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EXISTENCE OF SOLUTIONS FOR AN OLDROYD MODEL OF VISCOELASTIC FLUIDS

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Abstract. In this paper we investigate the unilateral problem for an Oldroyd model of a viscoelastic fluid. Using the penalty method, Faedo-Galerkin's approximation, and basic result from the theory of monotone operators, we establish the existence of weak solutions.

1. Introduction

It is well know that, the motion of incompressible fluids is described by the system of Cauchy equations

$$\begin{gathered} \frac{\partial u}{\partial t} + u_i \frac{\partial u}{\partial x_i} + \nabla p = \operatorname{div} \sigma + f \\ \operatorname{div} u = 0, \end{gathered} \tag{1.1}$$

where $u = (u_1, \dots, u_n)$ is the velocity, p is the pressure in the fluid, f is the density of external forces and σ is the deviator of the stress tensor, that is, σ has the purpose of letting us consider reactions arising in the fluid during its motion. The vector $(u_i \frac{\partial u_j}{\partial x_i})$, $j = 1, 2, \dots, n$, is denoted by $(u.\nabla)u$. The Hooke's Law establishes a relationship between the stress tensor σ and the deformation tensor $D_{ij}(u) = \frac{1}{2}\big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\big)$ and their derivatives. Therefore is the Hooke's Law that establishes the type of fluid. Such relationship is also called of *rheological equation* or *equation of state* (see Serrin [10] or Clifford [1]). For example, for an incompressible Stokes fluid the relationship has the form

$$\sigma = \alpha D + \beta D^2 \tag{1.2}$$

where α and β are scalar functions. If in (1.2) $\alpha = 2\nu$ positive constants. and $\beta \equiv 0$ we have the Newton's Law $\sigma = 2\nu D$, which substituting in (1.1) we obtain the equations of motion of Newtonian fluid, which is called the Navier-Stokes equations:

$$u' - \nu\Delta u + (u.\nabla)u + \nabla p = f, \quad \operatorname{div} u = 0,$$

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where ν is called the kinematic coefficient of viscosity. The Navier-Stokes model was studied from the mathematical point of view by Leray [15] and later by Ladyzhenskaya [9]. We mention other deep contributions by Lions [16], Temam [21], Tartar [19] and many others researchers.

The model studied in this work, introduced by Oldroyd [11, 12], was proposed for viscous incompressible fluids whose defining equations have the form

$$\big(1+\lambda\frac{\partial}{\partial t}\big)\sigma = 2\nu\big(1+k\nu^{-1}\frac{\partial}{\partial t}\big)D, \tag{1.3}$$

where λ, ν, k are positive constants with $\nu - \frac{k}{\lambda} > 0$. In this fluid the stress after instantaneous cessation of the motion die out like $e^{-\lambda^{-1}t}$, while the velocities of the flow after instantaneous removal of the stress die out like $e^{-k^{-1}t}$.

Assuming that $\sigma(0) = 0$ and $D(0) = 0$, we write the relationship (1.3) in the form of integral equation

$$\sigma(x,t) = 2k\lambda^{-1}D(x,t) + 2\lambda^{-1}(\nu - k\lambda^{-1})\int_0^t e^{-\frac{(t-\xi)}{\lambda}}D(x,\xi)d\xi. \tag{1.4}$$

Thus, the equation for the motion of Oldroyd fluid can be written by the system of integro-differential equations

$$\frac{\partial u}{\partial t} + (u.\nabla)u - \mu\Delta u - \int_0^t \beta(t-\xi)\Delta u(x,\xi)d\xi + \nabla p = f, \quad x\in\Omega,\ t>0 \tag{1.5}$$

and the incompressible condition

$$\operatorname{div} u = 0, \quad x\in\Omega,\ t>0,$$

with initial and boundary conditions

$$u(x,0) = u_0, \quad x\in\Omega, \quad \text{and} \quad u(x,t) = 0, \quad x\in\Gamma,\ t\ge 0.$$

Here, $\mu = k\lambda^{-1} > 0$ and $\beta(t) = \gamma e^{-\delta t}$, where $\gamma = \lambda^{-1}\left(\nu - k\lambda^{-1}\right)$ with $\delta = \lambda^{-1}$. For physical details and mathematical modelling see [2, 5, 11, 22].

The mixed problem above was investigated by Oskolkov [2], where he proves existence of weak solution for all $n\in\mathbb{N}$ in certain Sobolev class.

In Brézis [6] we find investigation for a unilateral problem for the case of the Navier-Stokes equations.

In the present work we consider a unilateral problem similar to Brézis [6], adding a memory term, that is $-\int_0^t g(t-\sigma)\Delta u(\sigma)d\sigma$. More precisely, in this paper we study a unilateral problem or a variational inequality, c.f. Lions [16], for the operator

$$L = \frac{\partial u}{\partial t} + (u.\nabla)u - \mu\Delta u - \int_0^t \beta(t-\xi)\Delta u(x,\xi)d\xi + \nabla p - f$$

under standard hypothesis on f and u_0. Making use of the penalty method and Galerkin's approximations, we establish existence and uniqueness of weak solutions.

This work is organized as follows: In Section 2, we introduce the notation and main results. In Section 3, we proof to the results. Finally, in Section 4, we prove an simple result of uniqueness.

2. Notation and Main Results

Let Ω be a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$ of class C^2. For $T>0$, we denote by Q_T the cylinder $(0,T)\times\Omega$, with lateral boundary $\Sigma_T=(0,T)\times\partial\Omega$. By $\langle .,.\rangle$ we will represent the duality pairing between X and X', X' being the topological dual of the space X, and by C we denote various positive constants. We propose the variational inequality

$$\begin{gathered} u'-\mu\Delta u+(u.\nabla)u-\int_0^t g(t-\sigma)\Delta u(\sigma)d\sigma+\nabla p\geq f \quad \text{in } Q_T\\ \operatorname{div} u=0 \quad \text{in } Q_T\\ u=0 \quad \text{on } \Sigma_T\\ u(x,0)=u_0(x) \quad \text{in } \Omega, \end{gathered} \tag{2.1}$$

where $g:[0,\infty)\to[0,\infty)$ is a function of $W^{1,1}(0,\infty)$ satisfying

$$\frac{\mu}{2}-2\int_0^\infty g(s)\,ds>0; \tag{2.2}$$

$$-C_1 g\leq g'\leq -C_2 g, \tag{2.3}$$

where C_1,C_2,C_3 are positive constants;

$$g(0)>0, \tag{2.4}$$

As an example, $g(s)=e^{-\frac{8}{\mu}s}$ satisfies the thhree conditions above.

To formulate problem (2.1) we need some notation about Sobolev spaces. We use standard natation of $L^2(\Omega)$, $L^p(\Omega)$, $W^{m,p}(\Omega)$ and $C^p(\Omega)$ for functions that are defined on Ω and range in $\mathbb{R}$, and the notation $L^2(\Omega)^n$, $L^p(\Omega)^n$, $W^{m,p}(\Omega)^n$ and $C^p(\Omega)^n$ for functions that range in $\mathbb{R}^n$. Besides, we work also with the spaces $L^p(0,T;H^m(\Omega))$ or $L^p(0,T;H^m(\Omega))^n$. To complete this recall on functional spaces, see for instance, Lions [16].

Also we define the following spaces

$$\mathcal{V}=\{\varphi\in\mathcal{D}(\Omega)^n:\operatorname{div}\varphi=0\},$$

$V=V(\Omega)$ is the closure of $\mathcal{V}$ in the space $H_0^1(\Omega)^n$ with inner product and norm denoted, respectively by

$$((u,z))=\sum_{i,j=1}^n\int_\Omega\frac{\partial u_i}{\partial x_j}(x)\frac{\partial z_i}{\partial x_j}(x)\,dx,\quad \|u\|^2=\sum_{i,j=1}^n\int_\Omega\Big(\frac{\partial u_i}{\partial x_j}(x)\Big)^2dx,$$

$H=H(\Omega)$ is the closure of $\mathcal{V}$ in the space $L^2(\Omega)^n$ with inner product and norm defined, respectively, by

$$(u,v)=\sum_{i=1}^n\int_\Omega u_i(x)v_i(x)\,dx,\quad |u|^2=\sum_{i=1}^n\int_\Omega|u_i(x)|^2\,dx$$

and V_2 is the closure of $\mathcal{V}$ in $H^2(\Omega)^n$ with inner product and norm denoted, respectively by

$$((u,z))_{V_2}=\sum_{i=1}^n(u_i,v_i)_{H^2(\Omega)},\quad \|u\|_{V_2}^2=((u,u))_{V_2},$$

Remark 2.1. V, H and V_2 are Hilbert's spaces, $V_2\hookrightarrow V\hookrightarrow H\hookrightarrow V'$ with embedding dense and continuous.

Let K be a closed and convex subset of $V \cap V_2$ with $0 \in K$. We introduce the following bilinear and the trilinear forms:

$$a(u,v) = \sum_{i,j=1}^{n} \int_\Omega \frac{\partial u_i}{\partial x_j}(x) \frac{\partial v_i}{\partial x_j}(x)\, dx = ((u,v)),$$

$$b(u,v,w) = \sum_{i,j=1}^{n} \int_\Omega u_i(x) \frac{\partial v_j}{\partial x_i}(x) w_j(x)\, dx,$$

We also assume that

$$a(v,v) + b(v,\varphi,v) + \int_0^t g(t-\sigma)((v,v))d\sigma \geq 0 \quad \forall \varphi \in K,\ \forall v \in V. \tag{2.5}$$

Next we shall state the main results of this paper.

Theorem 2.2. *If $f \in L^2(0,T;H)$ and hypotheses (2.5) holds, then there exists a function u such that*

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H) \tag{2.6}$$

$$u(t) \in K \quad a.e. \tag{2.7}$$

$$\begin{gathered} \int_0^T \langle \varphi', \varphi - u\rangle + \mu a(u,\varphi-u) + b(u,u,\varphi-u) \\ - \Big(\int_0^t g(t-\sigma)\Delta u(\sigma)d\sigma, \varphi - u\Big)dt \\ \geq \int_0^T \langle f, \varphi - u\rangle\, dt, \quad \forall \varphi \in L^2(0,T;V),\ \varphi' \in L^2(0,T;V'), \end{gathered} \tag{2.8}$$

$$\varphi(0) = 0, \quad \varphi(t) \in K \ a.e.$$

$$u(0) = u_0.$$

Theorem 2.3. *Assumption (2.5), $n = 2$, and*

$$f \in L^2(0,T;V), \quad f' \in L^2(0,T;V') \tag{2.9}$$

$$u_0 \in K. \tag{2.10}$$

Suppose also that

$$(f(0),v) - \mu a(u_0,v) - b(u_0,u_0,v) = (u_1,v) \quad \text{for all } v \in V \text{ some } u_1 \in V. \tag{2.11}$$

Then there exists a unique function u such that

$$u \in L^2(0,T;V\cap V_2), \quad u' \in L^2(0,T;V) \cap L^\infty(0,T;H) \tag{2.12}$$

$$u(t) \in K, \quad \forall t \in [0,T] \tag{2.13}$$

$$\begin{gathered} (u'(t), v-u(t)) + \mu a(u(t), v-u(t)) + b(u(t),u(t),v-u(t)) \\ + \int_0^T \int_0^t g(t-\sigma)((u(\sigma), v-u(t)))d\sigma dt \\ \geq (f(t), v-u(t)) \quad \forall v \in K,\ a.e. \text{ in } t, \end{gathered} \tag{2.14}$$

$$u(0) = u_0. \tag{2.15}$$

The proof of Theorems 2.2 and 2.3 will be given in Section 3 by the penalty method. It consists in considering a perturbation of the operator L adding a singular term called penalty, depending on a parameter $\epsilon > 0$. We solve the mixed problem in Q for the penalized operator and the estimates obtained for the local solution of the penalized equation, allow to pass to limits, when ϵ goes to zero, in order to obtain a function u which is the solution of our problem.

First of all, let us consider the penalty operator $\beta : V \to V'$ associated to the closed convex set K, c.f. Lions [16, p. 370]. The operator β is monotonous, hemicontinuous, takes bounded sets of V into bounded sets of V', its kernel is K and $\beta : L^2(0,T;V) \to L^2(0,T;V')$ is equally monotone and hemicontinous. The penalized problem associated with the variational inequalities (2.8) and (2.14) consists in, given $0 < \epsilon < 1$, find u_ϵ satisfying

$$\begin{gathered}(u_\epsilon', v) + \mu a(u_\epsilon, v) + b(u_\epsilon, u_\epsilon, v) - \int_0^t g(t-\sigma)(\Delta u_\epsilon(\sigma), v)d\sigma + \frac{1}{\epsilon}(\beta(u_\epsilon), v) = (f, v), \\ \forall\, v \in V, \quad u_\epsilon \in L^2(0,T;V), \quad u_\epsilon' \in L^2(0,T;V') \\ u_\epsilon(x,0) = u_{\epsilon_0}(x).\end{gathered} \tag{2.16}$$

We suppose $n = 2$. The solution of this problem is given by the followings theorems.

Theorem 2.4. *If $f \in L^2(0,T;H)$ and hypotheses* (2.2) *holds, then, for each $0 < \epsilon < 1$ and $u_{\epsilon_0} \in H$, there exists a function u_ϵ with $u_\epsilon \in L^2(0,T;V) \cap L^\infty(0,T;H)$, $u_\epsilon' \in L^2(0,T;V')$ solution of* (2.16).

Theorem 2.5. *If $f \in L^2(0,T;V)$ and $f' \in L^2(0,T;V')$ and hypotheses* (2.2) *holds, then for each $0 < \epsilon < 1$ and $u_{\epsilon_0} \in V$, there exists a function u_ϵ with $u_\epsilon \in L^\infty(0,T;V \cap V_2)$, $u_\epsilon' \in L^2(0,T;V) \cap L^\infty(0,T;H)$ satisfying* (2.16).

3. Proof of the Results

Proof of Theorem 2.2. We first prove Theorem 2.4 for the penalized problem. We employ the Faedo-Galerkin method. We note that the embedding $V \hookrightarrow V \overset{\text{comp}}{\hookrightarrow} H \hookrightarrow V'$ are continuous and dense and that V is compactly and densely embedded in H. Let $\{w_\nu, \lambda_\nu\}$, $\nu \in \mathbb{N}$, be solutions of the spectral problem

$$((w, v)) = \lambda(w, v), \quad \forall v \in V. \tag{3.1}$$

We consider $(w_\nu)_{\nu\in\mathbb{N}}$ a Hilbertian basis for Faedo-Galerkin method. We represent by $V_m = [w_1, w_2, \dots, w_m]$ the V subspace generated by the vectors $w_1, w_2, \dots, w_m$ and let us consider

$$u_{\epsilon_m}(t) = \sum_{j=1}^m g_{j\,m}(t) w_j$$

solution of approximate problem

$$\begin{aligned}&(u_{\epsilon_m}', w_j) + \mu a(u_{\epsilon_m}, w_j) + b(u_{\epsilon_m}, u_{\epsilon_m}, w_j) \\ &- \int_0^t g(t-\sigma)(\Delta u_{\epsilon_m}(\sigma), v)d\sigma + \frac{1}{\epsilon}\langle \beta u_{\epsilon_m}, w_j\rangle \\ &= \langle f(t), w_j\rangle, \quad j = 1, 2, \dots m \\ &\quad u_{\epsilon_m}(x,0) \to u_\epsilon(x,0) \quad \text{strongly in } V.\end{aligned} \tag{3.2}$$

This system of ordinary differential equations has a solution on a interval $[0, t_m[$, $0 < t_m < T$. The first estimate permits us to extend this solution to the whole interval $[0, T]$.

Remark 3.1. To obtain a better notation, we omit the parameter ϵ in the approximate solutions.

First estimate. Multiplying both sides of (3.2) by g_j and adding from $j = 1$ to $j = m$, we obtain

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \mu\|u_m(t)\|^2 + \int_0^t g(t-\sigma)(\nabla u_m(\sigma), \nabla u_m(t))d\sigma = (f(t), u_m(t)),$$

since $b(u_m, u_m, u_m) = 0$ (see Lions [16]) and
$(\beta u_m(t), u_m(t)) \geq 0$ because β is monotone and $0 \in K$. Its follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \mu\|u_m(t)\|^2 \\ &\leq \Big|\int_\Omega \nabla u_m(x,t)\Big(\int_0^t g(s-\sigma)\nabla u_m(x,\sigma)\Big)d\sigma dx\Big|_{\mathbb{R}} \\ &\quad + |f(t)||u_m(t)| + \int_\Omega |\nabla u_m(x,t)|_{\mathbb{R}}|g * \nabla u_m(x,t)|_{\mathbb{R}} dx, \end{aligned} \tag{3.3}$$

where $*$ denotes the convolution in t. It follows from (3.3) that

$$\begin{aligned} &\frac{d}{dt}|u_m(t)|^2 + 2\mu\|u_m(t)\|^2 \\ &\leq 2\int_\Omega |\nabla u_m(x,t)|_{\mathbb{R}}|g * \nabla u_m(x,t)|_{\mathbb{R}}\, dx + 2|f(t)|C\|u_m(t)\| \\ &= 2\int_\Omega |\nabla u_m(x,t)|_{\mathbb{R}}|g * \nabla u_m(x,t)|_{\mathbb{R}}\, dx + 2\sqrt{\frac{2}{3\mu}}C|f(t)|\sqrt{\frac{3\mu}{2}}\|u_m(t)\| \\ &= 2\int_\Omega |\nabla u_m(x,t)|_{\mathbb{R}}|g * \nabla u_m(x,t)|_{\mathbb{R}} dx + \frac{3\mu}{2}\|u_m(t)\|^2 + \frac{2}{3\mu}C^2|f(t)|^2. \end{aligned}$$

Remark 3.2. We note that from Cauchy-Schwarz inequality and Fubini's theorem we have

$$\|g * \nabla u_m\|_{L^2(Q)} \leq \|g\|_{L^1(0;\infty)}\|\nabla u_m\|_{L^2(Q)}.$$

Thus, integrating 0 to t the inequality above, using the Remark 3.2 and using Gronwall's inequality we obtain

$$|u_m|^2_{L^\infty(0,T;H)} + \Big(\frac{\mu}{2} - 2\|g\|_{L^1(0,\infty)}\Big)\|u_m\|^2_{L^2(0,T;V)} \leq \frac{2}{3\mu}C + C^2|f|^2_{L^2(0,T;H)}.$$

Integrating these last inequality in $t \in [0,T]$ and using (2.2), we have

$$u_m \quad \text{is bounded in } L^\infty(0,T;H) \tag{3.4}$$

$$u_m \quad \text{is bounded in } L^2(0,T;V). \tag{3.5}$$

From (3.5), we obtain

$$\beta(u_m) \quad \text{is bounded in } L^2(0,T;V') \tag{3.6}$$

Second estimate. By Remark 3.2, we observe that,

$$\text{if} \quad \xi \in L^2(0,T;H) \quad \text{then} \quad \int_0^t g(t-\sigma)\xi(\sigma)d\sigma \in L^2(0,T;H). \tag{3.7}$$

Similarly we obtain

$$\int_0^t g(t-\sigma)\xi(\sigma)d\sigma \in V \quad \text{if } \xi(t)\in V\,, \tag{3.8}$$

$$\int_0^t g(t-\sigma)\xi(\sigma)d\sigma \in V' \quad \text{if } \xi(t)\in V'\,. \tag{3.9}$$

We consider $\tilde{u}_m = u_m$, $\tilde{w} = w$ in $[0,T]$ and $\tilde{u}_m = 0$, $\tilde{w} = 0$ out of $[0,T]$, $\widetilde{g}(\xi) = g(\xi)$ if $\xi \geq 0$ and zero if $\xi < 0$. Therefore, $\nabla\widetilde{u}_m \in L^2(\mathbb{R};H)$, $\widetilde{w} \in L^2(\mathbb{R};V)$ and $\widetilde{g} \in L^1(\mathbb{R})$. This implies

$$\begin{aligned}
&\int_0^T\int_0^t g(t-\sigma)\left((u_m(\sigma),w(t))\right)\,d\sigma\,dt\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \widetilde{g}(t-\sigma)\int_\Omega \nabla\widetilde{u}_m(x,\sigma)\nabla\widetilde{w}(x,t)\,dx\,d\sigma\,dt\\
&= \int_{\mathbb{R}}\int_\Omega \widetilde{g}*\nabla\widetilde{u}_m(x,t)\nabla\widetilde{w}(x,t)\,dx\;dt\\
&= \int_{\mathbb{R}}\int_\Omega \nabla\widetilde{u}_m(x,\sigma)\breve{\widetilde{g}}*\nabla\widetilde{w}(x,\sigma)\,dx\,d\sigma,
\end{aligned}$$

where $\breve{\widetilde{g}}(x) = \widetilde{g}(-x)$. We observe that (3.5) implies that

$$\int_0^T ((u_m(t),w))dt \to \int_0^T ((u(t),w))dt,\;\; \forall\, w\in L^2(0,T;V). \tag{3.10}$$

From (3.7), we have that $\widetilde{g}*\widetilde{w}(t)\in V$, $\forall\, w\in L^2(0,T;V)$, therefore (3.10) yield

$$\int_{\mathbb{R}}\Big(\nabla\widetilde{u}_m(\sigma),\breve{\widetilde{g}}*\nabla\widetilde{w}(\sigma)\Big)\,dt \to \int_{\mathbb{R}}\Big(\nabla\widetilde{u}(\sigma),\breve{\widetilde{g}}*\nabla\widetilde{w}(\sigma)\Big)\,dt.$$

We note that

$$\begin{aligned}
\int_{\mathbb{R}}\Big(\nabla\widetilde{u}(\sigma),\breve{\widetilde{g}}*\nabla\widetilde{w}(\sigma)\Big)\;dt &= \int_{\mathbb{R}}\left(\widetilde{g}*\nabla\widetilde{u}(\sigma),\nabla\widetilde{w}(\sigma)\right)\;dt\\
&= \int_\Omega\int_{\mathbb{R}}\int_{\mathbb{R}} \widetilde{g}(t-\sigma)\nabla\widetilde{u}(x,\sigma)d\sigma\nabla\widetilde{w}(x,t)d\sigma\,dt\,dx\\
&= \int_0^T\int_0^t g(t-\sigma)\left((u(\sigma),w(t))\right)d\sigma dt\,.
\end{aligned}$$

Then

$$\int_0^T\int_0^t g(t-\sigma)\left((u_m(\sigma),w(t))\right)\,d\sigma\,dt \to \int_0^T\int_0^t g(t-\sigma)\left((u(\sigma),w(t))\right)d\sigma dt, \tag{3.11}$$

for all $w\in L^2(0,T;V)$.

Let P_m be the orthogonal projection $H\mapsto V_m$; that is,

$$P_m\varphi = \sum_{j=1}^m (\varphi,w_j)w_j,\quad \varphi\in H.$$

By the choice of $(w_\nu)_{\nu\in\mathbb{N}}$ we have

$$\|P_m\|_{\mathcal{L}(V,V)}\leq 1 \quad\text{and}\quad \|P_m^*\|_{\mathcal{L}(V',V')}\leq 1.$$

We note that $P_m u'_m = u'_m$. Multiplying both sides of the approximate equation (3.2) by the vector w_j and adding from $j = 1$ to $j = m$, we obtain using the notations and ideas of Lions [16, pages 75-76] and (3.7), that

$$(u'_m) \quad \text{is bounded in } L^2(0,T;V'). \tag{3.12}$$

The boundedness in (3.5), (3.12) and the Aubin-Lions compactness Theorem imply that there exists a subsequence from (u_m), still denoted by (u_m), such that

$$u_m \to u \quad \text{strongly in } L^2(0,T;H) \text{ and a. e. in } Q. \tag{3.13}$$

Returning to the notation u_{ϵ_m}, using (3.4), (3.5) and (3.13) (see Lions [16, pages 76-77]), (3.6) and (3.11) we obtain

$$\begin{gathered}(u'_\epsilon, v) + a(u_\epsilon, v) + b(u_\epsilon, u_\epsilon, v) - \int_0^t g(t-\sigma)(\Delta u(\sigma), v)d\sigma + \frac{1}{\epsilon}(\zeta, v) = (f, v),\\ \forall v \in V, \quad u_\epsilon \in L^2(0,T;V), \quad u'_\epsilon \in L^2(0,T;V')\\ u_\epsilon(x,0) = u_{\epsilon_0}(x).\end{gathered} \tag{3.14}$$

It is necessary to prove that $\zeta = \beta(u_\epsilon)$. We make this using the monotony of the operator β (see Lions [16, Chap. 2]). Therefore, we have proved the Theorem 2.4.

Proof of Theorem 2.2. From (3.4), (3.5), (3.13) and Banach-Steinhauss theorem, it follows that there exists a subnet $(u_\epsilon)_{0<\epsilon<1}$, such that it converges to u as $\epsilon \to 0$, in the weak sense . This function satisfies (2.6). On the other hand, we have from (3.14) that

$$\beta u_\epsilon = \epsilon[f - u'_\epsilon - Au_\epsilon - Bu_\epsilon - \int_0^t g(t-\sigma)\Delta u(\sigma)d\sigma]. \tag{3.15}$$

Where $\langle Au_\epsilon, v\rangle = a(u,v)$ and $\langle Bu_\epsilon, v\rangle = b(u_\epsilon, u_\epsilon, v)$.

Since $\int_0^t g(t-\sigma)\Delta u(\sigma)d\sigma \in V'$ and $[f - u'_\epsilon - Au_\epsilon - Bu_\epsilon]$ is bounded, we have

$$\beta u_\epsilon \to 0 \quad \text{in } \mathcal{D}'(0,T;V'). \tag{3.16}$$

Since βu_ϵ is bounded in $L^2(0,T;V')$, we have

$$\beta u_\epsilon \to 0 \ \text{ weak in } \ L^2(0,T;V'). \tag{3.17}$$

On the other hand we deduce from (3.14) that

$$0 \le \int_0^T \langle \beta u_\epsilon, u_\epsilon\rangle \, dt \le \epsilon\, C. \tag{3.18}$$

Thus $\int_0^T \langle \beta u_\epsilon, u_\epsilon\rangle dt \to 0$. We have that

$$\int_0^T \langle \beta u_\epsilon - \beta\varphi, u_\epsilon - \varphi\rangle \, dt \ge 0, \quad \forall \varphi \text{in } L^2(0,T;V),$$

because β is a monotonous operator. Thus,

$$\int_0^T \langle \beta u_\epsilon, u_\epsilon\rangle \, dt - \int_0^T \langle \beta u_\epsilon, \varphi\rangle \, dt - \int_0^T \langle \beta\varphi, u_\epsilon - \varphi\rangle \, dt \ge 0. \tag{3.19}$$

We have from (3.17) and (3.19) that

$$\int_0^T \langle \beta\varphi, u(t) - \varphi\rangle \, dt \le 0. \tag{3.20}$$

Taking $\varphi = u - \lambda v$, with $v \in L^2(0,T;V)$ and $\lambda > 0$, we deduce using the hemicontinuity of β that

$$\beta(u(t)) = 0, \tag{3.21}$$

and this implies that $u(t) \in K$ a. e.

Next, we prove that u is a solution of inequality (2.8). Let us consider $\mathbf{X}_\epsilon$ defined by

$$\begin{aligned} \mathbf{X}_\epsilon = &\int_0^T \langle \varphi', \varphi - u_\epsilon\rangle\,dt + \int_0^T a(u_\epsilon, \varphi - u_\epsilon)\,dt + \int_0^T b(u_\epsilon, u_\epsilon, \varphi - u_\epsilon)\,dt \\ &+ \int_0^T \int_0^t g(t-\sigma)((u_\epsilon(\sigma), \varphi - u_\epsilon))d\sigma\,dt, - \int_0^T \langle f, \varphi - u_\epsilon\rangle\,dt, \end{aligned} \tag{3.22}$$

with $\varphi \in L^2(0,T;V), \varphi' \in L^2(0,T;V'), \varphi(0) = 0$, $\varphi(t) \in K$ a.e. It follows from (3.22) that

$$\begin{aligned} \mathbf{X}_\epsilon = &\int_0^T \langle \varphi', \varphi\rangle\,dt - \int_0^T \langle \varphi', u_\epsilon\rangle\,dt + \int_0^T a(u_\epsilon, \varphi)\,dt - \int_0^T a(u_\epsilon, u_\epsilon)\,dt \\ &+ \int_0^T b(u_\epsilon, u_\epsilon, \varphi)\,dt - \int_0^T b(u_\epsilon, u_\epsilon, u_\epsilon)dt + \int_0^T \int_0^t g(t-\sigma)((u_\epsilon(\sigma), \varphi))d\sigma\,dt \\ &- \int_0^T \int_0^t g(t-\sigma)((u_\epsilon(\sigma), u_\epsilon))d\sigma\,dt - \int_0^T \langle f, \varphi\rangle\,dt + \int_0^T \langle f, u_\epsilon\rangle\,dt. \end{aligned} \tag{3.23}$$

On the other hand, taking $v = \varphi - u_\epsilon$ in (2.16) and integrating in Q_T, we obtain that

$$\begin{aligned} &- \int_0^T \langle u_\epsilon', \varphi\rangle\,dt + \int_0^T \langle u_\epsilon', u_\epsilon\rangle\,dt - \int_0^T a(u_\epsilon, \varphi)\,dt + \int_0^T a(u_\epsilon, u_\epsilon)\,dt \\ &- \int_0^T b(u_\epsilon, u_\epsilon, \varphi)\,dt + \int_0^T b(u_\epsilon, u_\epsilon, u_\epsilon)dt - \int_0^T \int_0^t g(t-\sigma)((u_\epsilon(\sigma), \varphi))d\sigma\,dt \\ &+ \int_0^T \int_0^t g(t-\sigma)((u_\epsilon(\sigma), u_\epsilon))d\sigma\,dt - \frac{1}{\epsilon}\int_0^T \langle \beta u_\epsilon - \beta\varphi, \varphi - u_\epsilon\rangle\,dt \\ &+ \int_0^T \langle f, \varphi\rangle\,dt - \int_0^T \langle f, u_\epsilon\rangle\,dt = 0, \end{aligned} \tag{3.24}$$

because $\beta\varphi = 0$. Adding member to member (3.23) and (3.24), we obtain

$$\begin{aligned} \mathbf{X}_\epsilon = &\int_0^T \langle \varphi', \varphi\rangle\,dt - \int_0^T \langle \varphi', u_\epsilon\rangle\,dt - \int_0^T \langle u_\epsilon', \varphi\rangle\,dt \\ &+ \int_0^T \langle u_\epsilon', u_\epsilon\rangle\,dt + \frac{1}{\epsilon}\int_0^T \langle \beta\varphi - \beta u_\epsilon, \varphi - u_\epsilon\rangle\,dt \geq 0, \end{aligned} \tag{3.25}$$

because

$$\begin{aligned} &\int_0^T \langle \varphi', \varphi\rangle\,dt - \int_0^T \langle \varphi', u_\epsilon\rangle\,dt - \int_0^T \langle u_\epsilon', \varphi\rangle\,dt + \int_0^T \langle u_\epsilon', u_\epsilon\rangle\,dt \\ &= \int_0^T \langle \varphi' - u_\epsilon', \varphi - u_\epsilon\rangle \geq 0. \end{aligned}$$

On the other hand, $b(u_\epsilon, u_\epsilon, u_\epsilon) = 0$. From (3.22)-(3.23) it follows that

$$\begin{aligned} \mathbf{X}_\epsilon &= \int_0^T \langle \varphi', \varphi - u_\epsilon \rangle \, dt + \int_0^T a(u_\epsilon, \varphi) \, dt \\ &\quad + \int_0^T \int_0^t g(t-\sigma)((u_\epsilon, \varphi)) d\sigma \, dt - \int_0^T \langle f, \varphi - u_\epsilon \rangle \, dt \\ &\geq \int_0^T a(u_\epsilon, u_\epsilon) dt + \int_0^T b(u_\epsilon, \varphi, u_\epsilon) \, dt + \int_0^T \int_0^t g(t-\sigma)((u_\epsilon, u_\epsilon)) d\sigma \, dt. \end{aligned} \tag{3.26}$$

Consider

$$\mathbf{Y}_\epsilon = \int_0^T a(u_\epsilon, u_\epsilon) dt + \int_0^T b(u_\epsilon, \varphi, u_\epsilon) \, dt + \int_0^T \int_0^t g(t-\sigma)((u_\epsilon, u_\epsilon)) d\sigma \, dt.. \tag{3.27}$$

It follows from (2.5) with $v = u - u_\epsilon$ that

$$a(u - u_\epsilon, u - u_\epsilon) + b(u - u_\epsilon, \varphi, u - u_\epsilon) + \int_0^t g(t-\sigma)((u - u_\epsilon, u - u_\epsilon)) d\sigma \geq 0.$$

On the other hand, we can write

$$\begin{aligned} \mathbf{Y}_\epsilon &= \int_0^T a(u_\epsilon - u, u_\epsilon - u) \, dt + \int_0^T b(u_\epsilon - u, \varphi, u_\epsilon - u) \, dt \\ &\quad + \int_0^T a(u, u_\epsilon - u) \, dt + \int_0^T a(u_\epsilon, u) \, dt + \int_0^T b(u, \varphi, u_\epsilon - u) \, dt \\ &\quad + \int_0^T b(u_\epsilon, \varphi, u) \, dt + \int_0^T \int_0^t g(t-\sigma)((u_\epsilon - u, u_\epsilon - u)) d\sigma \, dt \\ &\quad + \int_0^T \int_0^t g(t-\sigma)((u, u_\epsilon - u)) d\sigma \, dt + \int_0^T \int_0^t g(t-\sigma)((u_\epsilon, u)) d\sigma \, dt. \end{aligned}$$

This implies

$$\begin{aligned} \mathbf{Y}_\epsilon &\geq \int_0^T a(u_\epsilon, u) \, dt + \int_0^T a(u, u_\epsilon - u) \, dt + \int_0^T b(u, \varphi, u_\epsilon - u) \, dt \\ &\quad + \int_0^T b(u_\epsilon, \varphi, u) \, dt + \int_0^T \int_0^t g(t-\sigma)((u, u_\epsilon - u)) d\sigma \, dt \\ &\quad + \int_0^T \int_0^t g(t-\sigma)((u_\epsilon, u)) d\sigma \, dt. \end{aligned} \tag{3.28}$$

Taking lim sup in (3.28) we obtain

$$\limsup \mathbf{Y}_\epsilon \geq \int_0^T a(u, u) \, dt + \int_0^T b(u, \varphi, u) \, dt + \int_0^T \int_0^t g(t-\sigma)((u, u)) d\sigma \, dt. \tag{3.29}$$

It follows from (3.26) and (3.29) that

$$\begin{aligned} &\limsup \Big\{ \int_0^T \langle \varphi', \varphi - u_\epsilon \rangle \, dt + \int_0^T a(u_\epsilon, \varphi) \, dt \\ &+ \int_0^T \int_0^t g(t-\sigma)((u_\epsilon, u)) d\sigma \, dt - \int_0^T \langle f, \varphi - u_\epsilon \rangle \, dt \Big\} \\ &\geq \int_0^T a(u, u) \, dt + \int_0^T b(u, \varphi, u) \, dt + \int_0^T \int_0^t g(t-\sigma)((u, u)) d\sigma \, dt. \end{aligned} \tag{3.30}$$

It follows from (3.30) that

$$\begin{aligned}
&\int_0^T \langle \varphi', \varphi - u\rangle\,dt + \int_0^T a(u, \varphi - u)\,dt + \int_0^T b(u, u, \varphi - u)\,dt \\
&+ \int_0^T \int_0^t g(t-\sigma)((u, \varphi - u))d\sigma\,dt \\
&\geq \int_0^T \langle f, \varphi - u\rangle\,dt
\end{aligned}$$

for all $\varphi \in L^2(0,T;V)$, $\varphi' \in L^2(0,T;V')$, $\varphi(0) = 0$, $\varphi(t) \in K$ a.e.

Proof of Theorem 2.3. We first prove Theorem 2.5 for the penalized problem. As in the proof of Theorem 2.2, we employ the Faedo-Galerkin Method. Let $(w_\nu)_{\nu\in\mathbb{N}}$ be a Hilbertian basis of V. By $V_m = [w_1, w_2, \dots w_m]$ we represent the subspace generated by the m first vectors of (w_ν). Consider

$$u_{\epsilon_m} = \sum_{j=1}^{m} g_{j\,m} w_j$$

solution of approximate penalized problem

$$\begin{aligned}
&(u'_{\epsilon_m}, w_j) + \mu a(u_{\epsilon_m}, w_j) + b(u_{\epsilon_m}, u_{\epsilon_m}, w_j) \\
&\quad - \int_0^t g(t-\sigma)(\Delta u_{\epsilon m}(\sigma), v)d\sigma + \frac{1}{\epsilon}\langle \beta u_{\epsilon_m}, w_j\rangle \\
&= \langle f(t), w_j\rangle, \quad j = 1, 2, \dots m \\
&\qquad u_{\epsilon_m}(x,0) \to u_\epsilon(x,0) \quad \text{strongly in } V.
\end{aligned} \tag{3.31}$$

First estimate. As in the proof of Theorem 2.4, omitting the parameter ϵ and taking $v = u_m$ in the approximate equation (3.31) we obtain

$$(u_m) \quad \text{is bounded in } L^\infty(0,T;H), \tag{3.32}$$

$$(u_m) \quad \text{is bounded in } L^2(0,T;V), \tag{3.33}$$

Second estimate. In both sides of (3.31) we take the derivatives with respect t and consider $v = u'_m(t)$. We obtain

$$\begin{aligned}
&(u''_m(t), u'_m(t)) + \mu a(u'_m(t), u'_m(t)) \\
&+ b(u'_m(t), u_m(t), u'_m(t)) + b(u_m(t), u'_m(t), u'_m(t)) \\
&+ \frac{1}{\epsilon}((\beta u_m(t))', u'_m(t)) + \int_0^t g'(t-\sigma)((u_m(t), u'_m))d\sigma \\
&+ g(0)((u_m(t), u'_m(t))) + \frac{1}{\epsilon}\left((\beta u_m)'(t), u'_m(t)\right) \\
&= (f'(t), u'_m(t)),
\end{aligned} \tag{3.34}$$

because

$$\frac{d}{dt}\Big(\int_0^t g(t-\sigma)\Delta u_m(\sigma)\,d\sigma\Big) = g(0)\Delta u_m(t) + \int_0^t g'(t-\sigma)\Delta u_m(\sigma)\,d\sigma.$$

We note that

$$\begin{aligned}
u'_m(0) &\to u_1 \quad \text{strongly in } H, \\
u_m(0) &\to u_0 \quad \text{strongly in } V.
\end{aligned} \tag{3.35}$$

Indeed, $(3.35)_1$ is obtained using (3.31) with $t=0$ and (2.11). Note that $\beta(u_0)=0$. Then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_m'(t)|^2+\mu\|u_m'(t)\|^2+b(u_m'(t),u_m(t),u_m'(t))\\
&+\int_0^t g'(t-\sigma)((u_m(t),u_m'))d\sigma+g(0)\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2\\
&=(f'(t),u_m'(t)),
\end{aligned}\tag{3.36}$$

because $b(u_m(t),u_m'(t),u_m'(t))=0$ and $((\beta u_m)'(t),u_m'(t))\geq 0$ (see Lions [16, page 399]).

Remark 3.3. The derivative with respect to t of $(\beta(v(t)),w)$ is only formal. The correct method is to consider the difference equation in $t+h$ and t, divided by h and take the limits when $h\to 0$. Here is fundamental the operator β to be monotonous. This justify the formal procedure of taking the derivative with respect to t, on both sides of (3.31) and take $v=u_m'(t)$. See Brezis [6], Browder [8] or Lions [17] for details.

As $n=2$, we have (see Lions [16, page 70])

$$\|u\|^2_{L^4(\Omega)}\leq C\|u\||u|,\quad \forall\, u\in H_0^1(\Omega).\tag{3.37}$$

Moreover, $H_0^1(\Omega)\hookrightarrow L^2(\Omega)$; therefore,

$$|b(u,v,u)|\leq\sum_{i,j=1}^{2}\int_\Omega|u_i(x)|\Big|\frac{\partial v_j}{\partial x_i}(x)\Big||u_j(x)|\leq\|u\|_{(L^4(\Omega))^2}\|v\|.$$

This and (3.37) imply

$$\begin{aligned}
|b(u_m'(t),u_m(t),u_m'(t))|&\leq\sqrt{\frac{\mu}{2}}\|u_m'(t)\|C\sqrt{\frac{2}{\mu}}|u_m'(t)|\|u_m(t)\|\\
&\leq\frac{\mu}{4}\|u_m'(t)\|^2+\frac{C^2}{\mu}\|u_m(t)\|^2|u_m'(t)|^2.
\end{aligned}\tag{3.38}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_m'(t)|^2+\mu\|u_m'(t)\|^2+g(0)\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2\\
&\leq\Big|\int_0^t g'(t-\sigma)((u_m(t),u_m'))d\sigma\Big|_{\mathbb{R}}+|b(u_m'(t),u_m(t),u_m'(t))|_{\mathbb{R}}\\
&\quad+\sqrt{\frac{2}{\mu}}\|f'(t)\|_{V'}\sqrt{\frac{\mu}{2}}\|u_m'(t)\|,
\end{aligned}\tag{3.39}$$

Therefore, from (3.39), (3.37) and Remark 3.2 we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_m'(t)|^2+g(0)\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2+\mu\|u_m'(t)\|^2\\
&\leq\frac{\mu}{4}\|u_m'(t)\|^2+\frac{C^2}{2\mu}\|u_m(t)\||u_m'(t)|^2\\
&\quad+\|g'\|_{L^1(0,\infty)}\|u_m'(t)\|\|u_m(t)\|+\frac{C_1}{\mu}\|f'(t)\|^2+\frac{\mu}{4}\|u_m'(t)\|^2,
\end{aligned}\tag{3.40}$$

Integrating (3.40) from 0 to t and using the hypothesis (2.3), (2.4) we obtain

$$\begin{aligned}|u_m'(t)|^2 &+ \Big(\frac{\mu}{2} - 2\|g\|_{L^1(0,\infty)}\Big)\int_0^t \|u_m'(s)\|^2 ds\\ &\le C_2^2\|g\|_{L^1(0,\infty)}\int_0^T \|u_m(t)\|^2\,dt + C\int_0^t \|u_m(s)\|^2|u_m'(s)|^2 ds + C\int_0^T \|f'(t)\|^2\,dt.\end{aligned} \tag{3.41}$$

From (3.4) and hypothesis on f we obtain

$$|u_m'(t)|^2 + \Big(\frac{\mu}{2} - 2\|g\|_{L^1(0,\infty)}\Big)\int_0^t \|u_m'(s)\|^2 ds \le C + C\int_0^t \|u_m(s)\|^2|u_m'(s)|^2 ds. \tag{3.42}$$

Being (u_m) is bounded in $L^2(0,T;V)$ we have, using Gronwall's inequality in (3.41) and hypothesis $H1$, that

$$(u_m') \quad \text{is bounded in } L^2(0,T;V) \tag{3.43}$$

$$(u_m') \quad \text{is bounded in } L^\infty(0,T;H). \tag{3.44}$$

Third estimate. Let (w_ν) be the orthonormal system of $V \cap V_2$ formed by the eigenfunctions of the Laplace operator.

As in the proof of Theorem 2.4, omitting the parameter ϵ and taking $w_j = -\Delta u_m$ in the approximate equation (3.31) we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2 + \mu|\Delta u_m(t)|^2\\ &\le |b(u_m(t), u_m(t), -\Delta u_m(t))|_{\mathbb{R}}\\ &\quad + \Big|\int_\Omega \Delta u_m(x,t)\Big(\int_0^t g(t-\sigma)\Delta u_m(x,\sigma)d\sigma\Big)\Big|_{\mathbb{R}} dx\\ &\quad + \frac{1}{\mu}|f(t)|^2 + \frac{\mu}{4}|\Delta u_m(t)|^2\end{aligned} \tag{3.45}$$

because $\langle \beta u_m, -\Delta u_m\rangle \ge 0$ (see Haraux [4, page 58]). We note that

$$\begin{aligned}|b(u_m(t), u_m(t), -\Delta u_m(t)| &\le \sum_{i,j=1}^{2}\int |u_{m_j}(t)|\frac{\partial u_{m_j}}{\partial x_i}(t)||\Delta u_{m_j}(t)|\\ &\le \|u_m(t)\|^2_{(L^3(\Omega))^2}\Big\|\frac{\partial u_m}{\partial x_i}(t)\Big\||\Delta u_m(t)|,\end{aligned} \tag{3.46}$$

because $H_0^1(\Omega) \hookrightarrow L^3(\Omega)$, $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$, with $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$.

Substituting (3.46) in (3.45) and using the Remark 3.2, we obtain

$$\begin{aligned}&\frac{d}{dt}\|u_m(t)\|^2 + \Big(\frac{\mu}{2} - 2\|g\|_{L^1(0,\infty)}\Big)|\Delta u_m(t)|^2\\ &\le C\|f(t)\|^2 + \Big(C\|u_m(t)\|^2_{(L^3(\Omega))^2} - C\Big)\|u_m(t)\|^2.\end{aligned}$$

Integrating the above inequality from 0 to t, observing that $u_m \in L^2(0,T;V) \subset L^2(0,T;(L^3(\Omega))^2$ and using the Gronwall's Lemma, it follows that

$$u_m \quad \text{is bounded in } L^\infty(0,T;V) \tag{3.47}$$

$$u_m \quad \text{is bounded in } L^2(0,T;V_2). \tag{3.48}$$

To complete the proof of Theorem 2.5, we use the same argument used in the proof of Theorem 2.4.

We shall now prove Theorem 2.3. From the previous convergence, and Banach-Steinhauss theorem, it follows that there exists a subnet $(u_\epsilon)_{0<\epsilon<1}$, such that it converges to u as $\epsilon \to 0$, in the sense of previous convergence.

This function satisfies (2.12) and (2.13). Using the same arguments used in Theorem 2.2 we obtain that $\beta u = 0$. Therefore, u satisfy (2.15) of Theorem 2.3.

We need to show only that u is a solution of inequality (2.14) a.e. in t. In fact, we have that u_ϵ satisfies

$$\begin{gathered} (u'_\epsilon, \widetilde{v}) + \mu a(u_\epsilon, \widetilde{v}) + b(u_\epsilon, u_\epsilon, \widetilde{v}) + \int_0^t g(t-\sigma)((u_\epsilon, \widetilde{v}))d\sigma + \frac{1}{\epsilon}(\beta u_\epsilon, \widetilde{v}) = (f, \widetilde{v}), \\ u_\epsilon(0) = u_0. \end{gathered} \tag{3.49}$$

for all $\widetilde{v} \in V$. Then from (3.49), with $\widetilde{v} = v - u_\epsilon$, $v \in K$, we have

$$\begin{aligned} &(u'_\epsilon, v - u_\epsilon) + \mu a(u_\epsilon, v) + b(u_\epsilon, u_\epsilon, v) + \int_0^t g(t-\sigma)((u_\epsilon, v))d\sigma - (f, v - u_\epsilon) \\ &\geq \mu a(u_\epsilon, u_\epsilon) + \int_0^t g(t-\sigma)((u_\epsilon, u_\epsilon))d\sigma, \quad \forall v \in K, \end{aligned} \tag{3.50}$$

because $(\beta u_\epsilon - \beta v, u_\epsilon - v) \geq 0$. Let us denote

$$X_\epsilon^v = (u'_\epsilon, v - u_\epsilon) + \mu a(u_\epsilon, v) + b(u_\epsilon, u_\epsilon, v) + \int_0^t g(t-\sigma)((u_\epsilon, v))d\sigma - (f, v - u_\epsilon).$$

We obtain

$$X_\epsilon^v \geq \mu a(u_\epsilon, u_\epsilon) + \int_0^t g(t-\sigma)((u_\epsilon, u_\epsilon))d\sigma, \quad \forall v \in V. \tag{3.51}$$

Let $\psi \in C^0([0,T])$ with $\psi(t) \geq 0$. Then $v\psi \in C^0([0,T]; V)$ for all $v \in V$.

$$u_{\epsilon i} u_{\epsilon j} \to u_i u_j \text{ weakly in } L^2(0, T, L^2(\Omega))$$

It follows from (3.51) that

$$\begin{aligned} &\int_0^T \psi(u'_\epsilon, v - u_\epsilon)\,dt + \mu \int_0^T \psi a(u_\epsilon, v)\,dt + \int_0^T \psi b(u_\epsilon, u_\epsilon, v)\,dt \\ &+ \psi \int_0^T \int_0^t g(t-\sigma)((u_\epsilon, v))d\sigma\,dt - \int_0^T \psi(f, v - u_\epsilon)\,dt \\ &\geq \mu \int_0^T \psi a(u_\epsilon, u_\epsilon)\,dt + \int_0^t g(t-\sigma)((u_\epsilon, u_\epsilon))d\sigma. \end{aligned} \tag{3.52}$$

Taking lim sup in both side of inequality (3.52) we obtain

$$\begin{aligned} &\int_0^T \psi(u', v - u)\,dt + \mu \int_0^T \psi a(u, v)\,dt - \int_0^T \psi b(u, u, v)\,dt \\ &+ \int_0^T \int_0^t g(t-\sigma)((u, v))d\sigma\,dt - \int_0^T \psi(f, v - u)\,dt \\ &\geq \mu \int_0^T \psi a(u, u)\,dt + \int_0^T \int_0^t g(t-\sigma)((u, u))d\sigma\,dt, \end{aligned} \tag{3.53}$$

because

$$\limsup \mu \int_0^T \psi a(u_\epsilon, u_\epsilon)\,dt \geq \liminf \mu \int_0^T \psi a(u_\epsilon, u_\epsilon)\,dt \geq \mu \int_0^T \psi a(u, u)\,dt$$

and

$$\begin{aligned}\limsup \int_0^T \int_0^t g(t-\sigma)((u\epsilon, u_\epsilon))d\sigma\, dt &= \limsup \int_0^T \int_0^t g(t-\sigma)(-\Delta u_\epsilon, u_\epsilon)d\sigma\, dt \\ &= \int_0^T \int_0^t g(t-\sigma)(-\Delta u, u)d\sigma\, dt \\ &= \int_0^T \int_0^t g(t-\sigma)((u,u))d\sigma\, dt\end{aligned}$$

From (3.53) we obtain finally

$$\begin{aligned}&(u', v-u) + \mu a(u, v-u) + b(u,u,v-u) + \int_0^t g(t-\sigma)((u, v-u))d\sigma \\ &\geq (f, v-u) \quad \forall v \in K, \text{ a.e. in } t.\end{aligned} \tag{3.54}$$

4. Uniqueness

We now prove that when $n = 2$ we have uniqueness in Theorem 2.3. Indeed, suppose that u_1, u_2 are two solutions of (2.14) and set $w = u_2 - u_1$ and $t \in (0,T)$. Taking $v = u_1$ (resp. u_2) in the inequality (2.14) relative to v_2 (resp. v_1) and adding up the results we obtain

$$\begin{aligned}&-\int_0^t (w', w)\, dt - \mu \int_0^t a(w,w)\, dt + \int_0^t b(u_1, u_1, w)\, dt \\ &- \int_0^t b(u_2, u_2, w)\, dt - \int_0^t \int_0^t g(t-\sigma)((w,w)) \geq 0.\end{aligned}$$

Therefore,

$$\frac{1}{2}\int_0^t \frac{d}{dt}|w(t)|^2\, dt + \mu \int_0^t \|w(t)\|^2\, dt \leq \int_0^t |b(w, u_2, w)|\, dt, \tag{4.1}$$

because $\int_0^t \int_0^t g(t-\sigma)((w,w)) \geq 0$ and $b(u_2,u_2,w) - b(u_1,u_1,w) = b(w,u_2,w)$. On the other hand, if $n = 2$, we have (see Lions [16, page 70])

$$|b(w(t), u_2(t), w(t))| \leq C\|w(t)\||w(t)|\|u_2(t)\|. \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$|w(t)|^2 + \frac{\mu}{2}\int_0^t \|w(t)\|^2\, dt \leq C\int_0^t |w(t)|^2 \|u_2(t)\|^2\, dt.$$

This implies, using Gronwall's inequality that $w = 0$, because $u_2 \in L^2(0,T;V)$, therefore $u_1(t) = u_2(t)$, for all $t \in [0,T]$.

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