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POSITIVE PERIODIC SOLUTIONS FOR LIÉNARD TYPE p-LAPLACIAN EQUATIONS

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Abstract. Using topological degree theory, we obtain sufficient conditions for the existence and uniqueness of positive periodic solutions for Liénard type p-Laplacian differential equations.

1. Introduction

In recent years, the existence of periodic solutions for the Duffing equation, Rayleigh equation and Liénard type equation has received a lot of attention. We refer the reader to [3, 5, 6, 7, 8, 9] and the references cited therein. However, as far as we know, fewer papers discuss the existence and uniqueness of positive periodic solutions for Liénard type p-Laplacian differential equation.

In this paper we study the existence and uniqueness of positive T-periodic solutions of the Liénard type p-Laplacian differential equation of the form:

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + g(x(t)) = e(t), \tag{1.1}$$

where $p > 1$ and $\varphi_p : \mathbb{R} \to \mathbb{R}$ is given by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$, f and g are continuous functions defined on $\mathbb{R}$. e is a continuous periodic function defined on $\mathbb{R}$ with period T, and $T > 0$. By using topological degree theory and some analysis skill, we establish some sufficient conditions for the existence and uniqueness of T-periodic solutions of (1.1). The results of this paper are new and they complement previously known results.

2. Preliminaries

For convenience, let us denote

$$C_T^1 := \{x \in C^1(\mathbb{R}, \mathbb{R}) : x \text{ is T-periodic}\},$$

which is a Banach space endowed with the norm $\|x\| = \max\{|x|_\infty, |x'|_\infty\}$, and

$$|x|_\infty = \max_{t\in[0,T]} |x(t)|, \quad |x'|_\infty = \max_{t\in[0,T]} |x'(t)|, \quad |x|_k = \Big(\int_0^T |x(t)|^k dt\Big)^{1/k}.$$

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For the periodic boundary-value problem

$$(\varphi_p(x'(t)))' = \widetilde{f}(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T) \tag{2.1}$$

where $\widetilde{f}$ is a continuous function and T−periodic in the first variable, we have the following result.

Lemma 2.1 ([11]). *Let Ω be an open bounded set in C_T^1, if the following conditions hold*

(i) *For each $\lambda \in (0,1)$ the problem*

$$(\varphi_p(x'(t)))' = \lambda\widetilde{f}(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T)$$

has no solution on $\partial\Omega$;

(ii) *The equation*

$$F(a) := \frac{1}{T}\int_0^T \widetilde{f}(t, a, 0)\,dt = 0$$

has no solution on $\partial\Omega \cap \mathbb{R}$;

(iii) *The Brouwer degree of F satisfies*

$$\deg(F, \Omega \cap \mathbb{R}, 0) \neq 0,$$

Then the periodic boundary value problem (2.1) *has at least one $T-$periodic solution on $\overline{\Omega}$.*

Set

$$\Psi(x) = \int_0^x f(u)du, \quad y(t) = \varphi_p(x'(t)) + \Psi(x(t)). \tag{2.2}$$

We can rewrite (1.1) in the form

$$\begin{gathered} x'(t) = |y(t) - \Psi(x(t))|^{q-1}\operatorname{sign}(y(t) - \Psi(x(t))), \\ y'(t) = -g(x(t)) + e(t), \end{gathered} \tag{2.3}$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2.2. *Suppose that the following condition holds.*

(A1) *g is a continuously differentiable function defined on $\mathbb{R}$, and $g'_x(x) < 0$.*

Then (1.1) *has at most one T-periodic solution.*

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of (1.1). Then, from (2.3), we obtain

$$\begin{gathered} x_i'(t) = |y_i(t) - \Psi(x_i(t))|^{q-1}\operatorname{sign}(y_i(t) - \Psi(x_i(t))), \\ y_i'(t) = -g(x_i(t)) + e(t), \quad i = 1, 2. \end{gathered} \tag{2.4}$$

Set

$$v(t) = x_1(t) - x_2(t), \quad u(t) = y_1(t) - y_2(t), \tag{2.5}$$

it follows from (2.4) that

$$\begin{aligned} v'(t) &= |y_1(t) - \Psi(x_1(t))|^{q-1}\operatorname{sign}(y_1(t) - \Psi(x_1(t))) \\ &\quad - |y_2(t) - \Psi(x_2(t))|^{q-1}\operatorname{sign}(y_2(t) - \Psi(x_2(t))), \\ u'(t) &= -[g(x_1(t)) - g(x_2(t))], \end{aligned} \tag{2.6}$$

Now, we prove that $u(t) \le 0$ for all $t \in \mathbb{R}$. Contrarily, in view of $u \in C^2[0,T]$ and $u(t+T) = u(t)$ for all $t \in \mathbb{R}$, we obtain

$$\max_{t\in\mathbb{R}} u(t) > 0.$$

Then, there must exist $t^* \in \mathbb{R}$ (for convenience, we can choose $t^* \in (0,T)$) such that

$$u(t^*) = \max_{t\in[0,\ T]} u(t) = \max_{t\in\mathbb{R}} u(t) > 0,$$

which, together with $g'(x) < 0$, implies that

$$\begin{aligned} u'(t^*) &= -[g(x_1(t^*)) - g(x_2(t^*))] = 0, \quad x_1(t^*) = x_2(t^*), \\ u''(t^*) &= (-(g(x_1(t)) - g(x_2(t))))'|_{t=t^*} \\ &= -[g'_x(x_1(t^*))x'_1(t^*) - g'_x(x_2(t^*))x'_2(t^*)] \le 0. \end{aligned} \tag{2.7}$$

Then

$$\begin{aligned} u''(t^*) &= -g'_x(x_1(t^*))[x'_1(t^*) - x'_2(t^*)] \\ &= -g'_x(x_1(t^*))[|y_1(t^*) - \Psi(x_1(t^*))|^{q-1}\operatorname{sign}(y_1(t^*) - \Psi(x_1(t^*))) \\ &\quad - |y_2(t^*) - \Psi(x_2(t^*))|^{q-1}\operatorname{sign}(y_2(t^*) - \Psi(x_2(t^*)))] \\ &= -g'_x(x_1(t^*))[|y_1(t^*) - \Psi(x_1(t^*))|^{q-1}\operatorname{sign}(y_1(t^*) - \Psi(x_1(t^*))) \\ &\quad - |y_2(t^*) - \Psi(x_1(t^*))|^{q-1}\operatorname{sign}(y_2(t^*) - \Psi(x_1(t^*)))]. \end{aligned} \tag{2.8}$$

In view of

$$-g'_x(x_1(t^*)) > 0, \quad u(t^*) = y_1(t^*) - y_2(t^*) > 0, \tag{2.9}$$

and

$$\begin{aligned} &|y_1(t^*) - \Psi(x_1(t^*))|^{q-1}\operatorname{sign}(y_1(t^*) - \Psi(x_1(t^*))) \\ &- |y_2(t^*) - \Psi(x_1(t^*))|^{q-1}\operatorname{sign}(y_2(t^*) - \Psi(x_1(t^*))) > 0. \end{aligned}$$

It follows from (2.8) that

$$\begin{aligned} u''(t^*) &= -g'_x(x_1(t^*))[|y_1(t^*) - \Psi(x_1(t^*))|^{q-1}\operatorname{sign}(y_1(t^*) - \Psi(x_1(t^*))) \\ &\quad - |y_2(t^*) - \Psi(x_1(t^*))|^{q-1}\operatorname{sign}(y_2(t^*) - \Psi(x_1(t^*)))] > 0, \end{aligned} \tag{2.10}$$

which contradicts the second equation of (2.7). This contradiction implies that

$$u(t) = y_1(t) - y_2(t) \le 0 \quad \text{for all } t \in \mathbb{R}.$$

By using a similar argument, we can also show that

$$y_2(t) - y_1(t) \le 0 \quad \text{for all } t \in \mathbb{R}.$$

Therefore, we obtain $y_2(t) \equiv y_1(t)$ for all $t \in \mathbb{R}$. Then, from (2.6), we get

$$g(x_1(t)) - g(x_2(t)) \equiv 0 \quad \text{for all } t \in \mathbb{R},$$

again from $g'_x(x) < 0$, which implies that $x_2(t) \equiv x_1(t)$ for all $t \in \mathbb{R}$. Hence, (1.1) has at most one T-periodic solution. The proof is complete. □

3. Main Results

Using Lemmas 2.1 and 2.2, we obtain our main results:

Theorem 3.1. *Let* (A1) *hold. Suppose that there exists a positive constant* d *such that*

(A2) $g(x) - e(t) < 0$ *for* $x > d$ *and* $t \in \mathbb{R}$, $g(x) - e(t) > 0$ *for* $x \le 0$ *and* $t \in \mathbb{R}$.

Then (1.1) *has a unique positive* T*-periodic solution.*

Proof. Consider the homotopic equation of (1.1) as follows:

$$(\varphi_p(x'(t)))' + \lambda f(x(t))x'(t) + \lambda g(x(t)) = \lambda e(t), \quad \lambda \in (0,1) \tag{3.1}$$

By Lemma 2.2, and (A1), it is easy to see that (1.1) has at most one positive T-periodic solution. Thus, to prove Theorem 3.1, it suffices to show that (1.1) has at least one T-periodic solution. To do this, we shall apply Lemma 2.1. Firstly, we will claim that the set of all possible T-periodic solutions of (3.1) is bounded.

Let $x(t) \in C_T^1$ be an arbitrary solution of (3.1) with period T. By integrating two sides of (3.1) over $[0,T]$, and noticing that $x'(0) = x'(T)$, we have

$$\int_0^T (g(x(t)) - e(t))\, dt = 0. \tag{3.2}$$

As $x(0) = x(T)$, there exists $t_0 \in [0,T]$ such that $x'(t_0) = 0$, while $\varphi_p(0) = 0$ we see

$$\begin{aligned} |\varphi_p(x'(t))| &= |\int_{t_0}^t (\varphi_p(x'(s)))'\, ds| \\ &\le \lambda \int_0^T |f(x(t))||x'(t)|\, dt + \lambda \int_0^T |g(x(t))|\, dt + \lambda \int_0^T |e(t)|\, dt, \end{aligned} \tag{3.3}$$

where $t \in [t_0, t_0 + T]$.

From (3.2), there exists a $\bar{\xi} \in [0,T]$ such that $g(x(\bar{\xi})) - e(\bar{\xi}) = 0$. In view of (A2), we obtain $|x(\bar{\xi})| \le d$. Then, we have

$$|x(t)| = |x(\bar{\xi}) + \int_{\bar{\xi}}^t x'(s)ds| \le d + \int_{\bar{\xi}}^t |x'(s)|ds, \ t \in [\bar{\xi}, \quad \bar{\xi} + T],$$

and

$$|x(t)| = |x(t-T)| = |x(\bar{\xi}) - \int_{t-T}^{\bar{\xi}} x'(s)ds| \le d + \int_{t-T}^{\bar{\xi}} |x'(s)|ds, t \in [\bar{\xi}, \quad \bar{\xi} + T].$$

Combining the above two inequalities, we obtain

$$\begin{aligned} |x|_\infty &= \max_{t\in[0,T]} |x(t)| = \max_{t\in[\bar{\xi},\ \bar{\xi}+T]} |x(t)| \\ &\le \max_{t\in[\bar{\xi},\ \bar{\xi}+T]} \{d + \frac{1}{2}(\int_{\bar{\xi}}^t |x'(s)|ds + \int_{t-T}^{\bar{\xi}} |x'(s)|ds)\} \\ &\le d + \frac{1}{2}\int_0^T |x'(s)|ds. \end{aligned} \tag{3.4}$$

Denote

$$E_1 = \{t : t \in [0,T],\ |x(t)| > d\}, \quad E_2 = \{t : t \in [0,T], |x(t)| \le d\}.$$

Since $x(t)$ is T-periodic, multiplying $x(t)$ and (3.1) and then integrating it from 0 to T, in view of (A2), we get

$$\begin{aligned}\int_0^T |x'(t)|^p dt &= -\int_0^T (\varphi_p(x'(t)))'x(t)dt \\ &= \lambda\int_{E_1}[g(x(t))-e(t)]x(t)dt + \lambda\int_{E_2}[g(x(t))-e(t)]x(t)dt \\ &\le \int_0^T \max\{|g(x(t))-e(t)| : t\in\mathbb{R}, |x(t)|\le d\}|x(t)|dt \\ &\le DT|x|_\infty,\end{aligned} \tag{3.5}$$

where $D=\max\{|g(x)-e(t)| : |x|\le d,\ t\in\mathbb{R}\}$.

For $x(t)\in C(\mathbb{R},\mathbb{R})$ with $x(t+T)=x(t)$, and $0<r\le s$, by using Hölder inequality, we obtain

$$\begin{aligned}\Big(\frac{1}{T}\int_0^T |x(t)|^r dt\Big)^{1/r} &\le \Big(\frac{1}{T}(\int_0^T (|x(t)|^r)^{s/r}dt)^{r/s}(\int_0^T 1dt)^{\frac{s-r}{s}}\Big)^{1/r} \\ &= \Big(\frac{1}{T}\int_0^T |x(t)|^s dt\Big)^{1/s},\end{aligned}$$

this implies that

$$|x|_r \le T^{\frac{s-r}{rs}}|x|_s, \quad \text{for } 0<r\le s. \tag{3.6}$$

Then, in view of (3.4), (3.5) and (3.6), we can get

$$\begin{aligned}(\int_0^T |x'(t)|dt)^p &\le T^{p-1}|x'(t)|_p^p = T^{p-1}\int_0^T |x'(t)|^p dt \\ &\le T^{p-1}DT|x|_\infty \\ &\le T^pD(d+\frac{1}{2}\int_0^T |x'(s)|ds).\end{aligned} \tag{3.7}$$

Since $p>1$, the above inequality allows as to choose a positive constant M_1 such that

$$\int_0^T |x'(s)|ds \le M_1, \quad |x|_\infty \le d+\frac{1}{2}\int_0^T |x'(s)|ds \le M_1.$$

In view of (3.3), we have

$$\begin{aligned}|x'|_\infty^{p-1} &= \max_{t\in[0,T]}\{|\varphi_p(x'(t))|\} \\ &= \max_{t\in[t_0,t_0+T]}\{|\int_{t_0}^t (\varphi_p(x'(s)))'\,ds|\} \\ &\le \int_0^T |f(x(t))||x'(t)|\,dt + \int_0^T |g(x(t))|\,dt + \int_0^T |e(t)|\,dt \\ &\le [\max\{|f(x)| : |x|\le M_1\}]M_1 + T[\max\{|g(x)| : |x|\le M_1\} + |e|_\infty].\end{aligned} \tag{3.8}$$

Thus, we can get some positive constant $M_2 > M_1+1$ such that for all $t\in\mathbb{R}$, $|x'(t)|\le M_2$. Set $\Omega=\{x\in C_T^1 : \|x\|\le M_2+1\}$, then we know that (3.1) has no solution on $\partial\Omega$ as $\lambda\in(0,1)$ and when $x(t)\in\partial\Omega\cap\mathbb{R}$, $x(t)=M_2+1$ or

$x(t) = -M_2 - 1$, from (A_2) , we can see that

$$\frac{1}{T}\int_0^T \{-g(M_2+1)+e(t)\}\,dt = -\frac{1}{T}\int_0^T \{g(M_2+1)-e(t)\}\,dt > 0,$$
$$\frac{1}{T}\int_0^T \{-g(-M_2-1)+e(t)\}\,dt = -\frac{1}{T}\int_0^T \{g(-M_2-1)-e(t)\}\,dt < 0,$$

so condition (ii) is also satisfied. Set

$$H(x,\mu) = \mu x - (1-\mu)\frac{1}{T}\int_0^T \{g(x)-e(t)\}\,dt,$$

and when $x \in \partial\Omega \cap \mathbb{R}$, $\mu \in [0,1]$ we have

$$xH(x,\mu) = \mu x^2 - (1-\mu)x\frac{1}{T}\int_0^T \{g(x)-e(t)\}\,dt > 0.$$

Thus $H(x,\mu)$ is a homotopic transformation and

$$\deg\{F,\Omega\cap\mathbb{R},0\} = \deg\{-\frac{1}{T}\int_0^T \{g(x)-e(t)\}\,dt,\Omega\cap\mathbb{R},0\} = \deg\{x,\Omega\cap\mathbb{R},0\} \neq 0.$$

so condition (iii) is satisfied. In view of the previous Lemma 2.1, there exists at least one solution with period T.

Suppose that $x(t)$ is the T-periodic solution of (1.1). Let $\bar{t}$ be the global minimum point of $x(t)$ on $[0,T]$. Then $x'(\bar{t}) = 0$ and we claim that

$$(\varphi_p(x'(\bar{t})))' = (|x'(\bar{t})|^{p-2}x'(\bar{t}))' \geq 0. \tag{3.9}$$

Assume, by way of contradiction, that (3.9) does not hold. Then

$$(\varphi_p(x'(\bar{t})))' = (|x'(\bar{t})|^{p-2}x'(\bar{t}))' < 0,$$

and there exists $\varepsilon > 0$ such that $(\varphi_p(x'(t)))' = (|x'(t)|^{p-2}x'(t))' < 0$ for $t \in (\bar{t}-\varepsilon, \bar{t}+\varepsilon)$. Therefore, $\varphi_p(x'(t)) = |x'(t)|^{p-2}x'(t)$ is strictly decreasing for $t \in (\bar{t}-\varepsilon, \bar{t}+\varepsilon)$, which implies that $x'(t)$ is strictly decreasing for $t \in (\bar{t}-\varepsilon, \bar{t}+\varepsilon)$. This contradicts the definition of $\bar{t}$. Thus, (3.9) is true. From (1.1) and (3.9), we have

$$g(x(\bar{t})) - e(\bar{t}) \leq 0. \tag{3.10}$$

In view of (A2), (3.10) implies $x(\bar{t}) > 0$. Thus,

$$x(t) \geq \min_{t\in[0,\ T]} x(t) = x(\bar{t}) > 0, \quad \text{for all } t \in \mathbb{R},$$

which implies that (1.1) has at least one positive solution with period T. This completes the proof. □

4. An Example

As an application, let us consider the following equation

$$(\varphi_p x'(t))' + e^{x(t)}x'(t) - (x^9(t) + x(t) - 12) = \cos^2 t, \tag{4.1}$$

where $p = \sqrt{5}$. We can easily check the conditions (A1) and (A2) hold. By Theorem 3.1, equation (4.1) has a unique positive 2π-periodic solution.

Since the periodic solution of p-Laplacian equation (4.1) is positive, one can easily see that the results of this paper are essentially new.

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