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A COMBUSTION MODEL WITH UNBOUNDED THERMAL CONDUCTIVITY AND REACTANT DIFFUSIVITY IN NON-SMOOTH DOMAINS

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Abstract. In this article, we present a strongly coupled quasilinear parabolic combustion model with unbounded thermal conductivity and reactant diffusivity in arbitrary non-smooth domains. A priori estimates are obtained, and the existence of a unique global strong solution is proved using a Banach fixed point theorem.

1. Introduction

We consider the combustion model

$$\frac{\partial u}{\partial t} - \operatorname{div}(\phi\nabla u) = Qwf(u), \quad \text{in } \Omega\times[0,\infty), \tag{1.1}$$

$$\frac{\partial w}{\partial t} - \operatorname{div}(\psi\nabla w) = -wf(u), \quad \text{in } \Omega\times[0,\infty), \tag{1.2}$$

$$\frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega\times[0,\infty), \tag{1.3}$$

$$u(x,0) = u_0(x), \quad w(x,0) = w_0(x), \tag{1.4}$$

where

$$\phi = h^1(x,t,u,w), \quad \psi = h^2(x,t,u,w). \tag{1.5}$$

Here $Qwf(u)$ and $-wf(u)$ are the reaction kinetics determined by a positive, uniformly bounded, differentiable and Lipschitz continuous function $f(u)$. Further, $f'(u)$ is also assumed to be uniformly bounded and Lipschitz continuous. For our analysis, we shall take

$$0 \le f(u) \le B, \quad |f(u) - f(\tilde u)| \le L|u-\tilde u|, \tag{1.6}$$

$$|f'(u)| \le B', \quad |f'(u) - f'(\tilde u)| \le L'|u-\tilde u|. \tag{1.7}$$

The system (1.1)—(1.4) represents a one step irreversible reaction, reactant → product. $w(x,t)$ is assumed to be the mass fraction of the reactant, $1-w(x,t)$, the mass fraction of the product, $u(x,t)$, the temperature in the reaction vessel, $\phi(x,t,u,w)$, the thermal conductivity, and $\psi(x,t,u,w)$, the reactant diffusivity. We assume that Ω is an open and bounded arbitrary non-smooth domain in $\mathbb{R}^3$. In

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theory, the reactant decomposes at a rate which is proportional to $f(u)w(x,t)$, where $f(u)$ is the approximate number of molecules that have sufficient energy for the reaction to begin.

We remark that (1.1) and (1.2) represent an approximate set of conservation equations for a single-step irreversible reaction. For the complete set of conservation equations, we refer to Fitzgibbon and Martin [1]. We further refer the reader to Frank-Kamenetskii for more information on chemical kinetics and combustion [2].

The literature on combustion models is quite rich. We mention here the work of a few authors. Marion [3], Terman [4] and Wagner [5] considered the existence of travelling wave solutions, taking $\Omega = R$; while their stability versus instability were analysed by Clavin [6] and Sivashinky [7]. Avrin [8, 9] investigated nontravelling wave solutions to the semilinear, constant coefficient case, obtaining existence results on both $\Omega = R$ and $\Omega = \mathbb{R}^n$. With Ω a bounded C^∞ domain in $\mathbb{R}^n$, Fitzgibbon and Martin [1] proved global existence to the system (1.1)-(1.4), with bounded thermal conductivity $\phi(x,t,u)$ and bounded reactant diffusivity $\psi(x,t,w)$; both assumed to be $C^2(\bar{\Omega}\times[0,\infty)\times R; R)$ functions. Further, the reaction kinetics $Qwf(u)$ and $-wf(u)$ were determined by a modified Arrhenius rate function $f(u)$.

In this paper, we shall prove the existence of a unique, global strong solution to the strongly coupled system (1.1)-(1.4), with strictly positive, unbounded Lipschitz continuous (with respect to u and w) ϕ and ψ. We remark that a similar situation arises in [12], where in analysing Reynolds Averaged Navier-Stokes Problem model, the eddy viscosities in the fluid equation are assumed to be nondecreasing, smooth and unbounded functions of the turbulent kinetic energy.

For our analysis, we assume that, for all $(x,t,u.w), (x,t,\tilde{u},\tilde{w}) \in \bar{\Omega}\times[0,T)\times R\times R$,

$$\phi = h^1(x,t,u,w) \geq \sigma_1 > 0, \tag{1.8}$$

$$\psi = h^2(x,t,u,w) \geq \sigma_2 > 0, \tag{1.9}$$

$$|\phi(x,t,u,w) - \phi(x,t,\tilde{u},\tilde{w})| \leq A_1(|u-\tilde{u}| + w - \tilde{w}|), \tag{1.10}$$

$$|\psi(x,t,u,w) - \psi(x,t,\tilde{u},\tilde{w})| \leq A_2(|u-\tilde{u}| + |w - \tilde{w}|), \tag{1.11}$$

$$\|\frac{\partial\phi}{\partial t}\|_{L^2[0,T;L^\infty(\Omega)]} \leq M, \quad \|\frac{\partial\psi}{\partial t}\|_{L^2[0,T;L^\infty(\Omega)]} \leq M, \tag{1.12}$$

$$\|\phi\|_{L^\infty[0,T;H^2(\Omega)]} \leq N, \quad \|\psi\|_{L^\infty[0,T;H^2(\Omega)]} \leq N, \tag{1.13}$$

$$\|D^3\phi\|_{L^2[0,T;L^2(\Omega)]} \leq N_1, \quad \|D^3\psi\|_{L^2[0,T;L^2(\Omega)]} \leq N_1, \tag{1.14}$$

$$\|\frac{\partial\phi}{\partial t}\|_{L^\infty[0,T;H^1(\Omega)]} \leq M_1, \quad \|\frac{\partial\psi}{\partial t}\|_{L^\infty[0,T;H^1(\Omega)]} \leq M_1, \tag{1.15}$$

$$\|D^2\partial_t\phi\|_{L^2[0,T;L^2(\Omega)]} \leq M_2, \quad \|D^2\partial_t\psi\|_{L^2[0,T;L^2(\Omega)]} \leq M_2. \tag{1.16}$$

All first partial derivatives of $h^l(x,t,u,w)$ $(l=1,2)$ with respect to its arguments are assumed bounded by λ_1, and Lipschitz continuous (with respect to u and w) with Lipschitz constant L_1; while all second partial derivatives, except $\frac{\partial^2 h^l}{\partial t^2}$, are assumed to be bounded by λ_2, Lipschitz continuous (with respect to u and w) with Lipschitz constant L_2. For example,

$$|\frac{\partial h^l}{\partial t}| \leq \lambda_1, \quad |\frac{\partial^2 h^l}{\partial t\partial u}| \leq \lambda_2, \tag{1.17}$$

$$|\frac{\partial h^l}{\partial t} - \frac{\partial \tilde{h}^l}{\partial t}| \leq L_1(|u-\tilde{u}| + |w-\tilde{w}|), \tag{1.18}$$

$$|\frac{\partial^2 h^l}{\partial u^2} - \frac{\partial^2 \tilde{h}^l}{\partial u^2}| \le L_2(|u-\tilde{u}| + |w-\tilde{w}|), \tag{1.19}$$

where

$$\tilde{h} = h^l(x,t,\tilde{u},\tilde{w}). \tag{1.20}$$

The letter C written, with or without one or more arguments, shall represent constants which might differ from one step to the other. We write the system (1.1)-(1.4) in the equivalent form

$$\frac{\partial u}{\partial t} - \operatorname{div}(\phi_0 \nabla u) = \operatorname{div}(f_u) + g_u, \quad \text{in } \Omega \times [0,\infty), \tag{1.21}$$

$$\frac{\partial w}{\partial t} - \operatorname{div}(\psi_0 \nabla w) = \operatorname{div}(f_w) + g_w, \quad \text{in } \Omega \times [0,\infty), \tag{1.22}$$

$$\frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0, \quad \text{on } \partial\Omega \times [0,\infty), \tag{1.23}$$

$$u(x,0) = u_0(x), \quad w(x,0) = w_0(x), \tag{1.24}$$

where

$$\phi = h^1(x,t,u,w), \quad \psi = h^2(x,t,u,w), \tag{1.25}$$

$$\phi_0 = h^1(x,0,u_0,w_0), \quad \psi_0 = h^2(x,0,u_0,w_0), \tag{1.26}$$

$$f_u = \int_0^t \partial_s \phi ds \nabla u, \quad g_u = Qwf(u), \tag{1.27}$$

$$f_w = \int_0^t \partial_s \psi ds \nabla w, \quad g_w = -wf(u). \tag{1.28}$$

2. A priori estimates

First, we state and prove three useful Lemmas.

Lemma 2.1. *Let the conditions* (1.8)-(1.20) *hold. Then for* $\theta = \phi, \psi$,

$$|\nabla(\theta-\tilde{\theta})|^2 \le C(\lambda_1, L_1)\big(|u-\tilde{u}|^2 + |w-\tilde{w}|^2 + |\nabla(u-\tilde{u})|^2 + |\nabla(w-\tilde{w})|^2\big), \tag{2.1}$$

$$|\partial_t(\theta-\tilde{\theta})|^2 \le C(\lambda_1, L_1)\big(|u-\tilde{u}|^2 + |w-\tilde{w}|^2 + |\partial_t(u-\tilde{u})|^2 + |\partial_t(w-\tilde{w})|^2\big), \tag{2.2}$$

$$\begin{aligned} |D^2(\theta-\tilde{\theta})|^2 \le C(\lambda_1,\lambda_2,L_1,L_2)\big[&\big(1+|\nabla\tilde{u}|^2+|\nabla\tilde{u}|^4+|\nabla\tilde{u}|^2|\nabla\tilde{w}|^2 \\ &+|D^2\tilde{u}|^2+|\nabla\tilde{w}|^2+|\nabla\tilde{w}|^4+|D^2\tilde{w}|^2\big)\big(|u-\tilde{u}|^2+|w-\tilde{w}|^2\big) \\ &+\big(1+|\nabla u|^2+|\nabla\tilde{u}|^2+|\nabla w|^2+|\nabla\tilde{w}|^2\big)\big(|\nabla(u-\tilde{u})|^2 \\ &+\nabla(w-\tilde{w})|^2\big)+|D^2(u-\tilde{u})|^2+|D^2(w-\tilde{w})|^2\big], \end{aligned} \tag{2.3}$$

$$\begin{aligned} |\partial_t\nabla(\theta-\tilde{\theta})|^2 \le C\left(\lambda_1,\lambda_2,L_1,L_2\right)\big[&\big(1+|\partial_t\tilde{u}|^2+|\partial_t\tilde{w}|^2+|\nabla\tilde{u}|^2+|\nabla\tilde{w}|^2 \\ &+|\partial_t\tilde{u}|^2|\nabla\tilde{u}|^2+|\partial_t\tilde{w}|^2|\nabla\tilde{u}|^2+|\partial_t\tilde{u}|^2|\nabla\tilde{w}|^2+|\partial_t\tilde{w}|^2|\nabla\tilde{w}|^2 \\ &+|\partial_t\nabla\tilde{u}|^2+|\partial_t\nabla\tilde{w}|^2\big)\big(|u-\tilde{u}|^2+|w-\tilde{w}|^2\big)+\big(1+|\partial_t u|^2 \\ &+|\partial_t w|^2\big)\big(|\nabla(u-\tilde{u})|^2+|\nabla(w-\tilde{w})|^2\big)+\big(1+|\nabla\tilde{u}|^2 \\ &+|\nabla\tilde{w}|^2\big)\big(|\partial_t(u-\tilde{u})|^2+|\partial_t(w-\tilde{w})|^2+|\partial_t\nabla(u-\tilde{u})|^2 \\ &+|\partial_t\nabla(w-\tilde{w})|^2\big)\big]. \end{aligned} \tag{2.4}$$

Proof. The estimates are proved using the chain rule to obtain appropriate differential coefficients of $\theta - \tilde{\theta}$, rearranging and using the conditions (1.8)-(1.20). For brevity, we illustrate with the proof of (2.1).

$$\begin{aligned} |\nabla(\theta-\tilde{\theta})|^2 &= \Big|\frac{\partial h^l}{\partial x_i} - \frac{\partial \tilde{h}^l}{\partial x_i} + \frac{\partial h^l}{\partial u}\frac{\partial u}{\partial x_i} - \frac{\partial \tilde{h}^l}{\partial \tilde{u}}\frac{\partial \tilde{u}}{\partial x_i} + \frac{\partial h^l}{\partial w}\frac{\partial w}{\partial x_i} - \frac{\partial \tilde{h}^l}{\partial \tilde{w}}\frac{\partial \tilde{w}}{\partial x_i}\Big|^2 \\ &= \Big|\frac{\partial h^l}{\partial x_i} - \frac{\partial \tilde{h}^l}{\partial x_i} + \frac{\partial h^l}{\partial u}\big(\frac{\partial u}{\partial x_i} - \frac{\partial \tilde{u}}{\partial x_i}\big) + \frac{\partial \tilde{u}}{\partial x_i}\big(\frac{\partial h^l}{\partial u} - \frac{\partial \tilde{h}^l}{\partial \tilde{u}}\big) \\ &\quad + \frac{\partial h^l}{\partial w}\big(\frac{\partial w}{\partial x_i} - \frac{\partial \tilde{w}}{\partial x_i}\big) + \frac{\partial \tilde{w}}{\partial x_i}\big(\frac{\partial h^l}{\partial w} - \frac{\partial \tilde{h}^l}{\partial \tilde{w}}\big)\Big|^2 \\ &\le C(\lambda_1, L_1)\big(|u-\tilde{u}|^2 + |w-\tilde{w}|^2 + |\nabla(u-\tilde{u})|^2 + |\nabla(w-\tilde{w})|^2\big). \end{aligned} \tag{2.5}$$

□

Lemma 2.2. *Let $\Omega \in \mathbb{R}^3$, be open and bounded set with a C^1 boundary; and $u, \tilde{u}, v, \tilde{v}, w \in H^1(\Omega)$. Then there exists $\epsilon > 0$, $C = C(\Omega)$ such that*

$$\int_\Omega |u|^2|v|^2 dx \le C\|u\|^2_{H^1(\Omega)}\Big(\frac{1}{\epsilon}\|v\|^2_{L^2(\Omega)} + \epsilon\|v\|^2_{H^1(\Omega)}\Big), \tag{2.6}$$

$$\begin{aligned} \int_\Omega |uv - \tilde{u}\tilde{v}|^2 dx &\le C\|v\|^2_{H^1(\Omega)}\Big(\frac{1}{\epsilon}\|u-\tilde{u}\|^2_{L^2(\Omega)} + \epsilon\|u-\tilde{u}\|^2_{H^1(\Omega)}\Big) \\ &\quad + C\|\tilde{u}\|^2_{H^1(\Omega)}\Big(\frac{1}{\epsilon}\|v-\tilde{v}\|^2_{L^2(\Omega)} + \epsilon\|v-\tilde{v}\|^2_{H^1(\Omega)}\Big), \end{aligned} \tag{2.7}$$

$$\int_\Omega |u|^2|v|^2|w|^2 dx \le C\|u\|^2_{H^1(\Omega)}\|v\|^2_{H^1(\Omega)}\|w\|^2_{H^1(\Omega)}. \tag{2.8}$$

Proof. Estimates (2.6) and (2.7) are proved in [10]. We therefore render only the proof of (2.8). By Hölder and Sobolev's inequalities, we calculate

$$\begin{aligned} \int_\Omega |u|^2|v|^2|w|^2 dx &\le \Big(\int_\Omega |u|^6 dx\Big)^{1/3}\Big(\int_\Omega |v|^6 dx\Big)^{1/3}\Big(\int_\Omega |w|^6 dx\Big)^{1/3} \\ &\le C\|u\|^2_{H^1(\Omega)}\|v\|^2_{H^1(\Omega)}\|w\|^2_{H^1(\Omega)}. \end{aligned} \tag{2.9}$$

□

Lemma 2.3. *Let $u_0 \in H^3(\Omega)$; and* (1.1), (1.2), (1.4) *and* (1.5) *be satisfied. Then we have the estimate*

$$\begin{aligned} &\|\partial_t u(x,0)\|^2_{H^1(\Omega)} + \|\partial_t w(x,0)\|^2_{H^1(\Omega)} \\ &\le C\Big(\|u_0\|^2_{H^3(\Omega)} + \|w_0\|^2_{H^3(\Omega)} + \|u_0\|^2_{H^2(\Omega)}\|w_0\|^2_{H^1(\Omega)}\Big) \end{aligned} \tag{2.10}$$

where $C = C(Q, \Omega, B, B', N, N_1)$.

Proof. 1. Taking (1.1), (1.2), and (1.5) at $t = 0$ yield

$$\frac{\partial u_0}{\partial t} - \operatorname{div}(\phi_0 \nabla u_0) = Q w_0 f(u_0), \quad \text{in } \Omega, \tag{2.11}$$

$$\frac{\partial w_0}{\partial t} - \operatorname{div}(\psi_0 \nabla w_0) = -w_0 f(u_0), \quad \text{in } \Omega, \tag{2.12}$$

$$u(x,0) = u_0(x), \quad w(x,0) = w_0(x), \tag{2.13}$$

where $\phi_0 = h^1(x, 0, u_0, w_0)$, $\psi_0 = h^2(x, 0, u_0, w_0)$.

2. Using (2.11), we estimate

$$\begin{aligned}
\int_\Omega |\partial_t u_0|^2 dx &= \int_\Omega |\operatorname{div}(\phi_0 \nabla u_0) + Q w_0 f(u_0)|^2 dx \\
&= \int_\Omega |\nabla \phi_0 . \nabla u_0 + \phi_0 \Delta u_0 + Q w_0 f(u_0)|^2 dx \\
&\le C(\Omega)\Big(\|\phi_0\|^2_{H^2(\Omega)} \|u_0\|^2_{H^2(\Omega)} + \|\phi_0\|^2_{C^{0,\frac12}(\Omega)} \|u_0\|^2_{H^2(\Omega)} \\
&\quad + Q^2 |f(u_0)|^2 \|w_0\|^2_{L^2(\Omega)} \Big) \\
&\le C(Q,\Omega,B,N)\Big(\|u_0\|^2_{H^2(\Omega)} + \|w_0\|^2_{L^2(\Omega)} \Big),
\end{aligned} \tag{2.14}$$

where we have used (2.8) of Lemma 2.2 and the Sobolev's embedding $\|\phi_0\|_{C^{0,\frac12}(\Omega)} \le C(\Omega)\|\phi_0\|_{H^2(\Omega)}$.

3. Taking the gradient of (2.11), we obtain the estimate

$$\begin{aligned}
\int_\Omega |\nabla(\partial_t u_0)|^2 dx &= \int_\Omega |\nabla(\operatorname{div}(\phi_0 \nabla u_0)) + \nabla(Q w_0 f(u_0))|^2 dx \\
&= \int_\Omega \big| \nabla^2 \phi_0 . \nabla u_0 + \nabla \phi_0 . \nabla^2 u_0 + \nabla \phi_0 \Delta u_0 + \phi_0 \nabla(\Delta u_0) \\
&\quad + Q \nabla w_0 f(u_0) + w_0 \nabla u_0 f'(u_0) \big|^2 dx \\
&\le \int_\Omega C(\Omega) \Big(\|\phi_0\|^2_{H^3(\Omega)} \|u_0\|^2_{H^2(\Omega)} + \|\phi_0\|^2_{H^2(\Omega)} \|u_0\|^2_{H^3(\Omega)} \\
&\quad + \|\phi_0\|^2_{C^{0,\frac12}(\Omega)} \|u_0\|^2_{H^3(\Omega)} + Q^2 |f(u)|^2 \|w_0\|^2_{H^2(\Omega)} \\
&\quad + Q^2 |f'(u_0)|^2 \|u_0\|^2_{H^2(\Omega)} \|w_0\|^2_{H^1(\Omega)} \Big) \quad \text{(using (2.8))} \\
&\le C\Big(\|u_0\|^2_{H^3(\Omega)} + \|w_0\|^2_{H^2(\Omega)} + \|u_0\|^2_{H^2(\Omega)} \|w_0\|^2_{H^1(\Omega)} \Big)
\end{aligned} \tag{2.15}$$

where $C = C(Q,\Omega,B,B',N,N_1)$, and we have used the Sobolev's embedding $\|\phi_0\|_{C^{0,\frac12}(\Omega)} \le C(\Omega)\|\phi_0\|_{H^2(\Omega)}$. Combining (2.14) and (2.15) yields (2.10),

$$\|\partial_t u_0\|^2_{H^1(\Omega)} \le C\Big(\|u_0\|^2_{H^3(\Omega)} + \|w_0\|^2_{H^2(\Omega)} + \|u_0\|^2_{H^2(\Omega)} \|w_0\|^2_{H^1(\Omega)} \Big) \tag{2.16}$$

where $C = (Q,\Omega,B,B',N,N_1)$

4. Using (2.12), we have an analogous estimate to (2.16), namely

$$\|\partial_t w_0\|^2_{H^1(\Omega)} \le C\Big(\|w_0\|^2_{H^3(\Omega)} + \|w_0\|^2_{H^2(\Omega)} + \|u_0\|^2_{H^2(\Omega)} \|w_0\|^2_{H^1(\Omega)} \Big) \tag{2.17}$$

where $C = (\Omega,B,B',N,N_1)$. Hence, a combination of (2.16) and (2.17) gives (2.10). □

Theorem 2.4. *Let $u_o, w_o \in H^3(\Omega)$. Suppose there exists (u,w) which satisfies the system (1.21)-(1.28). Then $(u,w) \in X \times X$, where*

$$X = L^\infty[0,T;H^2(\Omega)] \cap L^2[0,T;H^3(\Omega)] \cap W^{1,\infty}[0,T;H^1(\Omega)] \cap H^1[0,T;H^2(\Omega)] \tag{2.18}$$

and we have the estimate

$$
\begin{aligned}
&\sup_{[0,T]}\Big(\|u\|^2_{H^2(\Omega)}+\|w\|^2_{H^2(\Omega)}+\|\partial_t u\|^2_{H^1(\Omega)}+\|\partial_t w\|^2_{H^1(\Omega)}\Big)\\
&\quad+\int_0^T\int_\Omega |D^3u|^2\,dx\,dt+\int_0^T\int_\Omega |D^3w|^2\,dx\,dt+\int_0^T\int_\Omega |D^2(\partial_t u)|^2\,dx\,dt\\
&\quad+\int_0^T\int_\Omega |D^2(\partial_t w)|^2\,dx\,dt\\
&\le C(Q,\Omega,B,B',N)\Big(\|u_0\|^2_{H^2(\Omega)}\|w_0\|^2_{H^1(\Omega)}+\|u_0\|^2_{H^3(\Omega)}\\
&\quad+\|w_0\|^2_{H^3(\Omega)}\Big)e^{CT(C_1+1)}=:\Lambda.
\end{aligned}
\tag{2.19}
$$

where $C=C(\Omega,Q,L,L',L_1,L_2,\lambda_1,\lambda_2,M,N,N_1,M_1,M_2)$ *and* C_1 *is the bound in* (2.28).

Proof. 1. We multiply (1.21) by u and integrate by parts over Ω; and use (1.8) and (1.23) to obtain

$$
\begin{aligned}
&\frac12\frac{d}{dt}\Big(\int_\Omega u^2dx\Big)+\sigma_1\int_\Omega|\nabla u|^2dx\\
&\le-\int_\Omega \nabla u f_u dx+\int_\Omega u g_u dx\\
&\le\frac{\sigma_1}{2}\int_\Omega|\nabla u|^2dx+\frac{1}{2\sigma_1}\int_\Omega f_u^2dx+\frac12\int_\Omega u^2dx+\frac12\int_\Omega g_u^2dx
\end{aligned}
$$

or

$$
\frac{d}{dt}\Big(\int_\Omega u^2dx\Big)+\int_\Omega|\nabla u|^2dx\le C(\sigma_1)\Big(\int_\Omega u^2dx+\int_\Omega f_u^2dx+\int_\Omega g_u^2dx\Big). \tag{2.20}
$$

Similarly, we may use (1.9), (1.22), (1.23) to obtain

$$
\frac{d}{dt}\Big(\int_\Omega w^2dx\Big)+\int_\Omega|\nabla w|^2dx\le C(\sigma_2)\Big(\int_\Omega w^2dx+\int_\Omega f_w^2dx+\int_\Omega g_w^2dx\Big). \tag{2.21}
$$

2. We represent the ith differential quotient of size h as

$$
D_i^h\zeta=\frac{\zeta(x+he_i,t)-\zeta(x,t)}{h},\quad (i=1,\dots,n)
$$

for $h\in R$, $0<|h|$; and define $D^h\zeta=(D_1^h\zeta,\dots,D_n^h\zeta)$. Hence, we apply D^h on (1.21) to get

$$
\frac{\partial}{\partial t}(D^hu)-\operatorname{div}\big[\phi_0^h\nabla(D^hu)\big]=\operatorname{div}\big(D^hf_u+D^h\phi_0\nabla u\big)+D^hg_u \tag{2.22}
$$

Multiplying this equality by D^hu, integrating over Ω and applying Young's inequality, we obtain

$$
\begin{aligned}
&\frac{\partial}{dt}\Big(\int_\Omega|D^hu|^2dx\Big)+\int_\Omega|\nabla(D^hu)|^2dx\\
&\le C(\sigma_1)\Big(\epsilon\int_\Omega|D^{-h}(D^hu)|^2dx+\epsilon\int_\Omega|\nabla(D^hu)|^2dx\\
&\quad+\frac1\epsilon\int_\Omega|D^hf_u+D^h\phi_0\nabla u|^2dx+\frac1\epsilon\int_\Omega g_u^2dx\\
&\quad+\int_{\partial\Omega}D^hu.\nabla(D^hu)\mathbf{n}dS+\int_{\partial\Omega}D^hu.\mathbf{n}Qwf(u)dS\Big)
\end{aligned}
\tag{2.23}
$$

Similarly, we have

$$
\begin{aligned}
&\frac{\partial}{dt}\Big(\int_\Omega |D^h w|^2 dx\Big) + \int_\Omega |\nabla(D^h w)|^2 dx \\
&\le C(\sigma_2)\Big(\epsilon\int_\Omega |D^{-h}(D^h w)|^2 dx + \epsilon\int_\Omega |\nabla(D^h w)|^2 dx \\
&\quad + \frac{1}{\epsilon}\int_\Omega |D^h f_w + D^h\phi_0\nabla w|^2 dx + \frac{1}{\epsilon}\int_\Omega g_w^2 dx \\
&\quad + \int_{\partial\Omega} D^h w.\nabla(D^h w)\mathbf{n} dS - \int_{\partial\Omega} D^h w.\mathbf{n} w f(u) dS\Big)
\end{aligned}
\tag{2.24}
$$

3. Using 2.20, 2.21, 2.23 and 2.24, we deduce, after taken the limit as $h \to 0$, chosen $\epsilon > 0$ sufficiently small and simplifying, that

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega u^2 dx + \int_\Omega w^2 dx + \int_\Omega |D_l u|^2 dx + \int_\Omega |D_l w|^2 dx\Big) \\
&\quad + \int_\Omega |\nabla(D_l u)|^2 dx + \int_\Omega |\nabla(D_l w)|^2 dx \\
&\le C\Big(\int_\Omega u^2 dx + \int_\Omega w^2 dx + \int_\Omega |f_u|^2 dx + \int_\Omega |g_u|^2 dx + \int_\Omega |f_w|^2 dx \\
&\quad + \int_\Omega |g_w|^2 dx + \int_\Omega |D_l f_u + \nabla\phi_0\nabla u|^2 dx + \int_\Omega |D_l f_w + \nabla\phi_0\nabla w|^2 dx\Big),
\end{aligned}
\tag{2.25}
$$

where $D_l q = \lim_{h\to 0} D^h q$ and $C = C(\sigma_1, \sigma_2)$.

4. Using (1.12), the definitions (1.27) and (1.28), we use Hölder inequality and (2.6) of Lemma 2.2 to evaluate some terms of (2.25), namely

$$
\begin{aligned}
&\int_\Omega |f_u|^2 dx + \int_\Omega |g_u|^2 dx + \int_\Omega |D_l f_u + \nabla\phi_0\nabla u|^2 dx + \int_\Omega |f_w|^2 dx \\
&+ \int_\Omega |g_w|^2 dx + \int_\Omega |D_l f_w + \nabla\phi_0\nabla w|^2 dx \\
&\le C(\Omega, M, N, B, Q)\Big[\int_\Omega w^2 dx + (\frac{1}{T} + T)\Big(\int_\Omega |D_l u|^2 dx \\
&\quad + \int_\Omega |D_l w|^2 dx\Big) + T\int_\Omega |\nabla(D_l u)|^2 dx + T\int_\Omega |\nabla(D_l w)|^2 dx\Big].
\end{aligned}
\tag{2.26}
$$

Using (2.26) in (2.25), choosing T sufficiently small and simplifying we obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega u^2 dx + \int_\Omega w^2 dx + \int_\Omega |D_l u|^2 dx + \int_\Omega |D_l w|^2 dx\Big) \\
&+ \int_\Omega |\nabla(D_l u)|^2 dx + \int_\Omega |\nabla(D_l w)|^2 dx \\
&\le C\Big(\int_\Omega u^2 dx + \int_\Omega w^2 dx + \int_\Omega |D_l u|^2 dx + \int_\Omega |D_l w|^2 dx\Big),
\end{aligned}
\tag{2.27}
$$

where $C = C(\sigma_1, \sigma_2, \Omega, M, N, B, Q)$. By applying the Gronwall's inequality and carrying out some simple calculations, we deduce from (2.27) the estimate

$$\begin{aligned}
&\sup_{[0,T]} \int_\Omega u^2 dx + \sup_{[0,T]} \int_\Omega w^2 dx + \sup_{[0,T]} \int_\Omega |D_l u|^2 dx + \sup_{[0,T]} \int_\Omega |D_l w|^2 dx \\
&+ \int_0^T \int_\Omega |\nabla(D_l u)|^2 \, dx \, dt + \int_0^T \int_\Omega |\nabla(D_l w)|^2 \, dx \, dt \\
&\leq C\Big(\|u_0\|^2_{H^1(\Omega)} + \|w_0\|^2_{H^1(\Omega)} \Big) =: C_1 < \infty,
\end{aligned} \tag{2.28}$$

where $C = C(\sigma_1, \sigma_2, \Omega, M, N, B, Q, T)$.

5. Let $Y = L^\infty[0, T; L^2(\Omega)] \cap L^2[0, T; H^1_0(\Omega)]$. For some $h > 0$, we infer from (2.28) that

$$\begin{aligned}
\|D^h(u, w)\|^2_{Y \times Y} &:= \sup_{[0,T]} \int_\Omega |D^h u|^2 dx + \sup_{[0,T]} \int_\Omega |D^h w|^2 dx \\
&\quad + \int_0^T \int_\Omega |\nabla(D^h u)|^2 \, dx \, dt + \int_0^T \int_\Omega |\nabla(D^h w)|^2 \, dx \, dt \\
&\leq C_1
\end{aligned} \tag{2.29}$$

Estimate (2.29) implies

$$\sup_h \|D^h(u, w)\|_{Y \times Y} < \infty. \tag{2.30}$$

The space $Y \times Y$ is reflexive. Therefore, there exists by the weak compactness theorem [11], a subsequence $h_k \to 0$, and a function $(U, W) \in Y \times Y$ such that

$$D^{h_k}(u, w) \rightharpoonup (U, W). \tag{2.31}$$

Thus, given a smooth function $\zeta \in C^\infty_c(\Omega \times [0, T))$, we calculate

$$\begin{aligned}
&\int_0^T \int_\Omega (u, w) \nabla \zeta \, dx \, dt \\
&= \lim_{h_k \to 0} \int_0^T \int_\Omega (u, w) D^{-h_k} \zeta \, dx \, dt - \lim_{h_k \to 0} \int_0^T \int_\Omega D^{h_k}(u, w) \zeta \, dx \, dt \\
&= -\int_0^T \int_\Omega (U, W) \zeta \, dx \, dt
\end{aligned}$$

Thus $(U, W) = (\nabla u, \nabla w)$ in the weak sense, and so $(\nabla u, \nabla w) \in Y \times Y$. Hence, by the above deduction, (2.28) implies

$$\begin{aligned}
&\sup_{[0,T]} \|u\|^2_{H^1(\Omega)} + \sup_{[0,T]} \|w\|^2_{H^1(\Omega)} + \int_0^T \int_\Omega |D^2 u|^2 \, dx \, dt + \int_0^T \int_\Omega |D^2 w|^2 \, dx \, dt. \\
&\leq C \left(\|u_0\|^2_{H^1(\Omega)} + \|w_0\|^2_{H^1(\Omega)} \right) =: C_1 < \infty
\end{aligned} \tag{2.32}$$

where $C = C(\sigma_1, \sigma_2, \Omega, M, N, B, Q, T)$

6. We now set out to obtain higher a priori estimates needed to complete part of the proof of Theorem (2.4). In view of estimates (2.32), we may take the limit

as $h \to 0$ in (2.22) and apply D^h on the ensuing equation to obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\nabla(D^h u)\right) - \operatorname{div}\left(\phi_0^h D^2(D^h u)\right) \\
&= \operatorname{div}\left[\nabla\left(D^h f_u\right) + \nabla\left(D^h \phi_0\right)\nabla u + \nabla\phi_0 \nabla(D^h u)\right] + D^h(\nabla g)
\end{aligned} \tag{2.33}
$$

We take the dot product of (2.33) with $\nabla(D^h u)$ to estimate

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega \left|\nabla(D^h u)\right|^2 dx\Big) + \int_\Omega |D^2(D^h u)|^2 dx \\
&\le C(\sigma_1)\Big(\epsilon\int_\Omega \left|D^2(D^h u)\right|^2 + \frac{1}{\epsilon}\int_\Omega \Big|\nabla\left(D^h f_u\right) + \nabla\left(D^h\phi_0\right)\nabla u \\
&\quad + \nabla\phi_0\nabla(D^h u)\Big|^2 dx + \frac{1}{\epsilon}\int_\Omega \left|D^h g_u\right|^2 dx\Big)
\end{aligned} \tag{2.34}
$$

Similarly, we have

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega \left|\nabla(D^h w)\right|^2 dx\Big) + \int_\Omega |D^2(D^h w)|^2 dx \\
&\le C(\sigma_1)\big(\epsilon\int_\Omega \left|D^2(D^h w)\right|^2 + \frac{1}{\epsilon}\int_\Omega \Big|\nabla\left(D^h f_w\right) + \nabla\left(D^h\psi_0\right)\nabla u \\
&\quad + \nabla\psi_0\nabla(D^h w)\Big|^2 dx + \frac{1}{\epsilon}\int_\Omega \left|D^h g_w\right|^2 dx\big)
\end{aligned} \tag{2.35}
$$

Combining the above inequality with (2.34), taking the limit as $h \to 0$, choosing $\epsilon > 0$ sufficiently small, and simplifying, we deduce

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega |\nabla(D_l u)|^2 dx + \int_\Omega |\nabla(D_l w)|^2 dx\Big) + \int_\Omega |D^2(D_l u)|^2 dx + \int_\Omega |D^2(D_l w)|^2 dx \\
&\le C(\sigma_1,\sigma_2)\Big[\int_\Omega |\nabla(D_l f_u) + D^2\phi_0\nabla u + \nabla\phi_0\nabla(D_l u)|^2 dx + \int_\Omega |D_l g_u|^2 dx \\
&\quad + \int_\Omega |\nabla(D_l f_w) + D^2\psi_0\nabla w + \nabla\psi_0\nabla(D_l w)|^2 dx + \int_\Omega |D_l g_w|^2 dx\Big]
\end{aligned} \tag{2.36}
$$

Note that this inequality is analogous to (2.25). Hence, by steps similar to steps (4) and (5), we obtain

$$
\begin{aligned}
&\sup_{[0,T]} \|D^2 u\|^2_{L^2(\Omega)} + \sup_{[0,T]} \|D^2 w\|^2_{L^2(\Omega)} + \int_0^T \int_\Omega |D^3 u|^2\,dx\,dt + \int_0^T\int_\Omega |D^3 w|^2\,dx\,dt. \\
&\le C\Big(\|D^2 u_0\|^2_{L^2(\Omega)} + \|D^2 w_0\|^2_{L^2(\Omega)} + C_1\Big) =: C_2 < \infty,
\end{aligned} \tag{2.37}
$$

where $C = C(\sigma_1, \sigma_2, \Omega, M, N, N_1, B, , B', Q, T)$ and C_1 is the bound in (2.32).

7. The last higher a priori estimates are obtained in this step. Define

$$
\partial^r := \frac{\theta(x,t) - \theta(x,t-r)}{r}, \quad \partial_l\theta := \lim_{r\to 0}\partial^r \tag{2.38}
$$

for all $r > 0$ such that $t - r \in [0, T)$. Applying ∂_l on the system (1.21)-(1.28) yields

$$
\frac{\partial}{\partial t}(\partial_l u) - \operatorname{div}\left(\phi_0\nabla(\partial_l u)\right) = \operatorname{div}(\partial_l f_u) + \partial_l g_u, \quad \text{in } \Omega\times[0,\infty), \tag{2.39}
$$

$$
\frac{\partial}{\partial t}(\partial_l w) - \operatorname{div}\left(\psi_0\nabla(\partial_l w)\right) = \operatorname{div}(\partial_l f_w) + \partial_l g_w, \quad \text{in } \Omega\times[0,\infty), \tag{2.40}
$$

$$\frac{\partial}{\partial n}(\partial_l u) = \frac{\partial}{\partial n}(\partial_l w) = 0 \quad \text{on } \partial\Omega \times [0,\infty), \tag{2.41}$$
$$\partial_l u(x,0) = \partial_l u_0(x), \quad \partial_l w(x,0) = \partial_l w_0(x), \tag{2.42}$$

where

$$\phi = h^1(x,t,u,w), \quad \psi = h^2(x,t,u,w),$$
$$\phi_0 = h^1(x,t,u_0,w_0), \quad \psi_0 = h^2(x,t,u_0,w_0),$$
$$f_u = \int_0^t \partial_s\phi ds \nabla u, \quad g_u = Qwf(u),$$
$$f_w = \int_0^t \partial_s\psi ds \nabla w, \quad g_w = -wf(u).$$

Note that the system (2.39)-(2) is analogous to (1.21)-(1.28). We therefore have an analogue to 2.25, namely

$$\begin{aligned} &\frac{d}{dt}\Big(\|\partial_l u\|^2_{H^1(\Omega)} + \|\partial_l w\|^2_{H^1(\Omega)}\Big) + \int_\Omega |D^2(\partial_l u)|^2 dx + \int_\Omega |D^2(\partial_l w)|^2 dx \\ &\le C(\sigma_1,\sigma_2)\Big(\|\partial_l u\|^2_{H^1(\Omega)} + \|\partial_l w\|^2_{H^1(\Omega)} + \int_\Omega |\partial_l f_u|^2 dx + \int_\Omega |\partial_l g_u|^2 dx \\ &\quad + \int_\Omega |\partial_l f_w|^2 dx + \int_\Omega |\partial_l g_w|^2 dx + \int_\Omega |\nabla(\partial_l f_u) + \nabla\phi_0 \nabla(\partial_l u)|^2 dx \\ &\quad + \int_\Omega |\nabla(\partial_l f_w) + \nabla\psi_0\nabla(\partial_l w)|^2 dx\Big) \end{aligned} \tag{2.43}$$

Hence, by calculations and deductions similar to that of steps (4)-(5), we obtain the estimates

$$\begin{aligned} &\sup_{[0,T]} \|\partial_t u\|^2_{H^1(\Omega)} + \sup_{[0,T]} \|\partial_t w\|^2_{H^1(\Omega)} \\ &+ \int_0^T \int_\Omega |D^2(\partial_t u)|^2 \,dx\,dt + \int_0^T \int_\Omega |D^2(\partial_t u)|^2 \,dx\,dt \\ &\le C\big(\|u_0\|^2_{H^3(\Omega)} + \|w_0\|^2_{H^3(\Omega)} + \|u_0\|^2_{H^2(\Omega)}\|w_0\|^2_{H^1(\Omega)} + C_1 + C_2\big) \\ &=: C_3 < \infty, \end{aligned} \tag{2.44}$$

where $C = C(\sigma_1,\sigma_2,M,N,M_1,M_2,B,B',Q,\Omega,T)$, C_1, and C_2 are the bounds in (2.28) and (2.37) respectively. Here we used the estimate (2.10) in Lemma 2.3.

8. Combining estimates (2.32), (2.37) and (2.44) we complete the proof of Theorem 2.4. □

3. Existence of Solutions

In this section, we shall prove the existence for a global unique strong solution to (1.21)-(1.28) in a subset of the space $X \times X$, which is equipped with the norm

$$\begin{aligned} \|(u,w)\|_{X\times X} = \Big[&\sup_{[0,T]}\Big(\|u\|^2_{H^2(\Omega)} + \|w\|^2_{H^2(\Omega)} + \|\partial_t u\|^2_{H^1(\Omega)} + \|\partial_t w\|^2_{H^1(\Omega)}\Big) \\ &+ \int_0^T\int_\Omega |D^3 u|^2 \,dx\,dt + \int_0^T\int_\Omega |D^3 w|^2\,dx\,dt \\ &+ \int_0^T\int_\Omega |D^2(\partial_t u)|^2\,dx\,dt + \int_0^T\int_\Omega |D^2(\partial_t w)|^2\,dx\,dt\Big]^{1/2} \end{aligned} \tag{3.1}$$

Theorem 3.1. *Let $u_0, w_0 \in H^3(\Omega)$. Then, there exists a global unique strong solution to* (1.21)-(1.28).

Proof. 1. The corresponding fixed point argument system to (1.21)-(1.28) is

$$\frac{\partial \chi}{\partial t} - \operatorname{div}(\phi_0 \nabla \chi) = \operatorname{div}(f_u) + g_u, \quad \text{in } \Omega \times [0, \infty), \tag{3.2}$$

$$\frac{\partial \tau}{\partial t} - \operatorname{div}(\psi_0 \nabla \tau) = \operatorname{div}(f_w) + g_w, \quad \text{in } \Omega \times [0, \infty), \tag{3.3}$$

$$\frac{\partial \chi}{\partial n} = \frac{\partial \chi}{\partial n} = 0, \quad \text{on } \partial\Omega \times [0, \infty), \tag{3.4}$$

$$\chi(x, 0) = u_0(x), \quad \tau(x, 0) = w_0(x), \tag{3.5}$$

where

$$\phi = h^1(x, t, u, w), \quad \psi = h^2(x, t, u, w), \tag{3.6}$$

$$\phi_0 = h^1(x, t, u_0, w_0), \quad \psi_0 = h^2(x, t, u_0, w_0). \tag{3.7}$$

$$f_u = \int_0^t \partial_s \phi ds \nabla u, \quad g_u = Qwf(u), \tag{3.8}$$

$$f_w = \int_0^t \partial_s \psi ds \nabla w, \quad g_w = -wf(u). \tag{3.9}$$

2. Define $A : X \times X \to X \times X$ by setting $A[(u, w)] = (\chi, \tau)$. We shall prove that if $T > 0$ is small enough, then A is a contraction mapping. We choose $(u, w), (\tilde{u}, \tilde{w}) \in X \times X$, and define $A[(u, w)] = (\chi, \tau)$, $A[(\tilde{u}, \tilde{w})] = (\tilde{\chi}, \tilde{\tau})$. For two solutions (χ, τ) and $(\tilde{\chi}, \tilde{\tau})$ of (3.2)- (3.9), we have

$$\frac{\partial}{\partial t}(\chi - \tilde{\chi}) - \operatorname{div}(\phi_0 \nabla (\chi - \tilde{\chi})) = \operatorname{div}\left(f_u - \tilde{f}_{\tilde{u}}\right) + g_u - \tilde{g}_{\tilde{u}}, \quad \text{in } \Omega \times [0, \infty) \tag{3.10}$$

$$\frac{\partial}{\partial t}(\tau - \tilde{\tau}) - \operatorname{div}(\phi_0 \nabla (\tau - \tilde{\tau})) = \operatorname{div}\left(f_w - \tilde{f}_{\tilde{w}}\right) + g_w - \tilde{g}_{\tilde{w}}, \quad \text{in } \Omega \times [0, \infty) \tag{3.11}$$

$$\frac{\partial}{\partial n}(\chi - \tilde{\chi}) = \frac{\partial}{\partial n}(\tau - \tilde{\tau}) = 0, \quad \text{on } \partial\Omega \times [0, \infty), \tag{3.12}$$

$$(\chi - \tilde{\chi})(x, 0) = 0, \quad (\tau - \tilde{\tau})(x, 0) = 0, \tag{3.13}$$

where

$$\phi = h^1(x, t, u, w), \quad \psi = h^2(x, t, u, w), \tag{3.14}$$

$$\tilde{\phi} = h^1(x, t, \tilde{u}, \tilde{w}), \quad \tilde{\psi} = h^2(x, t, \tilde{u}, \tilde{w}), \tag{3.15}$$

$$\phi_0 = h^1(x, t, u_0, w_0), \quad \psi_0 = h^2(x, t, u_0, w_0), \tag{3.16}$$

$$f_u = \int_0^t \partial_s \phi ds \nabla u, \quad g_u = Qwf(u), \tag{3.17}$$

$$\tilde{f}_{\tilde{u}} = \int_0^t \partial_s \tilde{\phi} ds \nabla \tilde{u}, \quad \tilde{g}_{\tilde{u}} = Q\tilde{w}f(\tilde{u}), \tag{3.18}$$

$$f_w = \int_0^t \partial_s \psi ds \nabla w, \quad g_w = -wf(u), \tag{3.19}$$

$$\tilde{f}_{\tilde{w}} = \int_0^t \partial_s \tilde{\psi} ds \nabla \tilde{w}, \quad \tilde{g}_{\tilde{w}} = -\tilde{w}f(\tilde{u}). \tag{3.20}$$

3. Now, the system (3.10)-(3.20) is analogous to the system (1.21)-(1.28). Consequently, we have analogous estimates to a combination of estimates (2.25), (2.36) and (2.43). From the ensuing analogues, choosing $\epsilon > 0$, simplifying and integrating with respect to t over $[0, T]$ we obtain

$$
\begin{aligned}
&\|\chi-\tilde{\chi}\|^2_{H^2(\Omega)} + \|\tau-\tilde{\tau}\|^2_{H^2(\Omega)} + \|\partial_t(\chi-\tilde{\chi})\|^2_{H^1(\Omega)} \\
&+ \|\partial_t(\tau-\tilde{\tau})\|^2_{H^1(\Omega)} + \|D^3(\chi-\tilde{\chi})\|^2_{L^2[0,T;L^2(\Omega)]} \\
&+ \|D^3(\tau-\tilde{\tau})\|^2_{L^2[0,T;L^2(\Omega)]} + \|D^2\partial_t(\chi-\tilde{\chi})\|^2_{L^2[0,T;L^2(\Omega)]} \\
&+ \|D^2\partial_t(\tau-\tilde{\tau})\|^2_{L^2[0,T;L^2(\Omega)]} \\
&\le \int_0^T \Big[\|\chi-\tilde{\chi}\|^2_{H^2(\Omega)} + \|\tau-\tilde{\tau}\|^2_{H^2(\Omega)} + \|\partial_t(\chi-\tilde{\chi})\|^2_{H^1(\Omega)} \\
&\quad + \|\partial_t(\tau-\tilde{\tau})\|^2_{H^1(\Omega)} + \int_\Omega |f_u-\tilde{f}_{\tilde{u}}|^2dx + \int_\Omega |g_u-\tilde{g}_{\tilde{u}}|^2dx \\
&\quad + \int_\Omega |f_w-\tilde{f}_{\tilde{w}}|^2dx + \int_\Omega |g_w-\tilde{g}_{\tilde{w}}|^2dx + \int_\Omega |\nabla(g_u-\tilde{g}_{\tilde{u}})|^2dx \\
&\quad + \int_\Omega |\nabla(g_w-\tilde{g}_{\tilde{w}})|^2dx + \int_\Omega |\nabla(f_u-\tilde{f}_{\tilde{u}}) + \nabla\phi_0\nabla(u-\tilde{u})|^2dx \\
&\quad + \int_\Omega |\nabla(f_w-\tilde{f}_{\tilde{w}}) + \nabla\psi_0\nabla(w-\tilde{w})|^2dx + \int_\Omega |\partial_t(f_u-\tilde{f}_{\tilde{u}})|^2dx + \\
&\quad + \int_\Omega |D^2(f_u-\tilde{f}_{\tilde{u}}) + D^2\phi_0\nabla(u-\tilde{u}) + \nabla\phi_0 D^2(u-\tilde{u})|^2dx \\
&\quad + \int_\Omega |D^2(f_w-\tilde{f}_{\tilde{w}}) + D^2\psi_0\nabla(w-\tilde{w}) + \nabla\psi_0 D^2(w-\tilde{w})|^2dx \\
&\quad + \int_\Omega |\nabla\partial_t(f_u-\tilde{f}_{\tilde{u}}) + \nabla\phi_0\nabla\partial_t(u-\tilde{u})|^2dx + \int_\Omega |\partial_t(f_w-\tilde{f}_{\tilde{w}})|^2dx \\
&\quad + \int_\Omega |\nabla\partial_t(f_w-\tilde{f}_{\tilde{w}}) + \nabla\psi_0\nabla\partial_t(w-\tilde{w})|^2dx + \int_\Omega |\partial_t(g_u-\tilde{g}_{\tilde{u}})|^2dx \\
&\quad + \int_\Omega |\partial_t(g_w-\tilde{g}_{\tilde{w}})|^2dx\Big]dt
\end{aligned} \tag{3.21}
$$

Calculations of the estimates of the necessary terms on the right side of (3.21) are rather lengthy; and not rendered here for brevity. The estimates can be obtained by using the conditions (1.6)-(1.16), Sobolev's embedding and Lemmas 2.1-2.2, after some suitable rearrangement. If the estimates are substituted into (3.21), we deduce, after an application of the integral form of the Gronwall's inequality, the estimates

$$
\begin{aligned}
&\|A[(u,w)] - A[(\tilde{u},\tilde{w})]\|_{X\times X} \\
&\le C\Big(\sqrt{T} + T\sqrt{T} + T\Big)^{1/2}\Big(1 + \|u\|^2_{L^\infty[0,T;H^2(\Omega)]} + \|\tilde{u}\|^2_{L^\infty[0,T;H^2(\Omega)]} \\
&\quad + \|w\|^2_{L^\infty[0,T;H^2(\Omega)]} + \|\tilde{w}|^2_{L^\infty[0,T;H^2(\Omega)]}\Big)^{1/2}\Big(1 + \|(\tilde{u},\tilde{w})\|^2_{X\times X} \\
&\quad + \|\partial_t u\|^2_{L^\infty[0,T;H^1(\Omega)]} + \|\partial_t w\|^2_{L^\infty[0,T;H^1(\Omega)]}\Big)\|(u,w) - (\tilde{u},\tilde{w})\|_{X\times X},
\end{aligned} \tag{3.22}
$$

where C depends on Ω, Q and all the bounds in (1.6)-(1.19).

4. We define a convex set

$$K = \{(u, w)|(u, w) - (u_0, w_0) \in X_0 \times X_0,\ \|(u, w)\|_{X\times X} \le 2\sqrt{\Lambda}\}, \tag{3.23}$$

where $X_0 \times X_0$ is the set where the initial and boundary values are zero; and $\Lambda =$ constant is the bound in (2.19). If $T > 0$ is sufficiently small, we shall show that

$$A[K] \subseteq K,\ \|A[(u, w)] - A[(\tilde{u}, \tilde{w})]\|_{X\times X} \le \frac{1}{2}\|(u, w) - (\tilde{u}, \tilde{w})\|_{X\times X} \tag{3.24}$$

for all $(u, w), (\tilde{u}, \tilde{w}) \in K$. Using (2.19), we have

$$\|A[(u_0, w_0)]\|_{X\times X} = \|\left(\chi(x, 0), \tau(x, 0)\right)\|_{X\times X} = \|(u_0, w_0)\|_{X\times X} \le \sqrt{\Lambda}. \tag{3.25}$$

Thus for $(u, w) \in K$,

$$\begin{aligned}
&\|A[(u, w)]\|_{X\times X}\\
&\le \|A[(u_0, w_0)]\|_{X\times X} + \|A[(u, w)] - A[(u_0, w_0)]\|_{X\times X}\\
&\le \sqrt{\Lambda} + C\left(\sqrt{T} + T\sqrt{T} + T\right)^{1/2}\\
&\quad \times \left(1 + \|u\|^2_{L^\infty[0,T;H^2(\Omega)]} + \|w\|^2_{L^\infty[0,T;H^2(\Omega)]} + \|u_0\|^2_{H^2(\Omega)} + \|w_0|^2_{H^2(\Omega)}\right)^{1/2}\\
&\quad \times \left(1 + \|(u_0, w_0)\|^2_{X\times X} + \|\partial_t u\|^2_{L^\infty[0,T;H^1(\Omega)]} + \|\partial_t w\|^2_{L^\infty[0,T;H^1(\Omega)]}\right)\\
&\quad \times \|(u, w) - (u_0, w_0)\|_{X\times X} \quad \text{(using (3.22), (3.25))}\\
&\le \sqrt{\Lambda} + C\left(\sqrt{T} + T\sqrt{T} + T\right)^{1/2}\left(1 + 4\Lambda + \|u_0\|^2_{H^2(\Omega)} + \|w_0|^2_{H^2(\Omega)}\right)^{1/2}\\
&\times \left(1 + 8\Lambda\right)\left(4\sqrt{\Lambda}\right)\\
&\le 2\sqrt{\Lambda}
\end{aligned}$$

where we used (3.23), for T sufficiently small, such that

$$4C\left(\sqrt{T} + T\sqrt{T} + T\right)^{1/2}\left(1 + 4\Lambda + \|u_0\|^2_{H^2(\Omega)} + \|w_0|^2_{H^2(\Omega)}\right)^{1/2}\left(1 + 8\Lambda\right) < 1.$$

Thus $A[(u, w)] \in K$ for all $(u, w) \in K$. Furthermore, if T is chosen sufficiently small, such that

$$C\left(\sqrt{T} + T\sqrt{T} + T\right)^{1/2}\left(1 + 4\Lambda + \|u_0\|^2_{H^2(\Omega)} + \|w_0|^2_{H^2(\Omega)}\right)^{1/2}\left(1 + 8\Lambda\right) < \frac{1}{2}.$$

Then (3.22) implies

$$\|A[(u, w)] - A[(\tilde{u}, \tilde{w})]\|_{X\times X} \le \frac{1}{2}\|(u, w) - (\tilde{u}, \tilde{w})\|_{X\times X}, \tag{3.26}$$

for all $(u, w), (\tilde{u}, \tilde{w}) \in K$. Thus the mapping A is a strict contraction for sufficiently small $T > 0$. Hence, there exists a unique fixed point $(u, w) \in K$ such that $A[(u, w)] = (u, w)$.

6. Given any $T > 0$, we select $T_1 > 0$ small so that

$$C\left(\sqrt{T_1} + T_1\sqrt{T_1} + T_1\right)^{1/2}\left(1 + 4\Lambda + \|u_0\|^2_{H^2(\Omega)} + \|w_0|^2_{H^2(\Omega)}\right)^{1/2}\left(1 + 8\Lambda\right) < \frac{1}{2}.$$

We can thus apply Banach's fixed point theorem to find a unique strong solution (u, w) of (1.21)-(1.28) existing on the interval $[0, T_1]$. Since $(u(x, t), w(x, t)) \in K$ for almost everywhere $0 \le t \le T_1$, we can upon redefining T_1 if necessary, assume $(u(x, T_1), w(x, T_1)) \in K$. The argument above can then be repeated to extend our unique strong solution to the interval $[T_1, 2T_1]$. Continuing after finitely many steps, we construct a unique strong solution of (1.21)-(1.28) existing on the full interval $[0, T]$. □

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