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NAVIER-STOKES EQUATION WITH SLIP-LIKE BOUNDARY CONDITION

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Abstract. The aim of this note is to investigate a time-discretized 2-dimensional Navier-Stokes equation with a slip-like boundary condition, which arises in the melting ice problem. We prove the existence and uniqueness of a weak solution.

1. Introduction

Consider an ice plate, placed upright, whose vertical face is exposed to the air and melting. So this face is covered by the layer of flowing water, and the shapes of the ice and the water-layer vary as time t goes on. Therefore, in the water region, this system can be described by Navier-Stokes equations with two free boundaries of the ice-water interface Γ_1 and the water-air interface Γ_2, whose movements would depend on the unknown functions. However, as a first step of analysis, we here consider the discretized Navier-Stokes equation in the time variable t with the discretizing parameter $\tau > 0$ in the fixed domain Ω with given interfaces Γ_1 and Γ_2. Experiments for this kind of problems can be found in [4] and mathematical treatments for problems similar to ours are discussed by several authors (see e.g. [1, 2]).

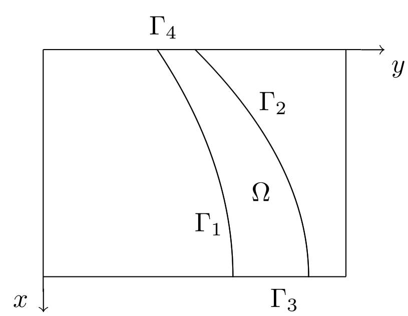

Figure 1. The water region and its boundaries

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Fix the x-axis vertically and downward, the y-axis in the direction of the thickness and outward, and the z-axis orthogonally to the x and y axes. The ice-water interface and the water-air interface are represented by $y = l(x,z)$ and $y = d(x,z)$ respectively. Further suppose that the size of ice plate in z-direction is so large that we can regard l and d as constant in z. So our problem can be formulated in the following 2-dimensional setting.

Define the domain Ω which is occupied by water by

$$\Omega = \{(x,y) : 0 < x < 1,\ l(x) < y < d(x)\},$$

where $l, d \in C^{0,1}([0,1])$; that is, l and d are Lipschitsz continuous on $[0,1]$ and

$$0 \le l(x) < d(x) \le 1 \quad \text{for all } 0 \le x \le 1. \tag{1.1}$$

Hence Ω is of class $C^{0,1}$. Define the ice-water interface Γ_1, the water-air interface Γ_2, the lower boundary Γ_3, and the upper boundary Γ_4 by

$$\begin{gathered}\Gamma_1 = \{(x,y) : 0 \le x \le 1,\ y = l(x)\},\\ \Gamma_2 = \{(x,y) : 0 \le x \le 1,\ y = d(x)\},\\ \Gamma_3 = \{(x,y) : x = 1,\ l(1) \le y \le d(1)\},\\ \Gamma_4 = \{(x,y) : x = 0,\ l(0) \le y \le d(0)\}\end{gathered}$$

respectively.

Our objective is to study the equations:

$$\begin{gathered}\frac{1}{\tau}(\mathbf{u} - \mathbf{u}_0) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p - \nu\Delta\mathbf{u} = \mathbf{g} \quad \text{in } \Omega,\\ \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega,\end{gathered} \tag{1.2}$$

for the fixed discretizing parameter $\tau > 0$ with the boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \tag{1.3}$$

$$U_Y = V_Y = 0 \quad \text{on } \Gamma_2, \tag{1.4}$$

$$v = 0 \quad \text{on } \Gamma_3, \tag{1.5}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_4. \tag{1.6}$$

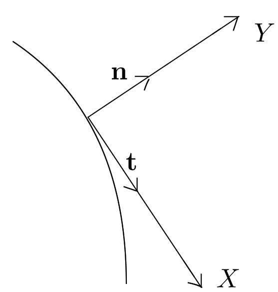

FIGURE 2. The local coordinates

Here, the velocity vector $\mathbf{u} = (u,v)$ and the pressure p are unknown functions of (x,y). The initial velocity $\mathbf{u}_0$, the gravity force $\mathbf{g}$, the density ρ, and the kinematic viscosity ν are given data. The unit time τ is to be determined later. Put $U = \mathbf{u} \cdot \mathbf{t}$, $V = \mathbf{u} \cdot \mathbf{n}$, where $\mathbf{n}$ designates the outer unit normal vector of Γ_2 and $\mathbf{t}$ designates the downward unit tangential vector of Γ_2. Denote by (X, Y) the local coordinate with directions $\mathbf{t}$ and $\mathbf{n}$. The original slip boundary condition is stated as

$$U_Y + V_X = 0 \quad \text{on } \Gamma_2,$$

(see [3]) and condition (1.4) is its linearized version. In the original problem, both Γ_1 and Γ_2 move after the unit time τ. But in our setting, the interfaces stay invariant.

2. Main results

Set

$$\mathbf{V} = \{\mathbf{u} \in \left(H^1(\Omega)\right)^2 : \operatorname{div} \mathbf{u} = 0,\ \mathbf{u} = \mathbf{0} \text{ on } \Gamma_1 \text{ and } \Gamma_4, v = 0 \text{ on } \Gamma_3\},$$
$$\mathbf{H} = \{\mathbf{u} \in (L_2(\Omega))^2 : \operatorname{div} \mathbf{u} = 0\},$$
$$P_\sigma \text{is the orthogonal projection from } (L_2(\Omega))^2 \text{ onto } \mathbf{H},$$
$$\mathbf{L}_4 = \{\mathbf{u} \in (L_4(\Omega))^2 : \operatorname{div} \mathbf{u} = 0\}$$

and let $(\cdot,\cdot)$ and $|\cdot|$ denote the inner product and the norm of the space $\mathbf{H}$.

Define a bounded positive bilinear form $a(\cdot,\cdot)$ on $\mathbf{V}$ by

$$a(\mathbf{u}, \delta\mathbf{u}) := \int_\Omega (\nabla u \cdot \nabla \delta u + \nabla v \cdot \nabla \delta v)\, d\mathbf{x}$$

for $\mathbf{u} = (u, v)$, $\delta\mathbf{u} = (\delta u, \delta v) \in \mathbf{V}$. Also define a trilinear form $b(\cdot,\cdot,\cdot)$ on $(\mathbf{L}_4)^2 \times \mathbf{V}$ by

$$b(\tilde{\mathbf{u}}, \mathbf{u}, \delta\mathbf{u}) := \int_\Omega (\tilde{u} u_x \delta u + \tilde{u} v_x \delta v + \tilde{v} u_y \delta u + \tilde{v} v_y \delta v)\, d\mathbf{x}$$

for $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v}) \in \mathbf{L}_4$, $\mathbf{u} = (u, v) \in \mathbf{V}$, $\delta\mathbf{u} = (\delta u, \delta v) \in \mathbf{L}_4$, where $\mathbf{x} = (x, y)$. We note that Hölder's inequality gives

$$|b(\tilde{\mathbf{u}}, \mathbf{u}, \delta\mathbf{u})| \le |\tilde{\mathbf{u}}|_4 |\nabla \mathbf{u}| |\delta\mathbf{u}|_4 \quad \text{for } \tilde{\mathbf{u}} \in \mathbf{L}_4,\ \mathbf{u} \in \mathbf{V},\ \delta\mathbf{u} \in \mathbf{L}_4. \tag{2.1}$$

Here $|\cdot|_4$ denotes the norm of $\mathbf{L}_4$ and

$$|\nabla \mathbf{u}| = |(|\nabla u|, |\nabla v|)|, \quad |\nabla u| = \Big(\Big|\frac{\partial u}{\partial x}\Big|^2 + \Big|\frac{\partial u}{\partial y}\Big|^2 \Big)^{1/2}.$$

In this paper, we are concerned with weak solutions of (1.2) with boundary conditions (1.3)-(1.6) in the following sense:

$\mathbf{u} \in \mathbf{V}$ is said to be *a weak solution* of (1.2) with boundary conditions (1.3)-(1.6), if the following relation holds.

$$\Big(\frac{1}{\tau}(\mathbf{u} - \mathbf{u}_0) - \mathbf{g}, \delta\mathbf{u}\Big) + b(\mathbf{u}, \mathbf{u}, \delta\mathbf{u}) + \nu a(\mathbf{u}, \delta\mathbf{u}) = 0 \quad \text{for all } \delta\mathbf{u} \in \mathbf{V}. \tag{2.2}$$

We remark that if a sufficiently smooth function $\mathbf{u}$, say in $(C^2(\bar{\Omega}))^2 \cap \mathbf{V}$, satisfies (2.2), then $\mathbf{u}$ should satisfy equation (1.2) and boundary condition (1.4) on Γ_2. In fact, let $\mathbf{u} \in (C^2(\bar{\Omega}))^2 \cap \mathbf{V}$ and let $\mathbf{f} = -\frac{1}{\nu}\big(\frac{1}{\tau}(\mathbf{u} - \mathbf{u}_0) - \mathbf{g} + P_\sigma(\mathbf{u} \cdot \nabla)\mathbf{u}\big) \in \mathbf{H}$, then (2.2) gives

$$a(\mathbf{u}, \delta\mathbf{u}) = (\mathbf{f}, \delta\mathbf{u}) \quad \text{for all } \delta\mathbf{u} \in \mathbf{V}.$$

Here we note that since $v \equiv 0$, $\operatorname{div} \mathbf{u} = u_x + v_y \equiv 0$ on Γ_3, $v_y \equiv 0$ and hence $u_x \equiv 0$ on Γ_3. Consequently, integration by parts yields

$$\begin{aligned} &a(\mathbf{u}, \delta\mathbf{u}) = (\mathbf{f}, \delta\mathbf{u}) \\ &\qquad = \int_\Omega (-\Delta u \delta u - \Delta v \delta v) dx + \int_{\Gamma_2} (u_Y \delta u + v_Y \delta v) dS + \int_{\Gamma_3} u_x \delta u dS \\ &\qquad = \int_\Omega (-\Delta u \delta u - \Delta v \delta v) dx + \int_{\Gamma_2} (U_Y \delta U + V_Y \delta V) dS, \end{aligned} \tag{2.3}$$

where δU and δV are $\mathbf{t}$ and $\mathbf{n}$ components of $\delta\mathbf{u}$ on Γ_2.

If we take $\delta\mathbf{u} \in (C_0^\infty(\Omega))^2 \cap \mathbf{V}$, then the term of the integration on Γ_2 in (2.3) vanishes, whence follows

$$(\mathbf{f}, \delta\mathbf{u}) = (-\Delta\mathbf{u}, \delta\mathbf{u}) \quad \text{for all } \delta\mathbf{u} \in (C_0^\infty(\Omega))^2 \cap \mathbf{V}.$$

This says that $\mathbf{f} = -P_\sigma \Delta\mathbf{u}$ in the sense of distribution. Hence $\mathbf{f} = -P_\sigma \Delta\mathbf{u}$ holds a.e. in Ω, which implies that $\mathbf{u}$ gives a solution of (1.2). Furthermore, plugging this relation into (2.3), we get

$$\int_{\Gamma_2} (U_Y \delta U + V_Y \delta V) dS = 0 \quad \text{for any } \delta\mathbf{u} \in \mathbf{V},$$

whence easily follows that $\mathbf{u}$ should satisfy (1.4).

Our main result is stated as follows.

Theorem 2.1. *Let $\mathbf{u}_0 \in \mathbf{V}$ and $\mathbf{g} \in \mathbf{H}$. There exists a positive number $\tau_0 = \tau_0(|\mathbf{g}|, |\nabla\mathbf{u}_0|)$ such that for all $\tau \in (0, \tau_0]$, (2.2) admits a unique weak solution $\mathbf{u} \in \mathbf{V}$.*

3. Proof of main theorem

By the Gagliardo-Nirenberg estimate and Poincare's lemma, there is a constant K_1 which depends only on Ω such that

$$|\mathbf{u}|_4 \le K_1 |\nabla\mathbf{u}|^{1/2} |\mathbf{u}|^{1/2} \quad \text{for all } \mathbf{u} \in \mathbf{V}. \tag{3.1}$$

Also, by the Sobolev embedding theorem, there is a constant K_2 such that

$$|\mathbf{u}|_4 \le K_2 |\nabla\mathbf{u}| \quad \text{for all } \mathbf{u} \in \mathbf{V}. \tag{3.2}$$

Lemma 3.1. *Let $\mathbf{u}_0, \tilde{\mathbf{u}} \in \mathbf{V}$ and $\mathbf{g} \in \mathbf{H}$. Define $\tau_1 = \tau_1(\nu, K_1, |\tilde{\mathbf{u}}|_4)$ by*

$$\tau_1 = \begin{cases} \frac{16}{27} K_1^{-4} \nu^3 |\tilde{\mathbf{u}}|_4^{-4} & \text{for } \tilde{\mathbf{u}} \neq \mathbf{0}, \\ \text{any positive number} & \text{for } \tilde{\mathbf{u}} = \mathbf{0}, \end{cases} \tag{3.3}$$

where K_1 is the constant appearing in (3.1). Then, for all $\tau \in (0, \tau_1]$, the problem

$$\Big(\frac{1}{\tau}(\mathbf{u} - \mathbf{u}_0) - \mathbf{g}, \delta\mathbf{u}\Big) + b(\tilde{\mathbf{u}}, \mathbf{u}, \delta\mathbf{u}) + \nu a(\mathbf{u}, \delta\mathbf{u}) = 0 \quad \text{for all } \delta\mathbf{u} \in \mathbf{V} \tag{3.4}$$

has a unique weak solution $\mathbf{u} \in \mathbf{V}$.

Proof. We show that the bilinear form

$$N(\mathbf{u}, \delta\mathbf{u}) := \nu a(\mathbf{u}, \delta\mathbf{u}) + \frac{1}{\tau}(\mathbf{u}, \delta\mathbf{u}) + b(\tilde{\mathbf{u}}, \mathbf{u}, \delta\mathbf{u})$$

is bounded and coercive on $\mathbf{V}$. In fact, the estimate

$$|b(\tilde{\mathbf{u}}, \mathbf{u}, \delta\mathbf{u})| \le |\tilde{\mathbf{u}}|_4 |\nabla\mathbf{u}| |\delta\mathbf{u}|_4 \le K_2 |\tilde{\mathbf{u}}|_4 |\nabla\mathbf{u}| |\nabla\delta\mathbf{u}|$$

assures the boundedness of $N(\cdot,\cdot)$ for any $\tau > 0$ and also the coerciveness of $N(\cdot,\cdot)$ for any $\tau > 0$ when $\tilde{\mathbf{u}} = \mathbf{0}$. For the case where $\tilde{\mathbf{u}} \neq \mathbf{0}$, (3.1) and Young's inequality give

$$\begin{aligned}
|b(\tilde{\mathbf{u}},\mathbf{u},\mathbf{u})| &\le |\tilde{\mathbf{u}}|_4 |\nabla \mathbf{u}||\mathbf{u}|_4 \\
&\le |\tilde{\mathbf{u}}|_4 K_1 |\nabla \mathbf{u}|^{\frac{3}{2}} |\mathbf{u}|^{1/2} \\
&\le |\tilde{\mathbf{u}}|_4 K_1 \Big(\frac{3}{4}\eta|\nabla \mathbf{u}|^2 + \frac{1}{4}\eta^{-3}|\mathbf{u}|^2\Big) \quad \forall \eta > 0.
\end{aligned}$$

Hence choosing $\eta = \frac{2\nu}{3K_1|\tilde{\mathbf{u}}|_4}$, we get

$$\begin{aligned}
N(\mathbf{u},\mathbf{u}) &= \nu a(\mathbf{u},\mathbf{u}) + \frac{1}{\tau}(\mathbf{u},\mathbf{u}) + b(\tilde{\mathbf{u}},\mathbf{u},\mathbf{u}) \\
&\ge \Big(\nu - |\tilde{\mathbf{u}}|_4 \frac{3}{4}\eta K_1\Big)|\nabla \mathbf{u}|^2 + \Big(\frac{1}{\tau} - |\tilde{\mathbf{u}}|_4 \frac{1}{4}\eta^{-3} K_1\Big)|\mathbf{u}|^2 \\
&= \frac{\nu}{2}|\nabla \mathbf{u}|^2 + \Big(\frac{1}{\tau} - \frac{27K_1^4|\tilde{\mathbf{u}}|_4^4}{32\nu^3}\Big)|\mathbf{u}|^2 \\
&\ge \frac{\nu}{2}|\nabla \mathbf{u}|^2 + \frac{1}{2\tau}|\mathbf{u}|^2 \quad \forall \tau \in (0,\tau_1],
\end{aligned}$$

which implies the coerciveness of $N(\cdot,\cdot)$. On the other hand,

$$\delta\mathbf{u} \mapsto \Big(\frac{1}{\tau}\mathbf{u}_0 + \mathbf{g}, \delta\mathbf{u}\Big)$$

is a bounded linear functional on $\mathbf{V}$. Therefore, the Lax-Milgram theorem assures the existence of a unique weak solution of (3.4). $\square$

Define an operator $F : \mathbf{L}_4 \to \mathbf{L}_4$ by $F(\tilde{\mathbf{u}}) = \mathbf{u}$, where $\mathbf{u}$ is the unique weak solution of (3.4), whose existence is assured in Lemma 3.1. Then F satisfies the following a priori estimates.

Lemma 3.2. *Let $\tilde{\mathbf{u}} \in \mathbf{L}_4$ and let $0 < \tau \le \tau_1^* := \min(1,\tau_1)$, then $\mathbf{u} = F(\tilde{\mathbf{u}})$ satisfies*

$$|\mathbf{u} - \mathbf{u}_0|_4 \le K_0 := 2K_2\Big(\frac{|\mathbf{g}|^2}{\nu} + \frac{|\nabla \mathbf{u}_0|^2}{2}\Big)^{1/2}, \tag{3.5}$$

$$|\nabla \mathbf{u}| \le \frac{K_0}{K_2}. \tag{3.6}$$

Proof. By substitution $\delta\mathbf{u} = \mathbf{u} - \mathbf{u}_0$ in (3.4), we have

$$0 = \frac{1}{\tau}|\mathbf{u} - \mathbf{u}_0|^2 + \nu a(\mathbf{u},\mathbf{u} - \mathbf{u}_0) + b(\tilde{\mathbf{u}},\mathbf{u},\mathbf{u} - \mathbf{u}_0) - (\mathbf{g},\mathbf{u} - \mathbf{u}_0). \tag{3.7}$$

Here, in view of the definition of τ_1, we get

$$\begin{aligned}
b(\tilde{\mathbf{u}},\mathbf{u},\mathbf{u} - \mathbf{u}_0) &\le |\tilde{\mathbf{u}}|_4|\nabla \mathbf{u}|K_1|\nabla \mathbf{u} - \nabla \mathbf{u}_0|^{1/2}|u - u_0|^{1/2} \\
&\le \frac{\nu}{4}|\nabla \mathbf{u}|^2 + \frac{1}{\nu}|\tilde{\mathbf{u}}|_4^2 K_1^2 |\nabla \mathbf{u} - \nabla \mathbf{u}_0||u - u_0| \\
&\le \frac{\nu}{4}|\nabla \mathbf{u}|^2 + \frac{\nu}{4}|\nabla \mathbf{u} - \nabla \mathbf{u}_0|^2 + \frac{1}{\nu^3}|\tilde{\mathbf{u}}|_4^4 K_1^4 |u - u_0|^2 \\
&\le \frac{\nu}{4}|\nabla \mathbf{u}|^2 + \frac{\nu}{4}|\nabla \mathbf{u} - \nabla \mathbf{u}_0|^2 + \frac{16}{27}\frac{1}{\tau_1}|u - u_0|^2.
\end{aligned}$$

Furthermore we note that

$$\begin{aligned} a(\mathbf{u}, \mathbf{u}-\mathbf{u}_0) &= \frac{1}{2} a((\mathbf{u}+\mathbf{u}_0)+(\mathbf{u}-\mathbf{u}_0), \mathbf{u}-\mathbf{u}_0) \\ &= \frac{1}{2}|\nabla \mathbf{u}|^2+\frac{1}{2}|\nabla \mathbf{u}-\nabla \mathbf{u}_0|^2-\frac{1}{2}|\nabla \mathbf{u}_0|^2, \\ |(\mathbf{g}, \mathbf{u}-\mathbf{u}_0)| &\leq \frac{1}{4\tau}|\mathbf{u}-\mathbf{u}_0|^2+\tau|\mathbf{g}|^2. \end{aligned}$$

Therefore, by (3.7), we have

$$\begin{aligned} &\frac{1}{\tau}|\mathbf{u}-\mathbf{u}_0|^2+\frac{\nu}{2}(|\nabla \mathbf{u}|^2+|\nabla \mathbf{u}-\nabla \mathbf{u}_0|^2) \\ &\leq \ \frac{\nu}{2}|\nabla \mathbf{u}_0|^2+\frac{\nu}{4}(|\nabla \mathbf{u}|^2+|\nabla \mathbf{u}-\nabla \mathbf{u}_0|^2)+\left(\frac{16}{27}+\frac{1}{4}\right) \ \frac{1}{\tau} \ |\mathbf{u}-\mathbf{u}_0|^2+\tau|\mathbf{g}|^2. \end{aligned}$$

Hence we get

$$\begin{aligned} &\frac{K_2^{-2}\ \nu}{4}|\mathbf{u}-\mathbf{u}_0|_4^2+\frac{\nu}{4}|\nabla \mathbf{u}|^2 \\ &\leq \frac{\nu}{4}\left(|\nabla \mathbf{u}-\nabla \mathbf{u}_0|^2+|\nabla \mathbf{u}|^2\right)+\frac{17}{108}\frac{1}{\tau}|\mathbf{u}-\mathbf{u}_0|^2 \leq \tau|\mathbf{g}|^2+\frac{\nu}{2}|\nabla \mathbf{u}_0|^2, \end{aligned}$$

whence follows

$$\begin{aligned} |\mathbf{u}-\mathbf{u}_0|_4 &\leq \Big(\frac{4K_2^2}{\nu}(|\mathbf{g}|^2+\frac{\nu}{2}|\nabla \mathbf{u}_0|^2)\Big)^{1/2} = K_0, \\ |\nabla \mathbf{u}| &\leq \Big(\frac{4}{\nu}|\mathbf{g}|^2+2|\nabla \mathbf{u}_0|^2\Big)^{1/2} = \frac{K_0}{K_2}. \end{aligned}$$

Thus, (3.5) and (3.6) are verified. □

Lemma 3.3. *Set*

$$\tau_0 := \frac{27}{16\cdot 17}\tau_0^*, \quad \tau_0^* := \min\Big(1, \ \frac{16}{27}\ \frac{\nu^3}{K_1^4}\ \frac{1}{(K_0+|\mathbf{u}_0|_4)^4}\Big). \tag{3.8}$$

Then, for any $\tau \in (0, \tau_0]$, $F(\cdot)$ becomes a contraction mapping from

$$\mathbf{L}_4^{K_0} := \{\mathbf{v} \in \mathbf{L}_4 : |\mathbf{v}-\mathbf{u}_0|_4 \leq K_0\}$$

into itself, where K_0 is the constant given in (3.5).

Proof. We first claim that F maps $\mathbf{L}_4^{K_0}$ into itself for all $\tau \in (0, \tau_0^*]$. In fact, for any $\tilde{\mathbf{u}} \in \mathbf{L}_4^{K_0}$, Lemma 3.2 assures that $F(\tilde{\mathbf{u}}) \in \mathbf{L}_4^{K_0}$, provided that $0 < \tau \leq \tau_1^* := \min(1, \tau_1)$ with $\tau_1 = \frac{16}{27}\nu^3 K_1^{-4}\ |\tilde{\mathbf{u}}|_4^{-4}$ for $\tilde{\mathbf{u}} \neq \mathbf{0}$ and $\tau_1 < +\infty$ for $\tilde{\mathbf{u}} = \mathbf{0}$. Since $\tilde{\mathbf{u}} \in \mathbf{L}_4^{K_0}$ implies $|\tilde{\mathbf{u}}|_4 \leq |\mathbf{u}_0|_4 + K_0$, we find

$$\tau_1 = \tau_1(|\tilde{\mathbf{u}}|_4) = \frac{16}{27}\nu^3\ K_1^{-4}\ |\tilde{\mathbf{u}}|_4^{-4} \geq \frac{16}{27}\nu^3\ K_1^{-4}\frac{1}{(K_0+|\mathbf{u}_0|_4)^4} \geq \tau_0^*.$$

Thus we conclude that $F(\tilde{\mathbf{u}}) \in \mathbf{L}_4^{K_0}$ for all $\tau \in (0, \tau_0^*]$.
Now we are going to show that $F(\cdot)$ is a contraction. Let $\tilde{\mathbf{u}}_i \in \mathbf{L}_4^{K_0}$, $\mathbf{u}_i = F(\tilde{\mathbf{u}}_i)$ $(i = 1, 2)$, then as in the proof of Lemma 3.2, it is easy to see that $\mathbf{u}_1 - \mathbf{u}_2$ satisfies

$$\frac{1}{\tau}|\mathbf{u}_1-\mathbf{u}_2|^2+b(\tilde{\mathbf{u}}_1, \mathbf{u}_1, \mathbf{u}_1-\mathbf{u}_2)-b(\tilde{\mathbf{u}}_2, \mathbf{u}_2, \mathbf{u}_1-\mathbf{u}_2)+\nu|\nabla \mathbf{u}_1-\nabla \mathbf{u}_2|^2 = 0. \tag{3.9}$$

We here note that

$$\begin{aligned}
&|b(\tilde{\mathbf{u}}_1, \mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) - b(\tilde{\mathbf{u}}_2, \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)| \\
&= |b(\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2, \mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) + b(\tilde{\mathbf{u}}_2, \mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)| \\
&\le |\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2|_4 |\nabla \mathbf{u}_1| |\mathbf{u}_1 - \mathbf{u}_2|_4 + |\tilde{\mathbf{u}}_2|_4 |\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2| |\mathbf{u}_1 - \mathbf{u}_2|_4
\end{aligned} \tag{3.10}$$

holds. Therefore, by (3.6) and (3.1), we get

$$\begin{aligned}
&|\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2|_4 |\nabla \mathbf{u}_1| |\mathbf{u}_1 - \mathbf{u}_2|_4 \\
&\le |\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2|_4 K_2^{-1} K_0 K_1 |\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2|^{1/2} |\mathbf{u}_1 - \mathbf{u}_2|^{1/2} \\
&\le \frac{K_2^{-2}\nu}{4} |\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2|_4^2 + \frac{1}{\nu} K_0^2 K_1^2 |\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2| \, |\mathbf{u}_1 - \mathbf{u}_2| \\
&\le \frac{K_2^{-2}\nu}{4} |\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2|_4^2 + \frac{\nu}{4} |\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2|^2 + \frac{1}{\nu^3} K_0^4 K_1^4 |\mathbf{u}_1 - \mathbf{u}_2|^2.
\end{aligned} \tag{3.11}$$

Since $|\tilde{\mathbf{u}}_2|_4 \le |\mathbf{u}_0|_4 + K_0$, (3.1) and Young's inequality yield

$$\begin{aligned}
&|\tilde{\mathbf{u}}_2|_4 |\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2| |\mathbf{u}_1 - \mathbf{u}_2|_4 \\
&\le (|\mathbf{u}_0|_4 + K_0) K_1 |\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2|^{\frac{3}{2}} |\mathbf{u}_1 - \mathbf{u}_2|^{1/2} \\
&\le \frac{3}{4}\frac{\nu}{4} |\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2|^2 + \frac{\nu^{-3}}{4} 4^3 (|\mathbf{u}_0|_4 + K_0)^4 K_1^4 |\mathbf{u}_1 - \mathbf{u}_2|^2.
\end{aligned} \tag{3.12}$$

Then, in view of (3.9)-(3.12), we obtain

$$\begin{aligned}
&\frac{1}{\tau} |\mathbf{u}_1 - \mathbf{u}_2|^2 + \nu |\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2|^2 \\
&\le \frac{K_2^{-2}\nu}{4} |\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2|_4^2 + \frac{\nu}{2} |\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2|^2 + \tilde{K} |\mathbf{u}_1 - \mathbf{u}_2|^2, \\
&\tilde{K} = \frac{1}{\nu^3} \left(K_0^4 K_1^4 + 16 (|\mathbf{u}_0|_4 + K_0)^4 K_1^4 \right).
\end{aligned}$$

Since $\tau \in (0, \tau_0]$ assures that $\frac{1}{\tau} \ge \frac{K_1^4}{\nu^3} 17 \left(|\mathbf{u}_0|_4 + K_0\right)^4 \ge \tilde{K}$, we obtain

$$\frac{K_2^{-2}\nu}{2} |\mathbf{u}_1 - \mathbf{u}_2|_4^2 \le \frac{\nu}{2} |\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2|^2 \le \frac{K_2^{-2}\nu}{4} |\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2|_4^2;$$

i.e., $|F(\tilde{\mathbf{u}}_1) - F(\tilde{\mathbf{u}}_2)|_4 \le \frac{1}{\sqrt{2}} |\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2|_4$. □

Proof of Theorem 2.1. It is clear that $\mathbf{L}_4^{K_0}$ is a closed convex subset of $\mathbf{L}_4$. Since $F(\cdot)$ is a contraction mapping from $\mathbf{L}_4^{K_0}$ into itself, F has a unique fixed point $\mathbf{u}$ in $\mathbf{L}_4^{K_0}$, which gives the desired solution of (2.2). □

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