$. In Section 3 we extend these results to general bounded domains in the constant-coefficient case. We employ the Perron process [15, 8] to obtain solutions to (1.1) when $f \in W^{1,\infty}(\Omega)$ and $v_0 = 0$, and then show how these can be used to obtain solutions for certain types of distributions. Section 4 discusses growth conditions on solutions to (1.1) when $\Omega = \mathbb{R}^n \times (0, T)$ that define distributions in the sense of (1.7). We make use of a technique of [6] to write the integral appearing in (1.7) as the difference of two other functionals, both of which have a limit as $t \to 0^+$. Using this, we obtain a sufficient growth criterion and explore its necessity in a few settings.

## 2. Weak $W^{1,2}$-solutions in small balls

We begin with establishing some basic existence and uniqueness results for solutions to divergence-form operators that are weak in a particular sense. Our methodology is based on that of [15, Chapter 3.3], , with minor generalizations to the hypotheses. This approach has the advantage in that it allows us to work with the relatively simple Sobolev spaces as opposed to the Hölder spaces, and also that it gives existence results in small balls $B$ that can be generalized to arbitrary bounded domains $\Omega$. To begin, we must describe the the type of weak solutions we are looking for: let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain, and define the diameter $2R = \operatorname{diam}(\Omega)$ by

$$2R = \sup_{(x,t),(y,s)\in\Omega} |x - y|.$$

For $1 \le i, j \le n$, let $a_{ij}$, $b_i$, and $c$ be elements of $C(\overline{\Omega})$ that satisfy (1.3)-(1.6), and assume in addition that the matrix $a_{ij}$ is symmetric. Then, for any fixed $\varepsilon, \eta \in (0, 1]$, we define divergence-form operator $L_{\varepsilon,\eta}$ as

$$L_{\varepsilon,\eta} v = \sum D_i(a_{ij} D_j v) + \sum b_i D_i v + cv + D_t(\varepsilon D_t v) - \eta D_t v.$$

Now consider the Sobolev space $W^{1,2}(\Omega)$, and let $W_0^{1,2}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ under the Sobolev norm $\|\cdot\|_{1,2}$. Choose any $f \in L^2(\Omega)$ and $v_0 \in W^{1,2}(\Omega)$. Using the terminology of [15], we say that $v$ is a weak $W^{1,2}$-solution of the problem

$$\begin{aligned} L_{\varepsilon,\eta} v &= f \quad \text{in } \Omega \\ v &= v_0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

if $v - v_0 \in W_0^{1,2}(\Omega)$ and, for all $\phi \in \mathcal{C}_0^2(\Omega)$,

$$\begin{aligned}&\int_\Omega -\sum_{ij} a_{ij}(D_j v)(D_i\phi) + \sum_i b_i(D_i v)(\phi) + cv\phi - \varepsilon(D_t v)(D_t\phi) - \eta(D_t v)\phi\,dx\,dt\\&= \int_\Omega f\phi\,dx\,dt.\end{aligned}$$

We begin with the following proposition concerning the existence and uniqueness of $W^{1,2}$-solutions to (2.1) in bounded domains; see also [12, Theorem 8.3] for an alternative proof that employs the Fredholm alternative for the operator $L_{\varepsilon,\eta}$:

**Proposition 2.1.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain and set $2R = \operatorname{diam}(\Omega)$. Assume $a_{ij}$, $b_i$, and $c$ are in $C(\overline{\Omega})$ and satisfy* (1.3)-(1.6) *with $a_{ij}$ symmetric. Then there exists a constant $K_{n,a,b}$ such that if $R < K$, then for any $f \in L^2(\Omega)$ and $v_0 \in W^{1,2}(\Omega)$ there is a unique $W^{1,2}$-solution of* (2.1).

*Proof.* We first prove the proposition for $v_0 = 0$. Assume, at first, that the $b_i$, $c$, and $\eta$ are all 0. As a consequence of (1.4), we may define an inner product on $W_0^{1,2}(\Omega)$ by

$$\langle \phi, \psi\rangle = \int_\Omega \sum_{ij} a_{ij}(D_j\phi)(D_i\psi) + \varepsilon(D_t\phi)(D_t\psi)\,dx\,dt$$

and observe that $W_0^{1,2}$ is complete with respect to this inner product. Now, $f \in L^2(\Omega)$ defines a linear functional on $W_0^{1,2}(\Omega)$ via the integral

$$F(\phi) = -\int_\Omega f\phi\,dx\,dt.$$

The Riesz Representation Theorem gives a unique function $v \in W_0^{1,2}(\Omega)$ such that $\langle v, \phi\rangle = F(\phi)$, and this is the unique solution of (2.1) for this case.

To extend this to nonzero $b_i$, $c$, and $\eta$, we use the method of continuity [13, 15]. For $h \in [0,1]$, define the operator $\mathcal{L}_h : W_0^{1,2}(\Omega) \mapsto W_0^{1,2}(\Omega)$ as follows: given $v \in W_0^{1,2}(\Omega)$ let $\mathcal{L}_h v(\phi)$ be the linear functional defined on $W_0^{1,2}(\Omega)$ by

$$\begin{aligned}\mathcal{L}_h v(\phi) = \int_\Omega &-\sum_{ij} a_{ij}(D_j v)(D_i\phi) + h\sum_i b_i(D_i v)(\phi)\\&+ hcv\phi - \varepsilon(D_t v)(D_t\phi) - h\eta(D_t v)\phi\,dx\,dt.\end{aligned}$$

Then set $\mathcal{L}_h(v) = g$ where $g \in W_0^{1,2}(\Omega)$ is the unique element for which $\langle g, \phi\rangle = \mathcal{L}_h v(\phi)$ under the Riesz Representation Theorem. Observe that $\mathcal{L}_h$ is linear and bounded for every $h$ and, by what we have just proved, $\mathcal{L}_0$ is invertible. Now, assume $\mathcal{L}_h(v) = g$. Then

$$\langle v, v\rangle = -\langle g, v\rangle + \int_\Omega h\sum_i b_i(D_i v)(v) + hcv^2 - h\eta(D_t v)v\,dx\,dt.$$

Since $c \le 0$ and $\int_\Omega (D_t v)v\,dx\,dt = \frac{1}{2}\int_\Omega D_t(v^2)\,dx\,dt = 0$, this implies

$$\begin{aligned}\langle v, v\rangle &\le -\langle g, v\rangle + h\int_\Omega \sum_i b_i(D_i v)(v)\,dx\,dt\\&\le \theta\langle v, v\rangle + \frac{1}{\theta}\langle g, g\rangle + \Big|\int_\Omega \sum_i b_i(D_i v)(v)\,dx\,dt\Big|\end{aligned} \tag{2.2}$$

for any $\theta > 0$.

Consider now the term $|\int_\Omega \sum_i b_i(D_i v)(v)\,dx\,dt|$. Let $a = \inf_{(x,t)\in\Omega} x_1$ and $b = \sup_{(x,t)\in\Omega} x_1$, so that $b - a \le 2R$ and $(x,t) \in \Omega$ implies $x_1 \in (a,b)$. Then, for $v \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}
\Big|\int_\Omega \sum_i b_i(D_i v)(v)\,dx\,dt\Big| &\le \int_\Omega \sum_i |b_i||D_i v||v|\,dx\,dt \\
&= \int_\Omega \sum_i |b_i||D_i v|\Big|\int_a^{x_1} D_1 v(s,x',t)ds\Big|\,dx\,dt
\end{aligned}$$

where we write $x'$ for the $n-1$-tuple $(x_2, \dots x_n)$. Using the Cauchy-Schwartz inequality for the $ds$ integral, this becomes

$$\begin{aligned}
&\int_\Omega \sum_i |b_i||D_i v| \int_a^b |D_1 v(s,x',t)ds|\,dx\,dt \\
&\le (2R)^{1/2} \int_\Omega \sum_i |b_i||D_i v|\Big(\int_a^b [D_1 v(s,x',t)]^2 ds\Big)^{1/2}\,dx\,dt.
\end{aligned}$$

We can then separate the terms in the sum to obtain

$$(2R)^{1/2}\Big[\theta' \int_\Omega \sum_i |b_i|^2 |D_i v|^2\,dx\,dt + \frac{n}{\theta'}\int_\Omega \int_a^b [D_1 v(s,x',t)]^2\,ds\,dx'dt\Big].$$

for any $\theta' > 0$. Setting $\theta' = 1$ and using the Fubini-Tonelli theorem for the second integral, we get the estimate

$$\begin{aligned}
&(2R)^{1/2}\Big[C_b \int_\Omega \sum_i |D_i v|^2\,dx\,dt + nR\int_\Omega [D_1 v(s,x',t)]^2\,ds\,dx'dt\Big] \\
&\le (2R)^{1/2}\Big[C_b\lambda \int_\Omega \frac{1}{\lambda}\sum_i |D_i v|^2\,dx\,dt + nR\lambda \int_\Omega \frac{1}{\lambda}\sum_i [D_i v(x,t)]^2\,dx\,dt\Big] \\
&\le C_{n,a,b}\big(R^{1/2} + R^{3/2}\big)\langle v, v\rangle
\end{aligned}$$

where the constant $C_{n,a,b}$ depends only on $n$, $a$ (through $\lambda$), and $b$. Hence,

$$\Big|\int_\Omega \sum_i b_i(D_i v)(v)\,dx\,dt\Big| \le C_{n,a,b}(R^{1/2} + R^{3/2})\langle v, v\rangle$$

for all $v \in C_0^\infty(\Omega)$, a result which extends to all $v \in W_0^{1,2}(\Omega)$ by density. Thus, we see that there is a $K_{n,a,b}$ such that $R < K$ implies

$$\Big|\int_\Omega \sum_i b_i(D_i v)(v)\,dx\,dt\Big| \le \frac{1}{2}\langle v, v\rangle.$$

Placing this into (2.2), it follows that with such $R$ we may choose $\theta > 0$ so that $\langle v, v\rangle \le \beta\langle g, g\rangle$ for some positive $\beta$ that is independent of $h$. The method of continuity then implies that $\mathcal{L}_h$ is invertible for all $h \in [0,1]$, and in particular for $h = 1$. Hence, given $f \in L^2(\Omega)$ we may use the Riesz Representation Theorem to find a $g \in W_0^{1,2}(\Omega)$ for which $\langle g, \phi\rangle = \int_\Omega f\phi\,dx\,dt$, and then use the invertibility of $\mathcal{L}_h$ to obtain the weak $W^{1,2}$-solution to (2.1) with $v_0 = 0$.

Finally, let $v_0 \in W^{1,2}(\Omega)$ be nonzero. Observe that $L_{\varepsilon,\eta}v_0(\phi)$ also defines a linear, continuous functional on $W_0^{1,2}(\Omega)$, and thus $L_{\varepsilon,\eta}(v_0)$ defines an element of

$W_0^{1,2}(\Omega)$ by the Riesz Representation Theorem, and in particular an element of $L^2(\Omega)$. Let $w$ be the unique weak $W^{1,2}$-solution to

$$\begin{gathered}L_{\varepsilon,\eta} w = g \quad \text{in } \Omega \\ w = 0 \quad \text{on } \partial\Omega\end{gathered}$$

where $g = f - L_{\varepsilon,\eta}(v_0)$. Then $v = w + v_0$ is the solution to (2.1).

It is possible to extend this existence result to $\varepsilon = 0$ if the coefficients $a_{ij}$ and $b_i$ are constant in addition to satisfying the hypotheses of Proposition 2.1. The basic strategy is to obtain a uniform estimate on the derivatives of solutions $v_\varepsilon$ to ( 2.1) with $\eta$ fixed and $\varepsilon \in (0, 1]$. This will require us to also strengthen our hypotheses on the $v_0$, $f$, and $\Omega$. The first result we need is a maximal property that holds when $v_0$ has a continuous extension to the boundary of $\Omega$. $\square$

**Lemma 2.2.** *Let $\Omega$ be a bounded domain, and assume $v_0 \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$ satisfies the inequality $v_0 \le M$ on $\partial\Omega$ for some constant $M \ge 0$. Assume further that $v \in W^{1,2}(\Omega)$ is such that $v - v_0 \in W_0^{1,2}(\Omega)$.*

- (a) *If $u = (v - M)^+$, then $u \in W_0^{1,2}(\Omega)$*
- (b) *If $R = \operatorname{diam}(\Omega) < K$ and $L_{\varepsilon,\eta} v(\phi) \ge 0$ for all nonnegative $\phi \in W_0^{1,2}(\Omega)$, then $v \le M$ in $\Omega$.*

*Proof.* (a) From of [15, Lemma 3.7], we have that if $f \in W^{1,2}(\Omega)$, then $f^+ \in W^{1,2}(\Omega)$ with

$$D_\alpha f^+ = \chi_A D_\alpha f,$$

where $|\alpha| = 1$ and $A = \{x : f(x) > 0\}$. Let $v_k \in C_0^\infty(\Omega)$ be such that $v_k \to v - v_0$ in $W^{1,2}(\Omega)$, and define $w = v_0 - M \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$. Then for every integer $k > 0$, the function $(v_k + w - \frac{1}{k})^+ \in W^{1,2}(\Omega)$ is compactly supported in $\Omega$, and so belongs to $W_0^{1,2}(\Omega)$ by convolution. Now $(v_k + w - \frac{1}{k})^+ \to (v - M)^+ \in L^2(\Omega)$ as $k \to \infty$. Furthermore, for $|\alpha| = 1$ we have

$$\|D_\alpha(v_k + w - \frac{1}{k})^+ - D_\alpha(v - M)^+\|_2 = \|\chi_{E_k} D_\alpha(v_k + v_0) - \chi_E D_\alpha v\|_2$$

where

$$E_k = \{x : v_k(x) + w(x) - \frac{1}{k} > 0\}, \quad E = \{x : v(x) - M > 0\}.$$

From this we obtain the estimate

$$\begin{aligned}&\|\chi_{E_k} D_\alpha(v_k + v_0) - \chi_E D_\alpha v\|_2 \\ &\le \|\chi_{E_k}[D_\alpha(v_k + v_0) - D_\alpha v]\|_2 + \|(\chi_{E_k} - \chi_E) D_\alpha v\|_2 \\ &\le \|\chi_{E_k}[D_\alpha(v_k + v_0) - D_\alpha v]\|_2 + \|\chi_B(\chi_{E_k} - \chi_E) D_\alpha v\|_2 \\ &\quad + \|\chi_{\Omega\setminus B}(\chi_{E_k} - \chi_E) D_\alpha v\|_2,\end{aligned}$$

where $B = \{x : v(x) = M\}$. Now, since $v_k + w + \frac{1}{k} \to v - M$ in $L^2(\Omega)$ it follows that $v_k + w + \frac{1}{k} \to v - M$ in measure, and so there is a subsequence $v_{k_n} + w + \frac{1}{k_n}$ that converges to $v - M$ pointwise a.e.. Since $\chi_{E_{k_n}} \to \chi_E$ a.e. on $\chi_{\Omega\setminus B}$ while $D_\alpha v = 0$ a.e. on $\chi_B$ (c.f. [15, Lemma 3.7] again), we conclude that

$$\|\chi_{E_{k_n}} D_\alpha(v_{k_n} + v_0) - \chi_E D_\alpha v\|_2 \to 0$$

as $n \to \infty$, and thus $(v - M)^+ \in W_0^{1,2}(\Omega)$.

(b) Let $u = (v - M)^+ \in W_0^{1,2}(\Omega)$. Then $L_{\varepsilon,\eta} v(u) \geq 0$, that is

$$\int_\Omega \sum a_{ij}(D_j v)(D_i u) - \sum b_i(D_i v)(u) - cvu + \varepsilon(D_t v)(D_t u) + \eta(D_t v)u \, dx\, dt \leq 0.$$

Observe, however, that the left hand side of this expression is equal to

$$\int_\Omega \sum_{ij} a_{ij}(D_j u)(D_i u) - \sum_i b_i(D_i u)(u) - cv(v-M)^+ + \varepsilon(D_t u)(D_t u) + \eta(D_t u)u \, dx\, dt.$$

We have that $cv(v - M)^+ \leq 0$ and and $\int_\Omega \eta(D_t u)u \, dx\, dt = 0$; so this implies

$$\int_\Omega \sum_{ij} a_{ij}(D_j u)(D_i u) - \sum_i b_i(D_i u)(u) + \varepsilon(D_t u)(D_t u) \, dx\, dt \leq 0.$$

However, since $R < K$, the proof of Lemma 2.1 gives

$$\int_\Omega \sum_{ij} a_{ij}(D_j u)(D_i u) - \sum_i b_i(D_i u)(u) + \varepsilon(D_t u)(D_t u) \, dx\, dt \geq \frac{1}{2}\langle u, u\rangle.$$

Thus, $\langle u, u\rangle \leq 0$; i.e., $u = 0$. □

Using Lemma 2.2, we can obtain the desired equicontinuity in the case that the domain $\Omega$ has the form of a small ball; i.e.,

$$\Omega = B(R) = \{(x, t) : |x|^2 + t^2 < R^2\}.$$

We will also require that the coefficients $a_{ij}$, $b_i$ of $L_{\varepsilon,\eta}$ be constant while $c \in C^1(\overline{\Omega})$. Furthermore, let $v_0 \in C^2(\overline{\Omega})$ and $f \in W^{1,\infty}(\Omega) \subset W^{1,2}(\Omega)$, so that there are constants $V$, $F$ for which

$$\begin{gathered} |v_0| + \sum_i |D_i v_0| + \sum_{ij} |D_{ij} v_0| + |D_t v_0| + |D_{tt} v_0| < V, \\ |f| + \sum_i |D_i f| + |f_t| < F. \end{gathered} \tag{2.3}$$

**Lemma 2.3.** *Let $\Omega = B(R)$ and assume that $L_{\varepsilon,\eta}$ satisfies the hypotheses of Proposition 2.1 in addition to the following: the coefficients $a_{ij}$, $b_i$ are constant and $c \in C^1(\overline{\Omega})$. Assume also that $v_0 \in C^2(\overline{\Omega})$, $f \in W^{1,\infty}(\Omega)$, and let $V$, $F$ be as in (2.3). Then there are constants $K'_{n,a,b,c,\eta}$ and $C_{n,a,b,c,\eta,V,F}$ such that if $R < K'_{n,a,b,c}$, then for any weak $W^{1,2}$-solution $v$ of* (2.1)*, we have*

$$\sum |D_i v| + |D_t v| \leq C \tag{2.4}$$

*where, in particular, $C$ is independent of $\varepsilon \in (0, 1]$.*

*Proof.* Let $w = R^2 - |x|^2 - t^2 \in W_0^{1,2}(\Omega)$. Then for any $\phi \in W_0^{1,2}(\Omega)$, we have

$$\begin{aligned} L_{\varepsilon,\eta} w(\phi) &= \int_\Omega \sum_{ij} a_{ij}(2x_j)(D_i\phi) + \sum_i b_i(-2x_i)\phi + c(R^2 - |x|^2 - t^2)\phi \\ &\quad + \varepsilon(2t)(D_t\phi) - \eta(-2t)\phi \, dx\, dt \\ &= \int_\Omega \sum_i -2a_{ii}\phi - \sum_i b_i(2x_i)\phi + c(R^2 - |x|^2 - t^2)\phi \\ &\quad - 2\varepsilon\phi + 2\eta t\phi \, dx\, dt. \end{aligned}$$

Thus, we may write $L_{\varepsilon,\eta}w = g \in L^2(\Omega)$, where

$$\begin{aligned} g &= -2\,\mathrm{trace}(a_{ij}) - \sum_i 2b_i x_i + c(R^2 - |x|^2 - t^2) - 2\varepsilon + 2\eta t \\ &\le -2n\lambda + nR\sup_i |b_i| + \|c\|_\infty R^2 + 2|\eta| R. \end{aligned}$$

Thus, it follows that we may choose $K'_{n,a,b,c,\eta} < K$ so that $R < K'_{n,a,b,\eta}$ implies $g \le -n\lambda$. Similarly, if $L_{\varepsilon,\eta}v_0 = h \in L^2(\Omega)$, then a straightforward calculation shows that for $R < K'$ we have $|h(x,t)| \le C_{n,a,b,c,\eta,V}$ for some constant $C$ independent of $\varepsilon$. In particular, since $L_{\varepsilon,\eta}v = f$, there is a constant $C'_{n,a,b,c,\eta,V,F}$ for which $|L_{\varepsilon,\eta}(v - v_0)| = |f - h| \le C'$. Now, for such $R$, if we define

$$u_\pm = \pm\frac{n\lambda}{C'}(v - v_0) - w \in W_0^{1,2}(\Omega),$$

then $L_{\varepsilon,\eta}u_\pm \ge 0$ in the sense of Lemma 2.2 and $u_\pm \le 0$ on $\partial\Omega$, so by Lemma 2.2 it follows that $u_\pm \le 0$ on $\Omega$; that is, $|v - v_0| \le \frac{C''}{n\lambda}w$.

Now, let $X = (x,t) \in \Omega$ and $Y = (y,s) \in \partial\Omega$, so that

$$\begin{aligned} |v(X) - v(Y)| = |v(X) - v_0(Y)| &\le |v(X) - v_0(X)| + |v_0(X) - v_0(Y)| \\ &\le \frac{C'}{n\lambda}w(X) + 2RV|X - Y| \end{aligned}$$

where the latter estimate follows from the Mean Value Theorem. The Mean Value Theorem also implies

$$w(X) = w(X) - w(Y) \le (\sup_\Omega |\nabla w|)|X - Y| \le C''_{n,R}|X - Y|,$$

and so, assuming $R < K'$, there is a constant $M_{n,a,b,c,\eta,V,F}$ for which

$$|v(X) - v(Y)| \le M|X - Y|.$$

In particular, for any $Y \in \partial\Omega$ and any $\tau \in \mathbb{R}^{n+1}$ such that $Y + \tau \in \Omega$, we have

$$|v(Y + \tau) - v(Y)| \le M|\tau|.$$

Our goal is to extend this Lipschitz condition to all $X \in \Omega$. Choose $\tau$ so that $\Omega_\tau = \{X \in \Omega : X + \tau \in \Omega\}$ is nonempty, and let $N$ be a constant to be determined later. We define $\rho_\pm \in W^{1,2}(\Omega_\tau)$ by

$$\rho_\pm(X) = \pm[v(X + \tau) - v(X)] - M|\tau| - N|\tau|w(X).$$

By a direct calculation, we find that for any $\phi \ge 0$ in $W_0^{1,2}(\Omega_\tau)$ that

$$\begin{aligned} L_{\rho_\pm}(\phi) = &\int_{\Omega_\tau} [\pm(f(X + \tau) - f(X)) - cM|\tau| - N|\tau|g(X)]\phi(X)dX \\ &- \int_{\Omega_\tau} [c(X + \tau) - c(X)]v(X + \tau)\phi(X)dX. \end{aligned}$$

Recall that $c \in C^1(\Omega)$. Observe also that given any $X \in \Omega$ and $Y \in \partial\Omega$, we have

$$\begin{aligned} |v(X)| &\le |v(X) - v(Y)| + |v(Y)| \\ &= |v(X) - v_0(Y)| + |v_0(Y)| \\ &\le M|X - Y| + V \\ &\le 2MR + V; \end{aligned}$$

i.e., $|v|$ is uniformly bounded on $\Omega$. Hence, with $R < K'$, there is a constant $C'''_{n,a,b,c,\eta,V,F}$ for which

$$\big|[c(X+\tau) - c(X)]v(X+\tau)\phi(X)\big| \le C'''|\tau|\phi(X),$$

and we may choose $N$ sufficiently large so that $L_{\rho_\pm}(\phi) \ge 0$ for nonnegative $\phi \in W_0^{1,2}(\Omega)$. Since $\rho_\pm \le 0$ on $\partial\Omega_\tau$, Lemma 2.2 again implies that $\rho_\pm \le 0$ on $\Omega_t$; that is,

$$|v(X+\tau) - v(X)| \le M|\tau| + N|\tau|w(X)$$

for all $X \in \Omega_\tau$. Choosing a final constant $C''''_{n,a,b,c,\eta,V,F}$ so that $M + Nw(X) < C''''$ on $\Omega$, we find that $v$ satisfies the Lipschitz condition

$$|v(X) - v(Y)| \le C''''|X - Y|$$

for all $X$, $Y \in \Omega$. By [15, Lemma 3.5], this implies the desired estimate (2.4). □

We now apply these results to find weak $W^{1,2}$-solutions of (2.1) with $\varepsilon = 0$ on sufficiently small balls $\Omega = B(R)$ by taking an appropriate subsequence of the family of solutions $v_\varepsilon$:

**Theorem 2.4.** *Let $\Omega$, $a_{ij}$, $b_i$, and $c$ satisfy the hypotheses of Lemma 2.3. Then for any $f \in W^{1,\infty}(\Omega)$ and $v_0 \in C^2(\overline{\Omega})$, there is a unique weak $W^{1,2}$-solution $v$ to (2.1) with $\varepsilon = 0$.*

*Proof.* For $\varepsilon \in (0,1]$, let $v_\varepsilon$ be the unique weak $W^{1,2}$-solution of (2.1) given by Proposition 2.1. Then by Lemma 2.3, the family $\{v_\varepsilon\}_{\varepsilon\in(0,1]}$ is uniformly bounded and equicontinuous, and so that there exists a uniformly convergent subsequence $v = \lim_m v_{\varepsilon_m}$. The estimate (2.4) implies also that $v \in W^{1,2}(\Omega)$ and satisfies (2.4) as well. To see that $v - v_0 \in W_0^{1,2}(\Omega)$, we note that since $v - v_0$ is equicontinuous and equal to 0 on $\partial\Omega$, it follows that $(v - v_0 - \frac{1}{k})^+ \in W^{1,2}(\Omega)$ is compactly supported in $\Omega$ for every integer $k > 0$. Hence, $(v - v_0 - \frac{1}{k})^+ \in W_0^{1,2}(\Omega)$, and since $(v - v_0 - \frac{1}{k})^+ \to (v - v_0)^+$ in $W^{1,2}(\Omega)$ (c.f. Lemma 2.2, part (a), it follows that $(v - v_0)^+ \in W_0^{1,2}(\Omega)$. Furthermore, the same argument holds for $v_0 - v$, so $(v - v_0)^- \in W_0^{1,2}(\Omega)$ and hence so does $v - v_0$. Finally, to show that $L_{0,\eta}u = f$, we have for any $\phi \in C_0^\infty(\Omega)$

$$\begin{aligned}
-\int_\Omega f\phi dx &= \int_\Omega \sum_{ij} a_{ij}(D_j v_{\varepsilon_m})(D_i\phi) - \sum_i b_i(D_i v_{\varepsilon_m})(\phi) - c v_{\varepsilon_m}\phi \\
&\quad + \varepsilon(D_t v)(D_t u) + \eta(D_t v)\phi \, dx\, dt \\
&= \int_\Omega v_{\varepsilon_m}\Big[\sum_{ij} -D_j(a_{ij}D_i\phi) + \sum_i D_i(b_i\phi) \\
&\quad - c\phi - \varepsilon_m D_{tt}\phi - \eta D_t\phi\Big]\, dx\, dt.
\end{aligned}$$

Since the integrand is uniformly bounded we obtain from Dominated Convergence that

$$-\int_\Omega f\phi dx = \int_\Omega v\Big[\sum_{ij} -D_j(a_{ij}D_i\phi) + \sum_i D_i(b_i\phi) - c\phi - \eta D_t\phi\Big]\, dx\, dt$$

and the theorem is proved. □

## 3. Weak solutions in general bounded domains and solutions involving derivatives

We will now use the Perron process in the same manner as [15] to extend this result to a general bounded domain $\Omega$. We begin with the following definitions: given $f \in C^1(\overline{\Omega})$ and $v_0 \in C^2(\overline{\Omega})$, we say that $u \in C(\overline{\Omega})$ is a subsolution of the problem

$$\begin{aligned} L_{0,\eta} v &= f \quad \text{in } \Omega \\ v &= v_0 \quad \text{on } \mathcal{P}\Omega \end{aligned} \tag{3.1}$$

if $u \leq v_0$ on $\mathcal{P}\Omega$ and if for any ball $B = B(R)$ with $R < K'$, the solution $\overline{u}$ of

$$\begin{aligned} L_{0,\eta} \overline{u} &= f \quad \text{in } B \\ \overline{u} &= u \quad \text{on } \partial B \end{aligned} \tag{3.2}$$

satisfies $\overline{u} \geq u$ in $B$. Supersolutions are defined similarly by reversing the inequalities. From the discussion in [15, Chapter 3.4], we see that subsolutions and supersolutions exhibit the following properties:

**Lemma 3.1.** *Consider the problem* (3.1)*:*

(a) *If $u$ is a subsolution and $w$ a supersolution, then $w \geq u$ in $\Omega$.*
(b) *Let $u$ be a subsolution and assume $B(R) \subset \Omega$ with $R < K'$. Then if $\overline{u}$ solves* (3.2)*, the function $U$ defined by*

$$U = \begin{cases} u & \text{on } \overline{\Omega} \backslash B \\ \overline{u} & \text{on } B \end{cases}$$

*is another subsolution, called the lift of $u$ relative to $B$.*
(c) *If $u$ and $w$ are subsolutions, then so is* $\max\{u, w\}$.

Recall from Theorem 2.4 that the derivatives of the solution $v$ to (2.1) satisfy the estimate (2.4) of Lemma 2.3. To apply the Perron process, we need a form of this estimate that does not make explicit use of the boundary function $v_0$. Corollary 3.20 of [15] provides such a result in the case that the coefficients $b_i$ and $c$ are 0. With some minor modifications, this estimate can be shown to hold when $b_i$ and $c$ are constant, and so we state the result without proof:

**Lemma 3.2.** *Let $\Omega = B(R)$ with $R < K'$, and assume $a_{ij}$, $b_i$, $f$, and $F$ are as in Theorem 2.4 while $c \leq 0$ is constant. Let $w$ be the function of Lemma 2.3. Then there is a constant $C_{n,a,b,c,\eta}$ such that if $v \in W^{1,2}(\Omega) \cap C(\Omega)$ satisfies $L_{0,\eta} v = f$ in the weak sense on $\Omega$, then*

$$w^2 \sum_i |D_i v|^2 + w^4 |D_t v|^2 \leq C\big(\sup |v|^2 + F\big).$$

Now, given a bounded domain $\Omega$, a function $f \in W^{1,\infty}(\Omega)$, and $v_0 \in C^2(\overline{\Omega})$, let $S$ be the set of all subsolutions $u$ of (3.1). The Perron process gives that $v(X) = \sup_{u \in S} u(X)$ defines an element of $C(\Omega)$ that satisfies $L_{0,\eta} v = f$ in the weak sense, though we cannot characterize its behavior at the boundary in the same way that we could the weak $W^{1,2}$-solutions. A proof that $v$ is a weak solution follows:

**Theorem 3.3.** *Let $\Omega$ be a bounded domain, and let $a_{ij}$, $b_i$, and $c$ satisfy the hypotheses of Lemma 3.2. Given any $f \in W^{1,\infty}(\Omega)$ and $v_0 \in C^2(\overline{\Omega})$, let $S$ be the set* *of all subsolutions of* (3.1) *and define* $v(X) = \sup_{u \in S} u(X)$. *Then* $v \in C(\Omega)$ *and* $v$ *satisfies* $L_{0,\eta} v = f$ *in the weak sense on* $\Omega$.

*Proof.* First, note from Lemma 2.2 that $u_0 = -\frac{1}{\eta}\|f\|_\infty t - \|v_0\|_\infty$ is a subsolution and $-u_0$ is a supersolution, hence $v$ is well-defined and bounded. To show that $v$ is a weak solution, let $X = (x,t) \in \Omega$ and $R < K'$ be such that $B_X(R) \subset \Omega$. Fix $X_1 = (x, t + R/8)$ and let $\{u_m\} \subset S$ be a sequence for which $u_m(X_1) \to v(X_1)$. Let $w_m = \max\{u_m, u_0\}$ so that the $w_m$ are increasing, and define $W_m$ to be the lift of $w_m$ relative to $B_X(R)$. By Lemma 3.2, there is a subsequence $W_{m_k}$ such that $W_{m_k}$ converges uniformly to a solution $w$ of $L_{0,\eta} w = f$ in $B_X(\frac{R}{2})$. That $w(X_1) = v(X_1)$ is clear; we now claim that $w = v$ for $Y$ sufficiently near $X$.

Indeed, let $X_2 \in B_X(\frac{R}{8})$, and choose a sequence $\{u'_m\} \subset S$ for which $u'_m(X_2) \to v(X_2)$. Let $w'_m = \max\{u'_m, w_m\}$ so that $w'_m$ is an increasing sequence for which $w'_m(X_1) \to v(X_1)$ and $w'_m(X_2) \to v(X_2)$. Let $W'_m$ be the lift of $w'_m$ relative to $B_X(\frac{R}{4})$, and let $W'_{m_k}$ be a subsequence which converges uniformly to a solution $w'$ of $L_{0,\eta} w = f$ in $B_X(\frac{R}{8})$. Then $w' \geq w$ in $B_X(\frac{R}{8})$ and $w'(X_2) = v(X_2)$. However, $w'(X_1) = w(X_1)$, so by the strong maximum principle it follows that $w' = w$ in $B_X(\frac{R}{8})$, and in particular $w(X_2) = w'(X_2) = v(X_2)$. Since $X_2$ was an arbitrary element of $B_X(\frac{R}{8})$, it follows that $w = v$ in $B_X(\frac{R}{8})$. Thus, $L_{0,\eta} v = f$ for functions $\phi \in C_0^\infty$ with support contained in $B_X(\frac{R}{8})$. Since $X \in \Omega$ was chosen arbitrarily, we can show that $L_{0,\eta} v = f$ in the weak sense for any $\phi \in C_0^\infty(\Omega)$ by taking an appropriate partition of unity, and the theorem is proved. □

**Remark 3.4.** We observe that proofs of Theorem 3.3 with more general conditions on the coefficients of $L_{0,\eta}$ are known, c.f. [14, Theorem 9.1]. However, the scheme given above for the constant-coefficient case is relatively straightforward and is all we require for the present discussion.

We may apply this result to obtain solutions when $f$ is a certain type of distribution. Let $\Omega$ be a bounded, convex domain with smooth boundary, and let $f \in \mathcal{D}'(\Omega)$ be of the form $f = D_\alpha g$ in the sense of distributions, where $g \in C(\overline{\Omega})$. Observe that if the coefficients $a_{ij}$, $b_i$, and $c$ are constant, then $L_{0,\eta}$ makes sense as a continuous map on the space $\mathcal{D}'(\Omega)$. We give the following existence result as a corollary to Theorem 3.3:

**Corollary 3.5.** *Let* $\Omega$, $f$ *be as above and assume that* $a_{ij}$, $b_i$, *and* $c$ *satisfy the hypotheses of Lemma 3.2. Then there is a* $w \in \mathcal{D}'(\Omega)$ *for which* $L_{0,\eta} w = f$.

*Proof.* Given $\phi \in \mathcal{D}(\Omega)$, we have for the action of $f$ on $\phi$ :

$$(f, \phi) = (-1)^{|\alpha|} \int_\Omega g D_\alpha \phi \, dx \, dt.$$

Since $g \in C(\overline{\Omega})$ and $\Omega$ is convex with a smooth boundary, we may integrate by parts to obtain

$$(f, \phi) = (-1)^{|\beta|} \int_\Omega G D_\beta \phi \, dx \, dt$$

where $G \in C^1(\overline{\Omega})$ and $\beta_i = \alpha_i + 1$. Now, let $v \in C(\Omega)$ be the weak solution of $L_{0,\eta} v = G$ on $\Omega$ from Theorem 3.3 and define $w \in \mathcal{D}'(\Omega)$ by

$$(w, \phi) = (-1)^{|\beta|} \int_\Omega v D_\beta \phi \, dx \, dt.$$

A straightforward calculation shows that $L_{0,\eta}w = f$ in the sense of distributions, and the result follows. □

## 4. Classical solutions defining distributions at their boundary

As mentioned in the Introduction, there has been an increasing interest in studying classical solutions to various differential equations whose boundary values define distributions in the sense of (1.7). Much of the work in this area has focused on differential equations arising from operator semigroups, such as the heat equation [17, 6, 9] and the Hermite heat equation ([7]). The characterizations take the form of growth conditions on solutions $u$ to these equations defined on $\mathbb{R}^n \times (0,T)$. Motivated by these results, we consider in this section sufficient growth conditions on classical solutions to parabolic equations on $\mathbb{R}^n \times (0,T)$ whose boundary values define distributions of the form $\sum_{|\alpha|\le M} D_\alpha(g_\alpha)$, where each $g_\alpha \in C(\mathbb{R}^n)$ is bounded. Our approach is based on [6, Theorem 2.4], which characterizes the growth of smooth solutions to the heat equation with boundary values in the space of infra-exponentially tempered distributions.

We begin with the following: let $L$ be an operator of the form

$$Lu = \sum_{ij} a_{ij} D_{ij} u + \sum_i b_i D_i u + cu$$

where $a_{ij}$, $b_i$, and $c$ belong to $C^\infty(\mathbb{R}^n)$ with bounded derivatives. Our interest lies in the behavior of solutions $u(x,t)$ to the problem

$$Lu - u_t = 0 \tag{4.1}$$

defined on $\mathbb{R}^n \times (0,T)$. Our first lemma concerns the existence of a "suitable" function $v \in C_0^\infty(\mathbb{R})$ that we will need in the proof.

**Lemma 4.1.** *Let $M \ge 0$ be an integer and $T > 0$. There is a function $v \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}(v) \subset [0, \frac{T}{2}]$ for which $v = \frac{t^M}{M!}$ on $(0, \frac{T}{4})$ and $v^{(M+1)} = \delta + w$ in the sense of distributions, where $w \in C^\infty(\mathbb{R})$ with $\operatorname{supp}(w) \subset [\frac{T}{4}, \frac{T}{2}]$.*

*Proof.* Define the function

$$f = \begin{cases} \frac{t^M}{M!} & \text{for } t > 0 \\ 0 & \text{for } t \le 0, \end{cases}$$

and let $\alpha \in C^\infty(\mathbb{R})$ be such that $\alpha(t) = 1$ for $t < \frac{5T}{16}$ and $\alpha(t) = 0$ for $t > \frac{7T}{16}$. Then $v = \alpha f$ is the desired function. □

Now, given a classical solution $u(x,t)$ to (4.1), we are interested in studying the behavior of $u$ on test functions $\phi \in \mathcal{D}(\Omega)$ in the sense of (1.7). This is done by using the function $v$ of Lemma 4.1 in conjunction with the operator $L$ to "split" the integral of (1.7) into two manageable parts:

**Proposition 4.2.** *Let $u(x,t)$ be a smooth solution to the parabolic equation (4.1) on $\mathbb{R}^n \times (0,T)$ such that $|u(x,t)| \le Ct^{-M}$ for some integer $M \ge 0$. Then, for any $\phi \in \mathcal{D}(\mathbb{R}^n)$, we have*

$$\lim_{t\to 0^+} \int_{\mathbb{R}^n} u(x,t)\phi(x)dx = \sum_{|\alpha|\le 2M+2} g_\alpha D_\alpha \phi$$

*where each $g_\alpha$ is continuous and bounded. In particular, the operation*

$$g(\phi) = \lim_{t\to 0^+} \int_{\mathbb{R}^n} u(x,t)\phi(x)dx$$

*defines an element of $\mathcal{D}'(\mathbb{R}^n)$.*

*Proof.* We define $\widetilde{u}(x,t)$ on $\mathbb{R}^n \times (0, \frac{T}{2})$ by

$$\widetilde{u}(x,t) = \int_{\mathbb{R}} u(x,t+s)v(s)ds.$$

From the bounds on $u$ and $v$ and their derivatives, we may take the derivative under the integral sign to conclude that $\widetilde{u}$ satisfies (4.1) on $\mathbb{R}^n \times (0, \frac{T}{2})$. In particular, since the derivative $D_t$ commutes with $L$, we have that $L^k\widetilde{u} = (D_t)^k\widetilde{u}$ for all integers $k \ge 0$. Now, for $\phi \in C_0^\infty(\mathbb{R}^n)$, consider

$$\int_{\mathbb{R}^n} \widetilde{u}(x,t)\phi(x)dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}} u(x,t+s)v(s)\phi(x)\,ds\,dx.$$

Observe that we may reverse the order of integration and differentiate under the integral sign to obtain

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}^n} [(-L)^{M+1}u](x,t+s)v(s)\phi(x)\,dx\,ds \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}} [(-D_t)^{M+1}u](x,t+s)v(s)\phi(x)\,ds\,dx. \end{aligned} \tag{4.2}$$

For the left hand side of (4.2), we may integrate by parts to obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} u(x,t+s)v(s)[(L^*)^{M+1}\phi](x)\,dx\,ds$$

where $L^*$ is the operator

$$\begin{aligned} L^*u = &-\sum_{ij}(D_{ij}a_{ij}u + D_ia_{ij}D_ju + D_ia_{ij}D_iu + a_{ij}D_{ij}u) \\ &+ \sum_i (D_ib_iu + b_iD_iu) - cu. \end{aligned}$$

As for the right hand side of (4.2), integrating by parts yields

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_{\mathbb{R}} u(x,t+s)v^{(M+1)}(s)\phi(x)\,ds\,dx \\ &= \int_{\mathbb{R}^n} u(x,t)\phi(x)dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}} u(x,t+s)w(s)\phi(x)\,ds\,dx. \end{aligned}$$

Substituting these two results into (4.2), we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} u(x,t)\phi(x)dx = &\int_{\mathbb{R}}\int_{\mathbb{R}^n} u(x,t+s)v(s)[(L^*)^{M+1}\phi](x)\,dx\,ds \\ &- \int_{\mathbb{R}^n}\int_{\mathbb{R}} u(x,t+s)w(s)\phi(x)\,ds\,dx. \end{aligned}$$

Thus, we find in the limit as $t \to 0^+$, that

$$\begin{aligned} \lim_{t\to 0^+}\int_{\mathbb{R}^n} u(x,t)\phi(x)dx = &\int_{\mathbb{R}^n}(\int_{\mathbb{R}} u(x,s)v(s)ds)[(L^*)^{M+1}\phi](x)dx \\ &- \int_{\mathbb{R}^n}(\int_{\mathbb{R}} u(x,s)w(s)ds)\phi(x)dx. \end{aligned}$$

Since the integrals in parentheses give continuous, bounded functions of $x$, the result follows. □

**Remark 4.3.** In the case that $L$ is the Laplacian $\Delta$, then the growth condition can be shown to be necessary in some sense. Indeed, let $g \in \mathcal{D}'(\mathbb{R}^n)$ have the form

$$(g, \phi) = \sum_{|\alpha| \le 2M+2} \int_{\mathbb{R}^n} g_\alpha(x) D_\alpha \phi(x) dx$$

where the $g_\alpha$ are continuous and bounded. We define

$$u(x,t) = (g_y, E_t(x-y))$$

on $\mathbb{R}^n \times (0,\infty)$. It can be shown (c.f. [1]) that $u(x,t)$ is a smooth solution to the heat equation on $\mathbb{R}^n \times (0,\infty)$ and satisfies

$$\lim_{t \to 0^+} \int_{\mathbb{R}^n} u(x,t)\phi(x) dx = (g, \phi)$$

for every $\phi \in \mathcal{D}(\mathbb{R}^n)$. Furthermore, each term $((g_\alpha)_y, (D_\alpha)_y E_t(x-y))$ appearing in $(g_y, E_t(x-y))$ is of the form

$$\begin{aligned}
&(-\sqrt{4t})^{|\alpha|} \int_{\mathbb{R}^n} g_\alpha(y) H_\alpha\big(\frac{x-y}{2\sqrt{t}}\big) E_t(x-y) dy \\
&= C_\alpha t^{-|\alpha|/2} \int_{\mathbb{R}^n} g_\alpha(x - 2z\sqrt{t}) H_\alpha(z) e^{-|z|^2} dz
\end{aligned}$$

where $H_\alpha$ is the Hermite polynomial of order $\alpha$. It follows that $|u(x,t)| \le Ct^{-M-1}$ for some constant $C$ depending on the $g_\alpha$, $M$, and the dimension $n$. We do not know if this can be sharpened to become $|u(x,t)| \le Ct^{-M}$.

**Remark 4.4.** In view of Remark 4.3, consider the case that $b_i$ and $c$ are all 0, and the matrix $a_{ij}$ is constant and satisfies the condition

$$\sum_{ij} a_{ij} x_i x_j \ge \lambda |x|^2$$

where $\lambda > 0$. Based on the discussion of [13, Lemma 8.9.1], we can find a nonsingular matrix $A_{ij}$ for which $AaA^T = I$. From Proposition 4.2, we see that if $u$ is smooth, solves $Lu = u_t$ and satisfies $|u(x,t)| \le Ct^{-m}$, then $u(x,t)$ defines a distribution of the form $g = \sum_{|\alpha| \le 2m+2} g_\alpha D_\alpha$ where each $g_\alpha$ is continuous and bounded. Conversely, given such $g_\alpha$ we define the distributions

$$v_\alpha = \sum \det(A)(A_{k_1^1,1} \dots A_{k_{\alpha_1}^1,1} \dots A_{k_1^n,n} \dots A_{k_{\alpha_n}^n,n}) D_{k_1^1 \dots k_{\alpha_1}^1 \dots k_1^n \dots k_{\alpha_n}^n} g_\alpha,$$

where the summation is taken from $k_1^1, \dots k_{\alpha_1}^1, \dots k_1^n, \dots k_{\alpha_n}^n = 1$ to $n$, as determined by the chain rule. Then each $v_\alpha$ satisfies the conditions of Remark 4.3,and so there are smooth solutions $u_\alpha$ of the heat equation on $\mathbb{R}^n \times (0,\infty)$ for which $u_\alpha(0,t) = v_\alpha$ in the sense of (1.7) and $|u_\alpha(x,t)| \le Ct^{-N}$ for some nonnegative integer $N$. Then, defining $v_\alpha(x,t) = u_\alpha(Ax,t)$, we see that $v_\alpha$ is a smooth solution to (4.1) on $\mathbb{R}^n \times (0,\infty)$ with $|v(x,t)| \le Ct^{-N}$, and a straightforward calculation yields

$$\lim_{t \to 0^+} \int_{\mathbb{R}^n} v(x,t)\phi(x) dx = (g_\alpha, \phi).$$

Hence, the conclusion of Remark 4.3 is also valid for such operators $L$.

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