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OBLIQUE DERIVATIVE PROBLEMS FOR GENERALIZED RASSIAS EQUATIONS OF MIXED TYPE WITH SEVERAL CHARACTERISTIC BOUNDARIES

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Abstract. This article concerns the oblique derivative problems for second-order quasilinear degenerate equations of mixed type with several characteristic boundaries, which include the Tricomi problem as a special case. First we formulate the problem and obtain estimates of its solutions, then we show the existence of solutions by the successive iterations and the Leray-Schauder theorem. We use a complex analytic method: elliptic complex functions are used in the elliptic domain, and hyperbolic complex functions in the hyperbolic domain, such that second-order equations of mixed type with degenerate curve are reduced to the first order mixed complex equations with singular coefficients. An application of the complex analytic method, solves (1.1) below with $m = n = 1$, $a = b = 0$, which was posed as an open problem by Rassias.

1. Formulation of oblique derivative problems

Tricomi problems for second-order equations of mixed type with parabolic degenerate lines possess important applications to gas dynamics, and have been discussed in [1]-[15], [19, 20]. In this article, we generalize those results to second-order equations of mixed type with parabolic degeneracy and several characteristic boundaries.

Let D be a simply connected bounded domain in the complex plane $\mathbb{C}$ with the boundary $\partial D = \Gamma \cup L$, where $\Gamma \subset \{\hat{y} = y - x^n > 0\}$ and is an element in C^2_μ with $0 < \mu < 1$ and with end points $z_* = -R - iR^n, z^* = R + iR^n$; and $L = L_1 \cup L_2 \cup L_3 \cup \cdots \cup L_{2N}$, where N is an odd positive integer, and for $l = 1, \dots, N$,

$$L_{2l-1} = \big\{x + \int_0^{y-x^n} \sqrt{|K(t)|}dt = a_{l-1},\ x \in [a_{l-1}, a_l]\big\},$$
$$L_{2l} = \big\{x - \int_0^{y-x^n} \sqrt{|K(t)|}dt = a_l,\ x \in [a_{l-1}, a_l]\big\}\,.$$

Herein $-R = a_0 < a_1 < \cdots < a_{N-1} < a_N = R$, $K(y - x^n) = \operatorname{sgn}(y - x^n)|y - x^n|^m$, R, m are positive constants, denote $D^+ = D \cap \{y - x^n > 0\}$, $D^- = D \cap \{y - x^n < 0\}$, and $G(y - x^n) = \int_0^{y-x^n} \sqrt{|K(t)|}dt$. Without loss of generality, we may assume that the boundary Γ possesses the form $x = -R + \tilde{G}(\hat{y})$ and $x = R - \tilde{G}(\hat{y})$ near z_* and

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z^* with the condition $d\tilde{G}(\hat{y})/d\hat{y} = \pm\tilde{H}(\hat{y}) = 0$ at $z = z_*, z^*$ respectively. Otherwise through a conformal mapping as stated in [20], this requirement can be realized. In this paper, we use the hyperbolic unit j with the condition $j^2 = 1$ in $\overline{D^-}$, and $x + jy$, $W(z) = U(z) + jV(z) = [H(\hat{y})u_x - ju_y]/2$ are called the hyperbolic number and hyperbolic complex function in D^-, and $x + iy$, $W(z) = U(z) + iV(z) = [H(\hat{y})u_x - iu_y]/2$ are called the complex number and elliptic complex function in $\overline{D^+}$ respectively (see [16]). Consider generalized Rassias equation of mixed type with parabolic degeneracy

$$K(y - x^n)u_{xx} + u_{yy} + au_x + bu_y + cu + d = 0 \quad \text{in } D, \tag{1.1}$$

where $\hat{y} = y - x^n$, a, b, c, d are real functions of $z \in \overline{D}$, $u, u_x, u_y \in \mathbb{R}$, and suppose that (1.1) satisfies the following conditions,

(C1) For continuously differentiable functions $u(z)$ in $D^* = \overline{D}\backslash\{\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_N\}$, the coefficients a, b, c, d satisfy

$$\begin{gathered}
\tilde{L}_\infty[\eta, D^+] = L_\infty[\eta, D^+] + L_\infty[\eta_x, D^+] \le k_0, \quad \eta = a, b, c, \\
\tilde{L}_\infty[d, D^+] \le k_1, \quad \tilde{C}[d, \overline{D^-}] = C[d, \overline{D^-}] + C[d_x, \overline{D-}] \le k_1, \\
\tilde{C}[\eta, \overline{D^-}] \le k_0, \quad \eta = a, b, c, \\
c \le 0 \quad \text{in } D^+, \\
|a(x, y)||\hat{y}|^{1-m/2} = \varepsilon_1(\hat{y}) \quad \text{as } \hat{y} \to 0, m \ge 2, z \in \overline{D^-},
\end{gathered} \tag{1.2}$$

where $\tilde{a}_l = a_l + ia_l^n, l = 0, 1, \dots, N$, $\hat{y} = y - x^n$, and $\varepsilon_1(\hat{y})$ is a non-negative function such that $\varepsilon_1(\hat{y}) \to 0$ as $\hat{y} \to 0$.

(C2) For any continuously differentiable functions $u_1(z), u_2(z)$ in D^*, the function $F(z, u, u_z) = au_x + bu_y + cu + d$ satisfies

$$\begin{aligned}
&F(z, u_1, u_{1z}) - F(z, u_2, u_{2z}) \\
&= \tilde{a}(u_1 - u_2)_x + \tilde{b}(u_1 - u_2)_y + \tilde{c}(u_1 - u_2) \quad \text{in } D,
\end{aligned} \tag{1.3}$$

in which $\tilde{a}, \tilde{b}, \tilde{c}$ satisfy the same conditions as those of a, b, c in (1.2), and k_0, k_1 are positive constants such that $k_0 \ge 2$, $k_1 \ge \max[1, 6k_0]$.

To write the complex form of the above equation, denote

$$\begin{aligned}
W(z) = U + iV &= \frac{1}{2}[H(y - x^n)u_x - iu_y] \\
&= u_{\tilde{z}} = \frac{H(y - x^n)}{2}[u_x - iu_Y] = H(y - x^n)u_Z, \\
H(y - x^n)W_{\overline{Z}} &= \frac{H(y - x^n)}{2}[W_x + iW_Y] \\
&= \frac{1}{2}[H(y - x^n)W_x + iW_y] = W_{\overline{\tilde{z}}} \quad \text{in } \overline{D^+},
\end{aligned}$$

where $Z(z) = x + iY = x + iG(\hat{y})$, $\hat{y} = y - x^n$ in $\overline{D^+}$. We have

$$\begin{aligned}
&K(y - x^n)u_{xx} + u_{yy} \\
&= H(y - x^n)[H(y - x^n)u_x - iu_y]_x + i[H(y - x^n)u_x - iu_y]_y - [iH_y + HH_x]u_x \\
&= 2\{H[U + iV]_x + i[U + iV]_y\} - [iH_y/H + H_x]Hu_x \\
&= 4H(y - x^n)W_{\overline{Z}} - [iH_y/H + H_x]Hu_x = -[au_x + bu_y + cu + d];
\end{aligned}$$

i.e.,

$$\begin{aligned} &H(y-x^n)W_{\overline{Z}}\\ &=H[W_x+iW_Y]/2\\ &=H[(U+iV)_x+i(U+iV)_Y]/2\\ &=\{(iH_y/H+H_x-a/H)Hu_x-bu_y-cu-d\}/4\\ &=\{(iH_y/H+H_x-a/H)(W+\overline{W})+ib(\overline{W}-W)-cu-d\}/4\\ &=A_1(z,u,W)W+A_2(z,u,W)\overline{W}+A_3(z,u,W)u+A_4(z,u,W)\\ &=g(Z)\quad\text{in } D_Z^+, \end{aligned} \tag{1.4}$$

in which $D_Z^+=D_Z$ is the image domains of D^+ with respect to the mapping $Z=Z(z)$. Moreover denote

$$\begin{gathered} W(z)=U+jV=\frac{1}{2}[H(y-x^n)u_x-ju_y]\\ =\frac{H(y-x^n)}{2}[u_x-ju_Y]=H(y-x^n)u_Z,\\ H(y-x^n)W_{\overline{Z}}=\frac{H(y-x^n)}{2}[W_x+jW_Y]=\frac{1}{2}[H(y-x^n)W_x+jW_y]=W_{\overline{z}}\quad\text{in } \overline{D^-}, \end{gathered}$$

in which $Z(z)=x+jY=x+jG(\hat{y})$, $\hat{y}=y-x^n$ in $\overline{D^-}$. Then we obtain

$$\begin{aligned} &-K(y-x^n)u_{xx}-u_{yy}\\ &=H(y-x^n)[H(y-x^n)u_x-ju_y]_x+j[H(y-x^n)u_x-ju_y]_y-[jH_y+HH_x]u_x\\ &=2\{H[U+jV]_x+j[U+jV]_y\}-[jH_y/H+H_x]Hu_x\\ &=4H(y-x^n)W_{\overline{Z}}-[jH_y/H+H_x]Hu_x\\ &=au_x+bu_y+cu+d, H(y-x^n)W_{\overline{Z}}\\ &=H[(U+jV)_x+j(U+jV)_Y]/2\\ &=\{(jH_y/H+H_x)Hu_x+au_x+bu_y+cu+d\}/4\\ &=\{(jH_y/H+H_x+a/H)(W+\overline{W})+jb(\overline{W}-W)+cu+d\}/4\\ &=H\{e_1[U_x+V_Y+V_x+U_Y]/2+e_2[U_x+V_Y-V_x-U_Y]/2\}\\ &=H\{e_1[(U+V)_x+(U+V)_Y]/2+e_2[(U-V)_x-(U-V)_Y]/2\}\\ &=H[e_1(U+V)_\mu+e_2(U-V)_\nu]\\ &=\frac{1}{4}\{(e_1-e_2)[H_y/H]Hu_x+(e_1+e_2)[(H_x+a/H)Hu_x+bu_y+cu+d]\}, \end{aligned}$$

and in D^-, we have

$$\begin{aligned} (U+V)_\mu&=\frac{1}{4H}\{2[H_y/H+H_x+a/H]U-2bV+cu+d\},\\ (U-V)_\nu&=\frac{1}{4H}\{-2[H_y/H-H_x-a/H]U-2bV+cu+d\}, \end{aligned} \tag{1.5}$$

where $e_1=(1+j)/2$, $e_2=(1-j)/2$, $2x=\mu+\nu$, $2Y=\mu-\nu$, $\partial x/\partial\mu=1/2=\partial Y/\partial\mu$ $\partial x/\partial\nu=1/2=-\partial Y/\partial\nu$. Hence the complex form of (1.1) can be written as

$$W_{\overline{z}}=A_1W+A_2\overline{W}+A_3u+A_4\quad\text{in } \overline{D},$$

$$u(z)=\begin{cases}2\operatorname{Re}\int_{z_*}^{z}[\frac{U(z)}{H(y-x^n)}+iV(z)]dz+c_0 & \text{in } \overline{D^+},\\ 2\operatorname{Re}\int_{z_*}^{z}[\frac{U(z)}{H(y-x^n)}-jV(z)]dz+c_0 & \text{in } \overline{D^-},\end{cases} \tag{1.6}$$

where $c_0=u(z_*)$, and the coefficients $A_l=A_l(z,u,W)$ are as follows

$$\begin{gathered}A_1=\begin{cases}\frac{1}{4}[-\frac{a}{H}+\frac{iH_y}{H}+H_x-ib],\\ \frac{1}{4}[\frac{a}{H}+\frac{jH_y}{H}+H_x-jb],\end{cases}\quad A_2=\begin{cases}\frac{1}{4}[-\frac{a}{H}+\frac{iH_y}{H}+H_x+ib],\\ \frac{1}{4}[\frac{a}{H}+\frac{jH_y}{H}+H_x+jb],\end{cases}\\ A_3=\begin{cases}-\frac{c}{4},\\ \frac{c}{4},\end{cases}\qquad A_4=\begin{cases}-\frac{d}{4} & \text{in } \overline{D^+},\\ \frac{d}{4} & \text{in } \overline{D^-}.\end{cases}\end{gathered} \tag{1.7}$$

For convenience, sometimes $\tilde a_l=a_l+ia_l^n$ $(l=0,1,\dots,N)$ in the $z=x+iy$-plane are replaced by $\hat t_1=a_0$, $\hat t_l=a_{l-2}$ $(l=3,\dots,N+1)$, $\hat t_2=a_N$ in $\hat z=x+i\hat y$-plane, and the hyperbolic complex number $\hat z=x+j\hat y$, the function $F[z(Z)]$ are simply written as $z=x+j\hat y, F(z)$ respectively.

The oblique derivative boundary-value problem for (1.1) may be formulated as follows:

Problem P. Find a continuous solution $u(z)$ of (1.1) in $\overline{D}$, where u_x,u_y are continuous in D^*, and satisfy the boundary conditions

$$\begin{gathered}\frac{1}{2}\frac{\partial u}{\partial\nu}=\frac{1}{H(y-x^n)}\operatorname{Re}[\overline{\lambda(z)}u_{\tilde z}]=\operatorname{Re}[\overline{\Lambda(z)}u_z]=r(z)\quad\text{on }\Gamma\cup\tilde L, u(\tilde a_0)=c_0,\\ \frac{1}{H(y-x^n)}\operatorname{Im}[\overline{\lambda(z)}u_{\tilde z}]|_{z=z_l}=\operatorname{Im}[\overline{\Lambda(z)}u_{\tilde z}]|_{z=z_l}=b_l, u(\tilde a_l)=c_l,\quad l=1,\dots,N.\end{gathered} \tag{1.8}$$

Herein $\tilde L=L_1\cup L_3\cup\dots\cup L_{2N-1}$, ν is a given vector at every point $z\in\Gamma\cup\tilde L$, $u_{\tilde z}=[H(y-x^n)u_x-iu_y]/2$, $\Lambda(z)=\cos(\nu,x)-i\cos(\nu,y)$, $\cos(\nu,x)$ means the cosine of angle between ν and x, $\lambda(z)=\operatorname{Re}\lambda(z)+i\operatorname{Im}\lambda(z)$, if $z\in\Gamma$, and $u_{\tilde z}=[H(y-x^n)u_x-ju_y]/2$, $\lambda(z)=\operatorname{Re}\lambda(z)+j\operatorname{Im}\lambda(z)$, if $z\in\tilde L$, $b_l,c_l\,(l=1,\dots,N),c_0$ are real constants, and $r(z),b_l,c_l\,(l=1\dots,N),c_0$ satisfy the conditions

$$\begin{gathered}C_\alpha^1[\lambda(z),\Gamma]\le k_0,\quad C_\alpha^1[\lambda(z),\tilde L]\le k_0,\quad C_\alpha^1[r(z),\Gamma]\le k_2,\\ C_\alpha^1[r(z),\tilde L_1]\le k_2,\quad \cos(\nu,n)\ge 0\quad\text{on }\Gamma,\\ \cos(\nu,n)<1\quad\text{on }\tilde L,\\ |b_l|,|c_l|,|c_0|\le k_2,\quad l=1,\dots,N,\\ \max_{z\in\tilde L}\frac{1}{|\operatorname{Re}\lambda(z)-\operatorname{Im}\lambda(z)|}\le k_0,\end{gathered} \tag{1.9}$$

in which n is the outward normal vector at every point on Γ, α,k_0,k_2 are positive constants with $0<\alpha<1$ and $k_2\ge k_0)$.

The number

$$K=\frac{1}{2}(K_1+K_2+\dots+K_{N+1})$$

is called the index of Problem P, where $K_l = \left[\frac{\phi_l}{\pi}\right] + J_l$, $\quad J_l = 0$ or 1,

$$e^{i\phi_l} = \frac{\lambda(\hat{t}_l - 0)}{\lambda(\hat{t}_l + 0)}, \quad \gamma_l = \frac{\phi_l}{\pi} - K_l, \quad l = 1, 2, \dots, N+1,$$

in which $\hat{t}_1 = a_0$, $\hat{t}_2 = a_N$, $\hat{t}_3 = a_1, \dots, \hat{t}_{N+1} = a_{N-1}$, $\lambda(t) = e^{i\pi/2}$ on L_0, $L_0 = D \cap \{y - x^n = 0\}$ on x-axis, and $\lambda(\hat{t}_1 + 0) = \lambda(\hat{t}_3 - 0) = \lambda(\hat{t}_3 + 0) = \dots = \lambda(\hat{t}_N - 0) = \lambda(\hat{t}_N + 0) = \lambda(\hat{t}_2 - 0) = \exp(i\pi/2)$. Here $K = -1/2$ or $(N-1)/2$ on the boundary ∂D^+ of D^+ can be chosen, in the last case we can add N point conditions $u(\tilde{a}_l) = c_l \, (l = 1, \dots, N)$. It is clear that we can require that $-1/2 \le \gamma_l < 1/2 \, (l = 0, 1, \dots, N)$. Moreover if $\cos(\nu, n) \equiv 0$ on Γ, the case is just the boundary condition of Tricomi problem, from (1.8), we can determine the value $u(z^*)$ by the value $u(z_*)$, namely

$$u(z^*) = 2\operatorname{Re}\int_{z_*}^{z^*} u_z dz + u(z_*) = 2\int_0^S \operatorname{Re}[z'(s)u_z]ds + c_0 = 2\int_0^S r(z)ds + c_0 = c_N,$$

and

$$u(z) = 2\operatorname{Re}\int_{\tilde{a}_0}^{z} u_z dz + u(\tilde{a}_0) = 2\int_0^s \operatorname{Re}[z'(s)u_z]ds + c_0 = 2\int_0^s r(z)ds + c_0 = \phi(z)$$

on Γ, and for $l = 0, 1, \dots, N-1$,

$$u(z) = 2\operatorname{Re}\int_{\tilde{a}_l}^{z} u_z d\bar{z} + u(\tilde{a}_l) = 2\int_0^{s_l} \operatorname{Re}[\overline{z'(s)}u_z]ds + c_l = 2\int_0^{s_l} r(z)ds + c_l = \psi(z)$$

on L_{2l+1}, in which $\overline{\Lambda(z)} = z'(s)$ on Γ, $z(s)$ is a parameter expression of arc length s of Γ with the condition $z(0) = z_*$, S is the length of the boundary Γ, and $\overline{\Lambda(z)} = \overline{z'(s)}$ on L_l, $z(s)$ is a parameter expression of arc length s of L_l with the condition $z(0) = \tilde{a}_l, l = 0, \dots, N-1$. If we consider

$$\operatorname{Re}[\overline{\lambda(z)}(U + jV)] = 0 \quad \text{on } L_0,$$

where $\lambda(z) = 1 = e^{i0\pi}$, then $\gamma_1 = \gamma_2 = -1/2$, $\gamma_l = 0$ $(l = 2, \dots, N+1)$ or $\gamma_1 = 1/2$, $\gamma_2 = -1/2$, $\gamma_l = 0$ $(l = 2, \dots, N+1)$, thus $K = 0$ or $-1/2$.

For (1.1) with $c = 0$, when $K = -1/2$ or $(N-1)/2$, the last point condition in (1.8) can be replaced by

$$Lu_{\bar{z}}(z_l') = \operatorname{Im}[\overline{\lambda(z)}u_{\bar{z}}]|_{z=z_l'} = H(y_l' - x_l'^n)c_l = c_l', \quad l = 1, \dots, N, \tag{1.10}$$

where $z_l' = x_l' + iy_l' = x_l' + ix_l'^n$ $(l = 1, \dots, N)$ are distinct points on $\Gamma\backslash\{\tilde{a}_0 \cup \tilde{a}_N\}$, and $c_l \, (l = 1, \dots, N)$ are real constants, in this case the condition $\cos(\nu, n) \ge 0$ on Γ in (1.9) can be cancelled. The boundary value problem is called Problem Q.

Noting that $\lambda(z), r(z) \in C^1_\alpha(\Gamma)$, $\lambda(z), r(z) \in C^1_\alpha(\tilde{L})$ $(0 < \alpha < 1)$, we can find two twice continuously differentiable functions $u_0^\pm(z)$ in $\overline{D^\pm}$, for instance, which are the solutions of the oblique derivative problem with the boundary condition in (1.8) for harmonic equations in $D^\pm$ (see [17]), thus the functions $v(z) = v^\pm(z) = u(z) - u_0^\pm(z)$ in $D^\pm$ is the solution of the following boundary value problem in the form

$$K(y - x^n)v_{xx} + v_{yy} + \hat{a}v_x + \hat{b}v_y + \hat{c}v + \hat{d} = 0 \quad \text{in } D, \tag{1.11}$$

$$\begin{gathered}\operatorname{Re}[\overline{\lambda(z)}v_{\bar{z}}(z)] = R(z) \quad \text{on } \Gamma \cup \tilde{L}, \\ v(\tilde{a}_0) = c_0, \\ \operatorname{Im}[\overline{\lambda(z_l)}v_{\bar{z}}(z_l)] = b_l', \quad v(\tilde{a}_l) = c_l \text{ or } \operatorname{Im}[\overline{\lambda(z_l')}v_{\bar{z}}(z_l')] = c_l', \quad l = 1, \dots, N.\end{gathered} \tag{1.12}$$

Herein $W(z) = U + iV = v^+_{\tilde z}$ in D^+, $W(z) = U + jV = v^-_{\tilde z}$ in $\overline{D^-}$, $R(z) = 0$ on $\Gamma \cup \tilde L$, $b_l = 0$, $c_0 = c_l = 0$, $l = 1, \dots, N$. Hence later on we only discuss the case of the homogeneous boundary condition and the index $K = (N-1)/2$, the other case can be similarly discussed. From $v(z) = v^\pm(z) = u(z) - u^\pm_0(z)$ in $\overline{D^\pm}$, we have $u(z) = v^+(z) + u^+_0(z)$ in $\overline{D^+}$, $u(z) = v^-(z) + u^-_0(z)$ in $\overline{D^-}$, $v^+(z) = v^-(z) - u^+_0(z) + u^-_0(z)$, $v^+_y = v^-_y - u^+_{0y} + u^-_{0y} = 2\hat R_0(x)$, and $v^-_y = 2\tilde R_0(x)$ on $L_0 = D \cap \{y = 0\}$, where $\hat R_0(x), \tilde R_0(x)$ are undetermined real functions. The boundary vale problem (1.11), (1.12) is called Problem $\tilde P$ or $\tilde Q$.

Here we mention that if the domain D is general, then we can choose a univalent conformal mapping, such that D is transformed onto a special domain with the partial boundary Γ as stated before, then the u_x in Conditions (C1),(C2) should be replaced by u_z. For the boundary condition (1.8) on the boundary ∂D of general domain D, we require that the boundary conditions about $u(z)$ and u_x in (1.8) satisfy the similar conditions.

2. Representation of solutions to oblique derivative problems

The representation of solutions of Problem P or Q for equation (1.1) is as follows.

Theorem 2.1. *Under Conditions* (C1), (C2), *any solution $u(z)$ of Problem P or Q for equation* (1.1) *in D^- can be expressed as*

$$\begin{gathered}
u(z) = \int_0^{y-x^n} V(z)dy + u(x) = 2\operatorname{Re}\int_{z_*}^{z} [\frac{\operatorname{Re} W}{H(\hat y)} + \begin{pmatrix} i \\ -j \end{pmatrix} \operatorname{Im} W]dz + c_0 \\
\text{in } \begin{pmatrix} \overline{D^+} \\ \overline{D^-} \end{pmatrix}, \\
W(z) = \Phi[Z(z)] + \Psi[(Z(z)] = \hat\Phi[Z(z)] + \hat\Psi[(Z(z)], \Psi(Z) = T(Z) - \overline{T(\overline{Z})}, \\
\hat\Psi(Z) = T(Z) + \overline{T(\overline{Z})}, \quad T(Z) = -\frac{1}{\pi}\int\int_{D^+_t} \frac{f(t)}{t - Z} d\sigma_t \quad \text{in } \overline{D^+_Z}, \\
W(z) = \phi(z) + \psi(z) = \xi(z)e_1 + \eta(z)e_2 \quad \text{in } \overline{D^-}, \\
\xi(z) = \zeta(z) + \int_0^{y-x^n} g_1(z)dt = \int_{S_1} g_1(z)dt + \int_0^{y-x^n} g_1(z)dt, \quad z \in s_1, \\
\eta(z) = \theta(z) + \int_0^{y-x^n} g_2(z)dt = \int_{S_2} g_2(z)dt + \int_0^{y-x^n} g_2(z)dt, \quad z \in s_2, \\
g_l(z) = \tilde A_l(U+V) + \tilde B_l(U-V) + 2\tilde C_l U + \tilde D_l u + \tilde E_l, \quad l = 1, 2.
\end{gathered} \tag{2.1}$$

Herein $Z = x + jG(y - x^n), f(Z) = g(Z)/H$, $U = Hu_x/2$, $V = -u_y/2$, $\begin{pmatrix} i \\ -j \end{pmatrix}$ is a 2×1 matrix, $\xi(z) = \int_{S_1} g_1(z)dt$ in D^-, $\zeta(x) + \theta(x) = 0$ on L_0, s_1, s_2 are two families of characteristics in D^-:

$$s_1 : \frac{dx}{dy} = H(y - x^n), \quad s_2 : \frac{dx}{dy} = -H(y - x^n) \tag{2.2}$$

passing through $z = x + j(y - x^n) \in \overline{D^-}$, S_1, S_2 are characteristic curves from the points on $\tilde L = L_1 \cup L_3 \cup \dots \cup L_{2N-1}, \tilde L' = L_2 \cup L_4 \cup \dots \cup L_{2N}$ to two points on L_0

respectively, and

$$\begin{gathered}
W(z)=U(z)+jV(z)=\frac{1}{2}Hu_x-\frac{j}{2}u_y,\\
\xi(z)=\operatorname{Re}W(z)+\operatorname{Im}W(z),\quad \eta(z)=\operatorname{Re}W(z)-\operatorname{Im}W(z),\\
\tilde{A}_1=\tilde{B}_2=-\frac{b}{2},\quad \tilde{A}_2=\tilde{B}_1=\frac{b}{2},\quad \tilde{C}_1=\frac{a}{2H}+\frac{m(1-nx^{n-1}H)}{4(y-x^n)},\\
\tilde{C}_2=-\frac{a}{2H}+\frac{m(1+nx^{n-1}H)}{4(y-x^n)},\quad \tilde{D}_1=-\tilde{D}_2=\frac{c}{2},\quad \tilde{E}_1=-\tilde{E}_2=\frac{d}{2},
\end{gathered}\tag{2.3}$$

in which we choose $H(y-x^n)=|y-x^n|^{m/2}$*, where* m *is as stated before.*

Proof. From (1.5), (1.6), we see that equation (1.1) in $\overline{D^-}$ can be reduced to the system of integral equations: (2.1). Moreover we can derive $H(0)u_x/2=U(x)=[\zeta(x)+\theta(x)]/2=0$, i.e. $\zeta(x)=-\theta(x)$ on L_0, and then $\zeta(z)=\int_{S_1}g_1(z)dt$, $\theta(z)=-\zeta(x+G(y-x^n))$ in $\overline{D^-}$. Here we mention that by using the way of symmetrical extension with respect to $L_l\,(l=1,2,\dots,2N)$, we can extend the function $W(Z),u(z)$ from $\overline{D^-}$ onto the exterior of D^-. □

In the following, we prove the uniqueness of solutions of Problem P for (1.1).

Theorem 2.2. *Suppose that* (1.1) *satisfies Condition* (C1), (C2). *Then Problem P for* (1.1) *in* D *has a unique solution.*

Proof. Let $u_1(z),u_2(z)$ be two solutions of Problem P for (1.1). Then $u(z)=u_1(z)-u_2(z)$ is a solution of the generalized Rassias homogeneous equation

$$K(y-x^n)u_{xx}+u_{yy}+\tilde{a}u_x+\tilde{b}u_y+\tilde{c}u=0\quad\text{in } D,\tag{2.4}$$

satisfying the boundary conditions

$$\begin{gathered}
\frac{1}{2}\frac{\partial u}{\partial\nu}=\frac{1}{H(\hat{y})}\operatorname{Re}[\overline{\lambda(z)}u_{\tilde{z}}(z)]=0\quad\text{on }\Gamma\cup\tilde{L},\\
u(\tilde{a}_0)=0,\quad \operatorname{Im}[\overline{\lambda(z_l)}u_{\tilde{z}}(z_l)]=0,\quad u(\tilde{a}_l)=0,\quad l=1,\dots,N,
\end{gathered}\tag{2.5}$$

where the function $W(z)=U(z)+jV(z)=[H(\hat{y})u_x-ju_y]/2$ in the hyperbolic domain D^- can be expressed in the form

$$\begin{gathered}
W(z)=\phi(x)+\psi(z)=\xi(z)e_1+\eta(z)e_2,\\
\xi(z)=\zeta(z)+\int_0^{y-x^n}[\tilde{A}_1(U+V)+\tilde{B}_1(U-V)+2\tilde{C}_1U+\tilde{D}_1u]dy,\quad z\in s_1,\\
\eta(z)=\theta(z)+\int_0^{y-x^n}[\tilde{A}_2(U+V)+\tilde{B}_2(U-V)+2\tilde{C}_2U+\tilde{D}_2u]dy,\quad z\in s_2,
\end{gathered}\tag{2.6}$$

where $\phi(z)=\zeta(z)e_1+\theta(z)e_2$ is a solution of equation $W_{\overline{\tilde{z}}}=0$ in D^-, and

$$u(z)=2\operatorname{Re}\int_{z_*}^{z}[\frac{\operatorname{Re}W(z)}{H(y-x^n)}+\begin{pmatrix} i\\ -j\end{pmatrix}\operatorname{Im}W]dz\quad\text{in }\begin{pmatrix}\overline{D^+}\\ \overline{D^-}\end{pmatrix}\tag{2.7}$$

By a similar way as in [20, Section 2, Chapter V], we can verify $u(z)=0$ in $\overline{D^-}$, especially $u_{\hat{y}}=0$ on L_0.

Now we verify that the above solution $u(z)\equiv 0$ in D^+. If the maximum $M=\max_{\overline{D^+}}u(z)>0$, it is clear that the maximum point $z'\notin D^+$. If the maximum M attains at a point $z'\in\Gamma$ and $\cos(\nu,n)>0$ at z', we get $\partial u/\partial\nu>0$ at z', which contradicts the first formula of (2.5). If $\cos(\nu, n) = 0$ at z', denote by Γ' the longest curve of Γ including the point z', so that $\cos(\nu, n) = 0$ and $u(z) = M$ on Γ', then there exists a point $z_0 \in \Gamma\backslash\Gamma'$, such that at z_0, $\cos(\nu, n) > 0, \partial u/\partial n > 0, \cos(\nu, s) > 0\,(< 0)$, $\partial u/\partial s \geq 0\,(\leq 0)$, hence the inequality

$$\frac{\partial u}{\partial \nu} = \cos(\nu, n)\frac{\partial u}{\partial n} + \cos(\nu, s)\frac{\partial u}{\partial s} > 0 \quad \text{at } z_0 \tag{2.8}$$

holds, in which s is the tangent vector at $z_0 \in \Gamma$, it is impossible. Thus $u(z)$ attains its positive maximum at a point $z = z' \in L_0$. By the Hopf Lemma, we can see that it is also impossible. Hence $u(z) = u_1(z) - u_2(z) = 0$ in $\overline{D^+}$, thus we have $u_1(z) = u_2(z)$ in $\overline{D}$. This completes the proof. □

3. Solvability of oblique derivative problems

In this section, we prove the existence of solutions of Problem P for equation (1.1). From the discussion in Section 1, we can only discuss the complex equation

$$W_{\tilde{\bar{z}}} = A_1(z, u, W)W + A_2(z, u, W)\overline{W} + A_3(z, u, W)u + A_4(z, u, W) \quad \text{in} \tag{3.1}$$

with the relation

$$u(z) = \begin{cases} 2\,\mathrm{Re}\displaystyle\int_{z_*}^{z}[\frac{\mathrm{Re}\,W(z)}{H(y-x^n)} + i\,\mathrm{Im}\,W(z)]dz + c_0 & \text{in } \overline{D^+}, \\ 2\,\mathrm{Re}\displaystyle\int_{z_*}^{z}[\frac{\mathrm{Re}\,W(z)}{H(y-x^n)} - j\,\mathrm{Im}\,W(z)]dz + c_0 & \text{in } \overline{D^-}, \end{cases} \tag{3.2}$$

and the homogeneous boundary conditions

$$\begin{gathered} \mathrm{Re}[\overline{\lambda(z)}W(z)] = R(z) \quad \text{on } \Gamma \cup \tilde{L}, \\ u(\tilde{a}_0) = c_0, \\ \mathrm{Im}[\overline{\lambda(z_l)}u_{\tilde{z}}(z_l)] = b'_l, u(\tilde{a}_l) = c_l \text{ or } \mathrm{Im}[\overline{\lambda(z'_l)}u_{\tilde{z}}(z'_l)] = c'_l, \quad l = 1, \dots, N, \end{gathered} \tag{3.3}$$

where $R(z) = 0$ on $\Gamma \cup L_1$ and $c_0 = b'_l = c_l = c'_l = 0$, $l = 1, \dots, N$. The boundary value problem (3.1), (3.2), (3.3) is called Problem $\tilde{A}$, which is corresponding to Problem $\tilde{P}$ or $\tilde{Q}$. It is clear that Problem $\tilde{A}$ can be divided into two problems, i.e. Problem A_1 of equation (3.1), (3.2) in D^+ and Problem A_2 of equation (3.1), (3.2) in D^-. The boundary conditions of Problems A_1 and A_2 as follows:

$$\begin{gathered} \mathrm{Re}[\overline{\lambda(z)}W(z)] = R(z) \quad \text{on } \Gamma \cup L_0, \\ u(\tilde{a}_l) = c_l \text{ or } \mathrm{Im}[\overline{\lambda(z'_l)}W(z'_l)] = c'_l, \quad l = 1, \dots, N, \end{gathered} \tag{3.4}$$

where $\lambda(z) = -i, R(x) = \hat{R}_0(x)$ on L_0, and

$$\begin{gathered} \mathrm{Re}[\overline{\lambda(z)}W(z)] = R(z) \quad \text{on } \tilde{L} \cup L_0, \\ \mathrm{Im}[\overline{\lambda(z_l)}W(z_l)] = b'_l, \quad l = 1, \dots, N, \end{gathered} \tag{3.5}$$

in which $\lambda(z) = a(z) + jb(z)$, $R(z) = 0$ on $\Gamma \cup \tilde{L}$ in (1.12), $\lambda(z) = 1 + j$, $R(z) = -\tilde{R}_0(x)$ on $L_0, \hat{R}_0(x), \tilde{R}_0(x)$ on L_0 are as stated in (1.12), because $\mathrm{Re}\,W(x) = 0$ on L_0, thus $1 + j$ can be replaced by j.

Introduce a function

$$X(Z) = \prod_{l=1}^{N+1}(Z - \hat{t}_l)^{\eta_l}, \tag{3.6}$$

where $\hat{t}_1 = -R$, $\hat{t}_2 = R$, $\hat{t}_l = a_{l-2}$, $l = 3, \dots, N+1$, the numbers $\eta_l = 1 - 2\gamma_l$ if $\gamma_l \ge 0$, $\eta_l = \max(-2\gamma_l, 0)$ if $\gamma_l < 0$, $\gamma_l\,(l = 1, 2)$ are as stated in Section 1, $\eta_3 = \dots = \eta_{N+1} = 1$, where we choose a branch of multi-valued function $X(Z)$ such that $\arg X(x) = \eta_2\pi/2$ on $L_0 \cap \{x > a_{N-1}\}$. Obviously that $X(Z)W[z(Z)]$ satisfies the complex equation

$$[X(Z)W]_{\overline{Z}} = X(Z)[A_1W + A_2\overline{W} + A_3u + A_4]/H = X(Z)g(Z)/H \quad \text{in} D_Z, \quad (3.7)$$

and the boundary conditions

$$\begin{gathered}\operatorname{Re}[\overline{\hat{\lambda}(z)}X(Z)W(z)] = R(z) = 0 \quad \text{on } \Gamma,\\ \operatorname{Re}[\overline{\hat{\lambda}(z)}X(Z)W(z)] = 0 \quad \text{on } \tilde{L},\\ u(\tilde{a}_0) = 0, \quad \operatorname{Im}[\overline{\lambda(z_l)}W(z_l)] = 0, \quad u(\tilde{a}_l) = 0, \quad l = 1, \dots, N,\end{gathered}$$

where $D_Z = Z_Z^+$, $\hat{\lambda}(z) = \lambda(z)e^{i\arg X(Z)}$. Noting that

$$\begin{gathered}e^{i\hat{\phi}_l} = \frac{\hat{\lambda}(\hat{t}_l - 0)}{\hat{\lambda}(\hat{t}_l + 0)} = \frac{\lambda(\hat{t}_l - 0)}{\lambda(\hat{t}_l + 0)}\frac{e^{i\arg X(\hat{t}_l - 0)}}{e^{i\arg X(\hat{t}_l + 0)}} = e^{i(\phi_l + \tilde{\eta}_l)},\\ \tau_l = \frac{\hat{\phi}_l}{\pi} - \hat{K}_l = 0, \quad l = 1, \dots, N+1,\end{gathered}$$

in which $\tilde{\eta}_l = \eta_l\pi/2$, $l = 1, 2$, $\tilde{\eta}_l = \eta_l\pi$, $l = 3, \dots, N+1$, which are corresponding to the numbers γ_l $(1 \le l \le N+1)$ in Section 1. If $\hat{K}_l = -1$, $\hat{K}_l = 1, l = 2, \dots, N+1$, or $\hat{K}_l = -1$, $\hat{K}_l = 0, l = 2, \dots, N+1$, then the index $\hat{K} = (\hat{K}_1 + \dots + \hat{K}_{N+1})/2 = (N-1)/2$ or $-1/2$ of $\hat{\lambda}(z)$ on $\Gamma \cup L_0$ is chosen. For the case $\hat{K} = (N-1)/2$, we need to add N point conditions $u(\tilde{a}_l) = c_l$ $(l = 1, \dots, N)$ in (1.8) and (1.10), such that Problem $\tilde{P}$ or $\tilde{Q}$ is well-posed.

Theorem 3.1. *Let* (1.1) *satisfy Conditions* (C1), (C2). *Then any solution of Problem A_1 for* (1.1) *in D^+ satisfies the estimate*

$$\begin{gathered}\hat{C}_\delta[W(z), \overline{D^+}] = C_\delta[X(Z)(\operatorname{Re} W(Z)/H + i\operatorname{Im} W(Z)), \overline{D^+}] + C_\delta[u(z), \overline{D^+}] \le M_1,\\ \hat{C}_\delta[W(z), \overline{D^+}] \le M_2(k_1 + k_2),\end{gathered} \quad (3.8)$$

where $X(Z)$ is as stated in (3.6), *$\delta < \min[2, m]/(m+2)$ is a sufficiently small positive constant, $M_1 = M_1(\delta, k, H, D^+)$, $M_2 = M_2(\delta, k_0, H, D^+)$ are positive constants, and $k = (k_0, k_1, k_2)$.*

Proof. We first assume that any solution $[W(z),\ u(z)]$ of Problem A_1 satisfies the estimate

$$\hat{C}[W(z), \overline{D^+}] = C[X(Z)(\operatorname{Re} W(Z)/H + i\operatorname{Im} W(Z), \overline{D_Z}] + C[u(z), \overline{D^+}] \le M_3, \quad (3.9)$$

where M_3 is a non-negative constant, and then give that $[W(z), u(z)]$ satisfy the Hölder continuous estimates in $\overline{D_Z}$.

Firstly, we verify the Hölder continuity of solutions $[W(z), u(z)]$ in

$$\overline{D_Z} \cap \{\operatorname{dist}(Z, \{\hat{t}_1 \cup \hat{t}_2 \cup \dots \cup \hat{t}_{N+1}\}) \ge \varepsilon\},$$

in which ε is a sufficiently small positive constant. Substituting the solution $[W(z), u(z)]$ into (3.7) and noting $\operatorname{Re} W(Z) = R(x) = 0$ on L_0, we can extend the function $X(Z)W[z(Z)]$ onto the symmetrical domain $\tilde{D}_Z$ of D_Z with respect to the real axis $\operatorname{Im} Z = 0$, namely set

$$\tilde{W}(Z) = \begin{cases} X(Z)W[z(Z)] & \text{in } D_Z, \\ -\overline{X(\overline{Z})W[z(\overline{Z})]} & \text{in } \tilde{D}_Z, \end{cases}$$

which satisfies the boundary conditions

$$\operatorname{Re}[\overline{\tilde{\lambda}(Z)}\tilde{W}(Z)] = 0 \quad \text{on } \Gamma \cup \tilde{\Gamma},$$

$$\tilde{\lambda}(Z) = \begin{cases} \lambda[z(Z)], \\ \overline{\lambda[z(\overline{Z})]}, \\ 1, \end{cases} \qquad \tilde{R}(Z) = \begin{cases} 0 & \text{on } \Gamma, \\ 0 & \text{on } \tilde{\Gamma}, \\ 0 & \text{on } L_0, \end{cases}$$

where $\tilde{\Gamma}$ is the symmetrical curve of Γ about $\operatorname{Im} Z = 0$. It is easy to see that the corresponding function $u(z)$ in (3.2) can be extended to the function $\tilde{u}(Z)$, where $\tilde{u}(Z) = u[z(Z)]$ in $D_Z(= D_Z^+)$ and $\tilde{u}(Z) = -u[z(\overline{Z})]$ in $\tilde{D}_Z$. Noting (C1), (C2) and the condition (3.9), we see that the function $\tilde{f}(Z) = X(Z)g(Z)/H$ in D_Z and $\tilde{f}(Z) = -\overline{X(\overline{Z})g(\overline{Z})}/H$ in $\tilde{D}_Z$ satisfies the condition $L_\infty[y^\tau H\tilde{f}(Z), D'_Z] \le M_4$, in which $D'_Z = D_Z \cup \tilde{D}_Z \cup L_0$, $\tau = \max(1 - m/2, 0), M_4 = M_4(\delta, k, H, D, M_3)$ is a positive constant. On the basis of [20, Lemma 2.1, Chapter I], we can verify that the function $\tilde{\Psi}(Z) = T(Z) - \overline{T(\overline{Z})}$ $(T(Z) = -1/\pi \int\int_{D_t}[\tilde{f}(t)/(t - Z)]d\sigma_t\}$ over $D_Z)$ satisfies the estimates

$$C_\beta[\tilde{\Psi}(Z), \overline{D_Z}] \le M_5, \quad \tilde{\Psi}(Z) - \tilde{\Psi}(\hat{t}_l) = O(|Z - \hat{t}_l|^{\beta_l}), \quad 1 \le l \le N + 1, \tag{3.10}$$

in which $\beta = \min(2, m)/(m+2) - 2\delta = \beta_l$ $(1 \le l \le N+1)$, δ is a constant as stated in (3.8), and $M_5 = M_5(\delta, k, H, D, M_3)$ is a positive constant. On the basis of Theorem 2.1, the solution $X(Z)\tilde{W}(z)$ can be expressed as $X(Z)\tilde{W}(Z) = \tilde{\Phi}(Z) + \tilde{\Psi}(Z)$, where $\tilde{\Phi}(Z)$ is an analytic function in D_Z satisfying the boundary conditions

$$\operatorname{Re}[\overline{\tilde{\lambda}(Z)}\tilde{\Phi}(Z)] = -\operatorname{Re}[\overline{\tilde{\lambda}(Z)}\tilde{\Psi}(Z)] = \hat{R}(Z) \quad \text{on } \Gamma \cup \tilde{L},$$
$$u(\tilde{a}_0) = 0, u(\tilde{a}_l) = 0 \text{ or } \operatorname{Im}[\overline{\lambda(z'_l)}\tilde{W}(z'_l)] = 0, \quad l = 1, \dots, N.$$

There is no harm in assuming that $\tilde{\Psi}(\hat{t}_l) = 0$, otherwise it suffices to replace $\tilde{\Psi}(Z)$ by $\tilde{\Psi}(Z) - \tilde{\Psi}(\hat{t}_l)$ $(1 \le l \le N + 1)$. For giving the estimates of $\tilde{\Phi}(Z)$ in $D_Z \cap \{\operatorname{dist}(Z, \Gamma) \ge \varepsilon(> 0)\}$, from the integral expression of solutions of the discontinuous Riemann-Hilbert problem for analytic functions, we can write the representation of the solution $\tilde{\Phi}(Z)$ of Problem A_1 for analytic functions, namely

$$\tilde{\Phi}[Z(\zeta)] = \frac{X_0(\zeta)}{2\pi i}\Big[\int_{\partial D_t} \frac{(t + \zeta)\tilde{\lambda}[Z(t)]\hat{R}[Z(t)]dt}{(t - \zeta)tX_0(t)} + Q(\zeta)\Big],$$

$$Q(\zeta) = i\sum_{k=0}^{[\hat{K}]}(c_k\zeta^k + \overline{c_k}\zeta^{-k}) + \begin{cases} 0, & \text{when } 2\hat{K} = N - 1 \text{ is even}, \\ ic_*\dfrac{\zeta_1 + \zeta}{\zeta_1 - \zeta},\ c_* = i\displaystyle\int_{\partial D_t} \frac{\tilde{\lambda}[Z(t)]\hat{R}[Z(t)]dt}{X_0(t)t}, & \text{when } 2\hat{K} = N - 1 \text{ is odd}, \end{cases}$$

(see [17, 18]), where $X_0(\zeta)=\Pi_{l=1}^{N+1}(\zeta-\hat{t}_l)^{\tau_l}$, τ_l $(l=1,\dots,N+1)$ are as before, $Z=Z(\zeta)$ is the conformal mapping from the unit disk $D_\zeta=\{|\zeta|<1\}$ onto the domain D_Z such that the three points $\zeta=-1,i,1$ are mapped onto $Z=-1,Z'(\in\Gamma),1$ respectively. Taking into account

$$|X_0(\zeta)|=O(|\zeta-\hat{t}_l|^{\tau_l}),\quad |\hat{\lambda}[Z(\zeta)]\hat{R}[Z(\zeta)]/X_0(\zeta)|=O(|\zeta-\hat{t}_l|^{\tilde{\eta}_l-\tau_l}),$$

and according to the results in [17], we see that the function $\tilde{\Phi}(Z)$ determined by the above integral in $D_Z\cap\{\operatorname{dist}(Z,\Gamma)\geq\varepsilon(>0)\}$ is Hölder continuous and $\tilde{\Phi}(\hat{t}_l)=0\,(1\leq l\leq N+1)$. Thus, from (3.10) and the above integral representation of $\tilde{\Phi}(Z)$, we can give the following estimates

$$C_\delta[\tilde{\Phi}(Z),D_\varepsilon]\leq M_6,\quad C_\delta[X(Z)u_x,D_\varepsilon]\leq M_6,\quad C_\delta[X(Z)u_y,D_\varepsilon]\leq M_6, \tag{3.11}$$

where $D_\varepsilon=\overline{D_Z}\cap\{\operatorname{dist}(Z,L_0)\geq\varepsilon\}$, ε is arbitrary small positive constant, $M_6=M_6(\delta,k,H,\,D_\varepsilon,M_3)$ is a non-negative constant. Similarly we can get

$$C_\delta[H(\hat{y})u_x,D'_\varepsilon]\leq M_7,\quad C_\delta[u_y,D'_\varepsilon]\leq M_7, \tag{3.12}$$

in which $D'_\varepsilon=\overline{D_Z}\cap\{\operatorname{dist}(Z,\Gamma\cup\tilde{\Gamma})\geq\varepsilon\}$, ε is arbitrary small positive constant, and $M_7=M_7(\delta,\,k,H,D'_\varepsilon,M_3)$ is a non-negative constant. □

Next, for giving the estimates of $X(Z)u_x,X(Z)u_x$ in $\tilde{D}_l=D_l\cap\overline{D_Z}$ $(D_l=\{|Z-\hat{t}_l|<\varepsilon(>0)\}$, $1\leq l\leq 2)$ separately, denote $X(Z)=\tilde{X}+i\tilde{Y}$ as in (3.6), we first conformally map the domain $D'_Z=D_Z\cup\tilde{D}_Z\cup L_0$ onto a domain D_ζ, such that L_0 is mapped onto himself, where D_ζ is a domain with the partial boundary $\Gamma\cup\tilde{\Gamma}$, and $\Gamma\cup\tilde{\Gamma}$ is a smooth curve including the line segment $\operatorname{Re}\zeta=\hat{t}_l$ near $\zeta=\hat{t}_l\,(1\leq l\leq 2)$. Through the above mapping, the index $\tilde{K}=(N-1)/2$ is not changed, and the function $\tilde{\Psi}[Z(\zeta)]$ in the neighborhood $\zeta(D_l)$ of $\hat{t}_l\,(1\leq l\leq 2)$ is Hölder continuous. For convenience denote by $D_Z,D_l,\tilde{W}(Z)$ the domains and function $D_\zeta,\zeta(D_l),\tilde{W}[Z(\zeta)]$ again. Secondly reduce the the above boundary condition to this case, i.e. the corresponding function $\tilde{\lambda}(Z)=1$ on $\Gamma\cup\tilde{\Gamma}$ near $Z=\hat{t}_l\,(1\leq l\leq 2)$. In fact there exists an analytic function $S(Z)$ in $D'_Z=D_Z\cup\tilde{D}_Z\cup L_0$ satisfying the boundary condition

$$\operatorname{Re}S(Z)=-\arg\tilde{\lambda}(Z)\quad\text{on }\Gamma\cup\tilde{\Gamma}\quad\text{near }\hat{t}_l,\quad \operatorname{Im}S(\hat{t}_l)=0,$$

and the estimate

$$C_\alpha[S(Z),D_l\cap D'_Z]\leq M_8=M_8(\delta,k,H,D,M_3)<\infty,$$

then the function $e^{jS(Z)}X(Z)W(Z)$ is satisfied the boundary condition

$$\operatorname{Re}[e^{iS(Z)}X(Z)W(Z)]=0\quad\text{on }\Gamma\cup\tilde{\Gamma}\quad\text{near }Z=\hat{t}_l\,(1\leq l\leq 2).$$

Next we symmetrically extend the function $\Phi^*(Z)$ in D'_Z onto the symmetrical domain D^*_Z with respect to $\operatorname{Re}Z=\hat{t}_l\,(1\leq l\leq 2)$, namely let

$$\hat{W}(Z)=\begin{cases}e^{iS(Z)}X(Z)W(Z) & \text{in }D'_Z,\\ -\overline{e^{iS(Z')}X(Z')W(Z')} & \text{in }D^*_Z,\end{cases}$$

where $Z'=-\overline{(Z-\hat{t}_l)}+\hat{t}_l$, later on we shall omit the secondary part $e^{iS(Z)}$.

After the above discussion, as stated in (2.1), the solution $X(Z)W(z)$ can be also expressed as $X(Z)W(Z)=\Phi(Z)+\Psi(Z)$, where $X(Z)=\tilde{X}+i\tilde{Y},X(Z)$ is as stated in (3.6), $\Psi(Z)$ in $\hat{D}_Z=\{D^*_Z\cup D'_Z\}\cap\{Y=G(y-x^n)>0\}$ is Hölder continuous, and $\Phi(Z)$ is an analytic function in $\hat{D}_Z$ satisfying the boundary conditions in the form

$$\begin{gathered}\operatorname{Re}[\overline{\tilde{\lambda}(Z)}\Phi(Z)] = \hat{R}(Z) \quad \text{on } \Gamma \cup L_0,\\ u(\hat{t}_l) = 0, \quad l = 1, \dots, N+1,\end{gathered}$$

because in the above case the index of $\tilde{\lambda}(Z)$ on ∂D_Z is $\tilde{K} = (N-1)/2$. Hence by the similar way as in the proof of (3.12), we have

$$C_\delta[X(Z)H(\hat{y})u_x, \tilde{D}_l] \le M_9, \quad C_\delta[X(Z)u_y, \tilde{D}_l] \le M_{10}, \quad 1 \le l \le 2,$$

where $M_l = M_l\ (\delta, k, H, D, M_3)(l = 9, 10)$ is a non-negative constant. As for the solution of Problem P in the neighborhood of $\hat{t}_l$ $(3 \le l \le N+1)$, we can use a similar way.

Finally we use the reduction to absurdity, suppose that (3.9) is not true, then there exist sequences of coefficients $\{A_l^{(m)}\}$ $(l = 1, 2, 3, 4)$, $\{\lambda^{(m)}\}$, $\{r^{(m)}\}$ and $\{c_l^{(m)}\}$ $(l = 0, 1, \dots, N)$, which satisfy the same conditions of coefficients as stated in (1.8), (1.9), such that $\{A_l^{(m)}\}$ $(l = 1, 2, 3, 4)$, $\{\lambda^{(m)}\}$, $\{r^{(m)}\}$, $\{c_l^{(m)}\}$ in $\overline{D^+}$, Γ, L_0 weakly converge or uniformly converge to $A_l^{(0)}$ $(l = 1, 2, 3, 4)$, $\lambda^{(0)}$, $r^{(0)}$, $\{c_l^{(0)}\}$ $(l = 0, 1, \dots, N)$ respectively, and the solutions of the corresponding boundary value problems

$$\begin{gathered}W_{\overline{Z}}^{(m)} = F^{(m)}(z, u^{(m)}, W^{(m)}), F^{(m)}(z, u^{(m)}, W^{(m)})\\ = A_1^{(m)}W^{(m)} + A_2^{(m)}\overline{W^{(m)}} + A_3^{(m)}u^{(m)} + A_4^{(m)} \quad \text{in } \overline{D^+},\\ \operatorname{Re}[\overline{\lambda^{(m)}(z)}W^{(m)}(z)] = R^{(m)}(z) \quad \text{on } \Gamma \cup L_0,\\ u^{(m)}(\tilde{a}_0) = c_0^{(m)}, \quad u^{(m)}(\tilde{a}_l) = c_l^{(m)} \text{ or } LW^{(m)}(z_l') = c_l'^{(m)}, \quad l = 1, \dots, N,\end{gathered}$$

and

$$\begin{aligned}u^{(m)}(z) &= u^{(m)}(x) - 2\int_0^y V^{(m)}(z)dy\\ &= 2\operatorname{Re}\int_{z_*}^z [\frac{\operatorname{Re} W^{(m)}}{H(\hat{y})} + i\operatorname{Im} W^{(m)}]dz + c_0^{(m)} \quad \text{in } \overline{D^+}\end{aligned}$$

have the solutions $[W^{(m)}(z), u^{(m)}(z)]$, but $\hat{C}[W^{(m)}(z), \overline{D^+}]$ $(m = 1, 2, \dots)$ are unbounded, hence we can choose a subsequence of $[W^{(m)}(z), u^{(m)}(z)]$ denoted by $[W^{(m)}(z), u^{(m)}(z)]$ again, such that $h_m = \hat{C}[W^{(m)}(z), \overline{D^+}] \to \infty$ as $m \to \infty$, we can assume $h_m \ge \max[k_1, k_2, 1]$. It is obvious that $[\tilde{W}^{(m)}(z), \tilde{u}^{(m)}(z)_m] = [W^{(m)}(z)/h_m,\ u^{(m)}(z)_m/h_m]$ are solutions of the boundary value problems

$$\begin{gathered}\tilde{W}_{\overline{Z}}^{(m)} = \tilde{F}^{(m)}(z, \tilde{u}^{(m)}, \tilde{W}^{(m)}),\\ \tilde{F}^{(m)}(z, \tilde{u}^{(m)}, \tilde{W}^{(m)}) = A_1^{(m)}\tilde{W}^{(m)} + A_2^{(m)}\overline{\tilde{W}^{(m)}} + A_3^{(m)}\tilde{u}^{(m)} + A_4^{(m)}/h_m \quad \text{in } \overline{D^+},\\ \operatorname{Re}[\overline{\lambda^{(m)}(z)}\tilde{W}^{(m)}(z)] = R^{(m)}(z)/h_m \quad \text{on } \Gamma \cup L_0,\\ \tilde{u}^{(m)}(\tilde{a}_0) = c_0^{(m)}/h_m,\\ \tilde{u}^{(m)}(\tilde{a}_l) = c_l^{(m)}/h_m \text{ or } L\tilde{W}^{(m)}(z_l') = c_l^{(m)}/h_m, \quad l = 1, \dots, N,\end{gathered}$$

and

$$\begin{aligned}\tilde{u}^{(m)}(z) &= \frac{u^{(m)}(x)}{h_m} - 2\int_0^{\hat{y}} \tilde{V}^{(m)}(z)dy \\ &= 2\operatorname{Re}\int_{z_*}^{z}[\frac{\operatorname{Re}\tilde{W}^{(m)}}{H(\hat{y})} + i\operatorname{Im}\tilde{W}^{(m)}]dz + \frac{c_0^{(m)}}{h_m} \quad \text{in } \overline{D^+}.\end{aligned}$$

We see that the functions in the above boundary value problems satisfy the same conditions. From the representation (2.1), the above solutions can be expressed as

$$\begin{gathered}\tilde{u}^{(m)}(z) = \frac{u^{(m)}(x)}{h_m} - 2\int_0^{y} \tilde{V}^{(m)}(z)dy \\ = 2\operatorname{Re}\int_{z_*}^{z}[\frac{\operatorname{Re}\tilde{W}^{(m)}}{H(\hat{y})} + i\operatorname{Im}\tilde{W}^{(m)}]dz + \frac{c_0^{(m)}}{h_m} \quad \text{in } \overline{D^+}, \\ \tilde{W}^{(m)}(z) = \tilde{\Phi}^{(m)}[Z(z)] + \tilde{\Psi}^{(m)}[Z(z)], \\ \tilde{\Psi}^{(m)}(Z) = T(Z) - \overline{T(\overline{Z})}, \quad T(Z) = -\frac{1}{\pi}\iint_{D^+}\frac{\tilde{f}^{(m)}(t)}{t-Z}d\sigma_t, \quad \text{in } \overline{D^+},\end{gathered}$$

As in the proof of (3.10), and notice that $y^\tau H(\hat{y})\tilde{f}^{(m)}(Z) = y^\tau X(Z)g^{(m)}(Z) \in L_\infty(D_Z)$, $\tau = \max(0, 1-m/2)$, we can verify that

$$C_\beta[\tilde{\Psi}(Z), \overline{D^+}] \le M_{11}, \quad \tilde{\Psi}(Z)|_{Z=t_j} = O(|Z-t_j|^{\beta_j}), \quad j = 1, 2,$$

where $M_{13} = M_{13}(\delta, k, H, D^+)$ is a non-negative constant.

Noting that Conditions (C1), (C2) and the complex equation and boundary conditions about $\tilde{W}_x^{(m)}$, which satisfy the conditions similar to those about $\tilde{W}^{(m)}(Z)$, we have

$$C[X(Z)\tilde{W}_x^{(m)}(Z), \overline{D^+}] \le M_{12} = M_{12}(\delta, k, H, \overline{D^+}).$$

Hence we can derive that sequence of functions:

$$\{X(Z)(\operatorname{Re}\tilde{W}^{(m)}(Z)/H(\hat{y}) + i\operatorname{Im}\tilde{W}^{(m)}(Z))\}$$

satisfies the estimate

$$\hat{C}_\delta[\tilde{W}^{(m)}(Z), \overline{D_Z}] \le M_{13} = M_{13}(\delta, k, H, D^+) < \infty.$$

Hence from $\{X(Z)[\operatorname{Re}\tilde{W}^{(m)}(z)/H + i\operatorname{Im}\tilde{W}^{(m)}(z)]\}$ and the sequence of corresponding functions $\{\tilde{u}^{(m)}(z)\}$, we can choose the subsequences denoted by

$$\{X(Z)[\operatorname{Re}\tilde{W}^{(m)}(z)/H + i\operatorname{Im}\tilde{W}^{(m)}(z)]\}, \quad \{\tilde{u}^{(m)}(z)\}$$

again, which uniformly converge to $X(Z)[\operatorname{Re}\tilde{W}^{(0)}(z)/H + i\operatorname{Im}\tilde{W}^{(0)}(z)]$, $\tilde{u}^{(0)}(z)$ respectively, it is clear that $[\tilde{W}^{(0)}(z), \tilde{u}^{(0)}(z)]$ is a solution of the homogeneous problem of Problem A_1. On the basis of Theorem 2.2, the solution $\tilde{W}^{(0)}(z) = 0$, $\tilde{u}^{(0)}(z) = 0$ in $\overline{D^+}$, however, from $\hat{C}[\tilde{W}^{(m)}(z), \overline{D^+}] = 1$, we can derive that there exists a point $z^* \in \overline{D^+}$, such that $\hat{C}[\tilde{W}^{(0)}(z^*), \overline{D^+}] = 1$, it is impossible. This shows that (3.9) is true, where the constant $M_3 = M_3(\delta, k, H, D^+)$, and then the first estimate in (3.8) can be derived. The second estimate in (3.8) is easily verified from the first estimate in (3.8).

Theorem 3.2. *Under the same conditions as in Theorem* 3.1*, Problem A_1 for* (3.1)*,* (3.2) *in D^+ is solvable, and then Problem Q for* (1.1) *with $c = 0$ in D^+ has a solution. Moreover, Problem P for* (1.1) *in D^+ is solvable.*

Proof. Applying using the estimates in Theorem 3.1 and the Leray-Schauder theorem, we can prove the existence of solutions of Problem A_1 for (3.1) with $A_3 = 0$ in D^+. We consider the equation and boundary conditions with the parameter $t \in [0,1]$:

$$W_{\bar{\bar{z}}} - tF(z,u,W) = 0, \quad F(z,u,W) = A_1 W + A_2 \overline{W} + A_4 \quad \text{in } \overline{D_Z}, \tag{3.13}$$

and introduce a bounded open set B_M of the Banach space $B = \hat{C}_\delta(\overline{D_Z})$, whose elements are functions $w(z)$ satisfying the condition

$$w(Z) \in \hat{C}_\delta(\overline{D^+}), \;\; \hat{C}_\delta[w(Z), \overline{D_Z}] < M_{14} = 1 + M_1, \tag{3.14}$$

where δ, M_1 are constants as stated in (3.8). We choose an arbitrary function $w(Z) \in B_M$ and substitute it in the position of W in $F(Z,u,W)$. By Theorem 2.1, we can find a solution $w(z) = \Phi(Z) + \Psi(Z) = w_0(Z) + T(tF)$ of Problem A_1 for the complex equation

$$W_{\overline{Z}} = tF(z,u,w). \tag{3.15}$$

Noting that $y^\tau HF[z(Z), u(z(Z)), w(z(Z))] \in L_\infty(\overline{D_Z})$, where $\tau = 1 - m/2$, from Theorem 2.2, we know that the above solution of Problem A_1 for (3.13) is unique. Denote by $W(z) = T[w,t]$ $(0 \le t \le 1)$ the mapping from $w(z)$ to $W(z)$. On the basis of Theorem 3.1, we know that if $W(z)$ is a solution of Problem A_1 for the equation

$$W_{\overline{Z}} = tF(Z,u,W) \quad \text{in } D_Z,$$

then the function $W(Z)$ satisfies the estimate

$$\hat{C}_\delta[W(Z), \overline{D_Z}] < M_{14}.$$

We can verify the three conditions of the Leray-Schauder theorem:

1. For every $t \in [0,1]$, $T[w,t]$ continuously maps the Banach space B into itself, and is completely continuous on B_M. In fact, arbitrarily select a sequence $w_n(z)$ in B_M, $n = 0,1,2,\dots$, such that $\hat{C}_\delta[w_n - w_0, \overline{D_Z}] \to 0$ as $n \to \infty$. By (C1), (C2), we can derive that $L_\infty[(y-x^n)^\tau X(Z)H(y-x^n)(F(z,u_n,w_n) - F(z,u_0,W_0)), \overline{D_Z}] \to 0$ as $n \to \infty$. Moreover, from $W = T[w_n,t]$, $W_0 = T[w_0,t]$, it is easy to see that $W_n - W_0$ is a solution of Problem A_1 for the complex equation

$$(W_n - W_0)_{\overline{Z}} = t[F(z,u_n,w_n) - F(z,u_0,w_0)] \text{ in } D_Z,$$

and then we can obtain the estimate

$$\hat{C}_\delta[W_n - W_0, \overline{D_Z}] \le 2k_0 \hat{C}[w_n(z) - w_0(z), \overline{D_Z}].$$

Hence $\hat{C}_\delta[W - W_0, \overline{D_Z}] \to 0$ as $n \to \infty$. Afterwards for $w_n(z) \in B_M$, $n = 1,2,\dots$, we can choose a subsequence $\{w_{n_k}(z)\}$ of $\{w_n(z)\}$, such that $\hat{C}[w_{n_k} - w_0, \overline{D_Z}] \to 0$ as $k \to \infty$, where $w_0(z) \in B_M$. Let $W_{n_k} = T[w_{n_k}, t]$ with $n = n_k$, $k = 1,2,\dots$, and $W_0 = T[w_0,t]$, we can verify that

$$\hat{C}_\delta[W_{n_k} - W_0, \overline{D_Z}] \to 0 \quad \text{as } k \to \infty.$$

This shows that $W = T[w,t]$ is completely continuous in B_M. Applying the similar method, we can also prove that for $w(Z) \in B_M$, $T[w,t)$ is uniformly continuous with respect to $t \in [0,1]$.

2. For $t = 0$, it is evident that $W = T[w,0] = \Phi(Z) \in B_M$.

3. From the estimate (3.8), we see that $W = T[w,t](0 \le t \le 1)$ does not have a solution $W(z)$ on the boundary $\partial B_M = \overline{B_M} \backslash B_M$.

Hence there exists a function $W(z) \in B_M$, such that $W(z) = T[W(z), 1]$, and the function $W(z) \in \hat{C}_\delta(\overline{D_Z})$ is just a solution of Problem A_1 for the complex equation (3.1) with $A_3 = 0$.

Next, substituting the solution $W(z)$ into the formula (3.2), it is clear that the function $u(z)$ is a solution of the corresponding Problem Q for linear equation (1.1) in D^+ with $c = 0$. Let $u_0(z)$ be a solution of Problem Q for the linear equation (1.1) with $c = 0$, if $u_0(z)$ satisfies the last N point conditions in (1.8), then the solution is also a solution of Problem P for the equation. Otherwise we can find N solutions $[u_1(z), \dots, u_N(z)]$ of Problem Q for the homogeneous linear equation with $c = 0$ satisfying the boundary conditions

$$\begin{gathered}\operatorname{Re}[\overline{\lambda(z)} u_{l\bar{z}}] = 0, \quad z \in \Gamma,\ u_l(\tilde{a}_0) = 0, \\ \operatorname{Im}[\overline{\lambda(z)} u_{l\bar{z}}]|_{z=z'_k} = \delta_{lk}, \quad l, k = 1, \dots, N.\end{gathered}$$

It is obvious that $U(z) = \sum_{k=1}^N u_k(z) \not\equiv 0$ in D^+, moreover we can verify that

$$J = \begin{vmatrix} u_1(\tilde{a}_1) & \dots & u_N(\tilde{a}_1) \\ \vdots & \ddots & \vdots \\ u_1(\tilde{a}_N) & \dots & u_N(\tilde{a}_N) \end{vmatrix} \neq 0,$$

thus there exist N real constants $d_1, \dots, d_N$, which are not all equal to zero, such that

$$d_1 u_1(\tilde{a}_l) + \dots + d_N u_N(\tilde{a}_l) = u_0(\tilde{a}_l) - c_l, \quad l = 1, \dots, N,$$

thus the function

$$u(z) = u_0(z) - \sum_{k=1}^N d_k u_k(z) \quad \text{in } D^+$$

is just a solution of Problem P for the linear equation (1.1) with $c = 0$. Moreover by using the method of parameter extension and the Schauder fixed-point theorem as stated in [20, Chapter II], we can find a solution of Problem P for the general equation (1.1). □

Theorem 3.3. *Suppose that* (1.1) *satisfies* (C1), (C2)*. Then the oblique derivative problem* (*Problem* P) *for* (1.1) *is solvable.*

Sketch of Proof. The solvability of Problem A_2 can be obtained by the similar methods as in [19, 20], and then the solution $u(z) = v(z) + u_0(z)$ of Problem P for (1.1) in D^- is found. The boundary value $u_y(x)/2 = -\operatorname{Im} W$ of above Problem P on L_0 can be as a part of boundary value of Problem A_1 for (3.1),(3.2), thus from Theorem 3.2, we can find the solution of Problem A_1 for (3.1), (3.2) in $\overline{D^+}$. Hence the existence of Problem P for (1.1) in D is proved. □

Finally we mention that the boundary conditions in (1.8) on $\tilde{L} = L_1 \cup L_3 \cup \dots \cup L_{2N-1}$ are replaced by the corresponding boundary conditions on $\tilde{L}' = L_2 \cup L_4 \cup \dots \cup L_{2N}$, we can also derive the similar results, and the coefficient $K(y - x^n)$ in equation (1.1) can be replaced by the generalized Rassias-Gellerstadt function

$$K(x, y) = \operatorname{sgn}(y - x^n)|y - x^n|^m h(x, y),$$

where the positive numbers m, n are as stated before, and $h(x, y)$ is continuously differentiable positive function.

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