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OSCILLATION AND NONOSCILLATION CRITERIA FOR TWO-DIMENSIONAL TIME-SCALE SYSTEMS OF FIRST-ORDER NONLINEAR DYNAMIC EQUATIONS

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Abstract. Oscillation criteria for two-dimensional difference and differential systems of first-order linear difference equations are generalized and extended to nonlinear dynamic equations on arbitrary time scales. This unifies and extends under one theory previous linear results from discrete and continuous systems. An example is given illustrating that a key theorem is sharp on all time scales.

1. Prelude

Jiang and Tang [14] establish sufficient conditions for the oscillation of the linear two-dimensional difference system

$$\Delta x_n = p_n y_n, \quad \Delta y_{n-1} = -q_n x_n, \quad n \in \mathbb{Z}, \tag{1.1}$$

where $\{p_n\}$, $\{q_n\}$ are nonnegative real sequences and Δ is the forward difference operator given via $\Delta x_n = x_{n+1} - x_n$; see also Li [16]. The system (1.1) may be viewed as a discrete analogue of the differential system

$$x'(t) = p(t)y(t), \quad y'(t) = -q(t)x(t), \quad t \in \mathbb{R}, \tag{1.2}$$

investigated by Lomtatidze and Partsvania [17].

Oscillation questions in difference and differential equations are an interesting and important area of study in modern mathematics. Furthermore, within the past two decades, these two related but distinct areas have begun to be combined under a powerful, more robust and general theory titled dynamic equations on time scales, a theory introduced by Hilger [13]. We wish to generalize (1.1) and (1.2) to the nonlinear time-scale system of the form

$$x^{\Delta}(t) = p(t)f\big(y(t)\big), \quad y^{\Delta}(t) = -q(t)g\big(x(t)\big), \quad t \in \mathbb{T}, \tag{1.3}$$

where $\mathbb{T}$ is an arbitrary time scale (any nonempty closed set of real numbers) unbounded above, with the special cases of $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$ yielding systems

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related to (1.1) and (1.2), respectively, as important corollaries. In this general time-scale setting, Δ represents the delta (or Hilger) derivative [4, Definition 1.10],

$$z^{\Delta}(t) := \lim_{s\to t}\frac{z(\sigma(t))-z(s)}{\sigma(t)-s} = \lim_{s\to t}\frac{z^{\sigma}(t)-z(s)}{\sigma(t)-s},$$

where $\sigma(t) := \inf\{s\in\mathbb{T} : s>t\}$ is the forward jump operator, $\mu(t) := \sigma(t)-t$ is the forward graininess function, and $z\circ\sigma$ is abbreviated as z^{σ}. In particular, if $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=t$ and $x^{\Delta}=x'$, while if $\mathbb{T}=h\mathbb{Z}$ for any $h>0$, then $\sigma(t)=t+h$ and

$$x^{\Delta}(t)=\frac{x(t+h)-x(t)}{h}.$$

A function $f:\mathbb{T}\to\mathbb{R}$ is right-dense continuous provided it is continuous at each right-dense point $t\in\mathbb{T}$ (a point where $\sigma(t)=t$) and has a left-sided limit at each left-dense point $t\in\mathbb{T}$. The set of right-dense continuous functions on $\mathbb{T}$ is denoted by $C_{\rm rd}(\mathbb{T})$. It can be shown that any right-dense continuous function f has an antiderivative (a function $\Phi:\mathbb{T}\to\mathbb{R}$ with the property $\Phi^{\Delta}(t)=f(t)$ for all $t\in\mathbb{T}$). Then the Cauchy delta integral of f is defined by

$$\int_{t_0}^{t_1} f(t)\Delta t = \Phi(t_1)-\Phi(t_0),$$

where Φ is an antiderivative of f on $\mathbb{T}$. For example, if $\mathbb{T}=\mathbb{Z}$, then

$$\int_{t_0}^{t_1} f(t)\Delta t = \sum_{t=t_0}^{t_1-1} f(t),$$

and if $\mathbb{T}=\mathbb{R}$, then

$$\int_{t_0}^{t_1} f(t)\Delta t = \int_{t_0}^{t_1} f(t)dt.$$

Throughout we assume that $t_0<t_1$ are points in $\mathbb{T}$, and define the time-scale interval $[t_0,t_1]_{\mathbb{T}}=\{t\in\mathbb{T} : t_0\le t\le t_1\}$. Other time-scale intervals are defined similarly.

Time scales and time-scale notation are introduced well in the fundamental texts by Bohner and Peterson [4, 5]. For related oscillation and nonoscillation results for dynamic equations on time scales, please see some of the many recent papers in this area, including Akin-Bohner, Bohner, and Saker [1], Bohner, Erbe, and Peterson [3], Bohner and Saker [6, 7], Bohner and Tisdell [8], Erbe and Peterson [9], Erbe, Peterson, and Saker [10, 11, 12], and Saker [18]. Recent papers on extensions of second-order self-adjoint equations to dynamic systems on time scales include Anderson and Hall [2], and Xu and Xu [19].

2. Preliminary results on oscillation

Let $\mathbb{T}$ be a time scale that is unbounded above, and let $t_0\in\mathbb{T}$. In (1.3), assume $p:\mathbb{T}\to\mathbb{R}$ is right-dense continuous with $p>0$ on $[t_0,\infty)_{\mathbb{T}}$, and $q:\mathbb{T}\to\mathbb{R}$ is a right-dense continuous function satisfying $q\ge 0$ on $[t_0,\infty)_{\mathbb{T}}$ with q nonzero and not eventually zero; note that p and q are delta integrable. Moreover, we assume that $f,g:\mathbb{R}\to\mathbb{R}$ are nondecreasing continuous functions that satisfy $zf(z), zg(z)>0$ for $z\ne 0$, and that there exist positive real numbers F and G such that $f(y)/y\ge F$ and $g(x)/x\ge G$.

A solution (x,y) of (1.3) is oscillatory if both component functions x and y are oscillatory, that is to say neither eventually positive nor eventually negative;

otherwise, the solution is nonoscillatory. The nonlinear dynamic system (1.3) is oscillatory if all its solutions are oscillatory.

Lemma 2.1. *The component functions x and y of a nonoscillatory solution (x, y) of* (1.3) *are themselves nonoscillatory.*

Proof. Assume to the contrary that x oscillates but y is eventually positive. Then $x^\Delta = pf(y) > 0$ eventually, so that $x(t) > 0$ or $x(t) < 0$ for all large $t \in \mathbb{T}$, a contradiction. The case where y is eventually negative is similar. Likewise, assuming that y oscillates while x is eventually positive or eventually negative leads to comparable contradictions. □

Lemma 2.2. *If*

$$\int_{t_0}^{\infty} p(r)\Delta r = \infty \quad \text{and} \quad \int_{t_0}^{\infty} q(s)\Delta s = \infty, \tag{2.1}$$

then each solution of nonlinear system (1.3) *is oscillatory.*

Proof. Let (x, y) be a nonoscillatory solution of (1.3). First assume that $x > 0$; then $y^\Delta = -qg(x) \le 0$, and in view of Lemma 2.1, y must be of constant sign eventually. If $y(t_1) < 0$ for some $t_1 \in [t_0, \infty)_\mathbb{T}$, then $y < 0$ on $[t_1, \infty)_\mathbb{T}$ and $x^\Delta = pf(y) < 0$ on $[t_1, \infty)_\mathbb{T}$; after delta integrating from t_1 to t, we have

$$x(t) = x(t_1) + \int_{t_1}^{t} p(r) f\big(y(r)\big)\Delta r. \tag{2.2}$$

Since y is negative and nonincreasing, and $yf(y) > 0$ with f nondecreasing, we know $f(y) < 0$, and by the first assumption in (2.1) the right-hand side of (2.2) tends to $-\infty$, a contradiction of $x > 0$. Consequently, $y > 0$ with $y^\Delta \le 0$ on $[t_0, \infty)_\mathbb{T}$, and $x^\Delta > 0$ on $[t_0, \infty)_\mathbb{T}$ by the first equation of (1.3). Thus there exists a constant $c > 0$ and $t_1 \in [t_0, \infty)_\mathbb{T}$ such that $x(t) \ge c$ for $t \in [t_1, \infty)_\mathbb{T}$. Delta integrating the second equation of (1.3), we obtain

$$g(c) \int_{t_1}^{\infty} q(s)\Delta s \le y(t_1) < \infty,$$

and this contradicts the second assumption in (2.1). Similar contradictions are reached for $x < 0$. □

Lemma 2.3. *If*

$$\int_{t_0}^{\infty} p(r)\Delta r < \infty \quad \text{and} \quad \int_{t_0}^{\infty} q(s)\Delta s < \infty, \tag{2.3}$$

then nonlinear system (1.3) *is nonoscillatory.*

Proof. Suppose that (2.3) holds. Then there exists $t_1 \in [t_0, \infty)_\mathbb{T}$ such that

$$\int_{t_1}^{\infty} p(r) f\Big(1 + g(2) \int_{r}^{\infty} q(s)\Delta s\Big)\Delta r < 1. \tag{2.4}$$

Let $\mathcal{B} = \mathrm{C}_{\mathrm{rd}}(\mathbb{T})$ be the Banach space of right-dense continuous functions on $\mathbb{T}$, with norm $\|x\| = \sup_{t \ge t_1, t \in \mathbb{T}} |x(t)|$ and the usual pointwise ordering $\le$. Define a subset $\mathcal{S}$ of $\mathcal{B}$ as follows:

$$\mathcal{S} = \{x \in \mathcal{B} : 1 \le x(t) \le 2,\ t \in [t_1, \infty)_\mathbb{T}\}.$$

For any subset $\mathcal{Q}$ of $\mathcal{S}$, we have that $\inf\mathcal{Q} \in \mathcal{S}$ and $\sup\mathcal{Q} \in \mathcal{S}$. Let $L : \mathcal{S} \to \mathcal{B}$ be the functional given via

$$(Lx)(t) = 1 + \int_{t_1}^{t} p(r) f\Big(1 + \int_{r}^{\infty} q(s) g\big(x(s)\big)\Delta s\Big)\Delta r, \quad t \in [t_1,\infty)_{\mathbb{T}}.$$

By the assumptions on $x \in \mathcal{S}$ and p and q and the fact that f and g are nondecreasing, $(Lx)(t) \ge 1$ for all $t \in [t_1,\infty)_{\mathbb{T}}$, and

$$(Lx)(t) \le 1 + \int_{t_1}^{t} p(r) f\Big(1 + \int_{r}^{\infty} q(s) g(2)\Delta s\Big)\Delta r \le 2$$

by (2.4). Moreover,

$$(Lx)^{\Delta}(t) = p(t) f\Big(1 + \int_{t}^{\infty} q(s) g\big(x(s)\big)\Delta s\Big) > 0, \tag{2.5}$$

ensuring that $L : \mathcal{S} \to \mathcal{S}$ is increasing. By Knaster's fixed-point theorem [15], we can conclude that there exists an $x \in \mathcal{S}$ such that $x = Lx$. If we let

$$y(t) = 1 + \int_{t}^{\infty} q(s) g\big(x(s)\big)\Delta s, \quad t \in [t_1,\infty)_{\mathbb{T}}$$

using the fixed point $x \in \mathcal{S}$, then we have

$$x^{\Delta}(t) = (Lx)^{\Delta}(t) = p(t) f\big(y(t)\big) \quad \text{and} \quad y^{\Delta}(t) = -q(t) g\big(x(t)\big)$$

for $t \in [t_1,\infty)_{\mathbb{T}}$ by using (2.5). Thus (x, y) is a nonoscillatory solution of (1.3). $\square$

In view of Lemmas 2.2 and 2.3, respectively, we could assume that either

$$\int_{t_0}^{\infty} p(r)\Delta r = \infty \quad \text{and} \quad \int_{t_0}^{\infty} q(s)\Delta s < \infty, \quad \text{or} \tag{2.6}$$

$$\int_{t_0}^{\infty} p(r)\Delta r < \infty \quad \text{and} \quad \int_{t_0}^{\infty} q(s)\Delta s = \infty; \tag{2.7}$$

in fact, we will focus on (2.6). Moreover, in preparation for what follows, we introduce the following notation. Let

$$P(t) := \int_{t_0}^{t} p(r)\Delta r. \tag{2.8}$$

Lemma 2.4. *Assume that* (2.6) *holds, P is given by* (2.8)*, and $\lambda \in [0,1)$ is a real number. If*

$$\lim_{t\to\infty} \frac{\mu(t)p(t)}{P(t)} = 0, \quad \Big(\textit{equivalently, } \lim_{t\to\infty} \frac{P^{\sigma}(t)}{P(t)} = 1\Big) \tag{2.9}$$

then given $\epsilon > 0$ there exists a $t_1 \equiv t_1(\epsilon) \in (t_0,\infty)_{\mathbb{T}}$ such that for any $t \in [t_1,\infty)_{\mathbb{T}}$,

$$\int_{t}^{\infty} \frac{\big[\big(P^{\lambda}\big)^{\Delta}(r)\big]^2}{p(r)P^{\lambda}(r)}\Delta r \le \frac{\lambda^2}{1-\lambda}\left(1+\epsilon\right)^{2-\lambda} P^{\lambda-1}(t), \quad \textit{and} \tag{2.10}$$

$$\int_{t}^{\infty} \frac{p(r)}{P^{2-\lambda}(r)}\Delta r \le \frac{(1+\epsilon)^{2-\lambda}}{1-\lambda} P^{\lambda-1}(t). \tag{2.11}$$

Proof. For $r \in (t_0,\infty)_{\mathbb{T}}$, by the chain rule [4, Theorem 1.90] we have

$$\big(P^{\lambda}\big)^{\Delta}(r) = \begin{cases} \dfrac{P^{\lambda}(\sigma(r)) - P^{\lambda}(r)}{\mu(r)} & : \mu(r) > 0, \\ \lambda p(r) P^{\lambda-1}(r) & : \mu(r) = 0. \end{cases}$$

By [4, Theorem 1.16 (iv)], $\mu P^{\Delta} = P^{\sigma} - P$, so that $\mu p = P^{\sigma} - P$ on $\mathbb{T}$. If $r \in (t_0, \infty)_{\mathbb{T}}$ is a right-scattered point, then $\mu(r) > 0$ and, suppressing the r,

$$\begin{aligned}
\frac{\left[\left(P^{\lambda}\right)^{\Delta}\right]^2}{pP^{\lambda}} &= \frac{p}{\mu^2 p^2 P^{\lambda}} \left(\left(P^{\sigma}\right)^{\lambda} - P^{\lambda}\right)^2 \\
&= \frac{p}{P^{\lambda}} \Big(\frac{\left(P^{\sigma}\right)^{\lambda} - P^{\lambda}}{P^{\sigma} - P}\Big)^2 \\
&\overset{\text{MVT}}{=} \frac{p}{P^{\lambda}} \left(\lambda \xi^{\lambda-1}\right)^2, \quad \xi \in \big(P(r), P^{\sigma}(r)\big)_{\mathbb{R}} \\
&\le \frac{p\lambda^2}{P^{\lambda}} P^{2\lambda-2}, \quad \lambda - 1 < 0 \\
&= \lambda^2 p P^{\lambda-2}.
\end{aligned}$$

If $r \in (t_0, \infty)_{\mathbb{T}}$ is a right-dense point, then $\mu(r) = 0$ and

$$\frac{\left[\left(P^{\lambda}\right)^{\Delta}\right]^2}{pP^{\lambda}} = \frac{\left[\lambda p P^{\lambda-1}\right]^2}{pP^{\lambda}} = \lambda^2 p P^{\lambda-2}.$$

It follows that in either case,

$$\frac{\Big[\left(P^{\lambda}\right)^{\Delta}(r)\Big]^2}{p(r)P^{\lambda}(r)} \le \lambda^2 p(r) P^{\lambda-2}(r), \quad r \in (t_0, \infty)_{\mathbb{T}}. \tag{2.12}$$

Similarly, if $r \in (t_0, \infty)_{\mathbb{T}}$ is a right-scattered point, then once again $\mu(r) > 0$ and, suppressing the r,

$$\begin{aligned}
-\left(P^{\lambda-1}\right)^{\Delta} &= \frac{-p}{\mu p} \left(\left(P^{\sigma}\right)^{\lambda-1} - P^{\lambda-1}\right) \\
&= -p\Big(\frac{\left(P^{\sigma}\right)^{\lambda-1} - P^{\lambda-1}}{P^{\sigma} - P}\Big) \\
&\overset{\text{MVT}}{=} p(1-\lambda)\eta^{\lambda-2}, \quad \eta \in \big(P(r), P^{\sigma}(r)\big)_{\mathbb{R}} \\
&\ge p(1-\lambda)\left(P^{\sigma}\right)^{\lambda-2}.
\end{aligned}$$

If r is a right-dense point, then $P^{\sigma} = P$, $\mu(r) = 0$, and $p(1-\lambda)P^{\lambda-2} = -\left(P^{\lambda-1}\right)^{\Delta}$. Summarizing, in either case we have

$$-\left(P^{\lambda-1}\right)^{\Delta} \ge p(1-\lambda)\left(P^{\sigma}\right)^{\lambda-2}, \quad r \in (t_0, \infty)_{\mathbb{T}}. \tag{2.13}$$

Combining (2.12) and (2.13), we see that

$$\frac{\left[\left(P^{\lambda}\right)^{\Delta}(r)\right]^2}{p(r)P^{\lambda}(r)} \le \frac{\lambda^2}{1-\lambda}\Big(\frac{P(r)}{P^{\sigma}(r)}\Big)^{\lambda-2}\left[-\left(P^{\lambda-1}\right)^{\Delta}(r)\right].$$

By (2.9), given $\epsilon > 0$ there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $P^{\sigma}/P \le (1+\epsilon)$ on $[t_1, \infty)_{\mathbb{T}}$. Consequently, for any $t \in [t_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned}
\int_t^{\infty} \frac{\left[\left(P^{\lambda}\right)^{\Delta}(r)\right]^2}{p(r)P^{\lambda}(r)} \Delta r &\le \frac{\lambda^2}{1-\lambda}(1+\epsilon)^{2-\lambda} \int_t^{\infty} \left[-\left(P^{\lambda-1}\right)^{\Delta}(r)\right] \Delta r \\
&\overset{(2.6),(2.8)}{=} \frac{\lambda^2}{1-\lambda}(1+\epsilon)^{2-\lambda} P^{\lambda-1}(t),
\end{aligned}$$

which is (2.10). Moreover, again for any $r \in [t_1,\infty)_{\mathbb{T}}$,

$$\begin{aligned}\frac{p(r)}{P^{2-\lambda}(r)} &= \frac{p(r)}{P^{2-\lambda}(\sigma(r))}\frac{P^{2-\lambda}(\sigma(r))}{P^{2-\lambda}(r)} \le (1+\epsilon)^{2-\lambda}\frac{p(r)}{P^{2-\lambda}(\sigma(r))} \\ &\overset{(2.13)}{\le} \frac{(1+\epsilon)^{2-\lambda}}{\lambda-1}\big(P^{\lambda-1}\big)^{\Delta}(r).\end{aligned} \tag{2.14}$$

Delta integrating (2.14) from t to infinity, we obtain

$$\int_t^\infty \frac{p(r)}{P^{2-\lambda}(r)}\Delta r \le \frac{(1+\epsilon)^{2-\lambda}}{\lambda-1}\int_t^\infty \big(P^{\lambda-1}\big)^{\Delta}(r)\Delta r \overset{(2.6),(2.8)}{=} \frac{(1+\epsilon)^{2-\lambda}}{1-\lambda}P^{\lambda-1}(t),$$

which is (2.11). □

Note that if $\mathbb{T}=\mathbb{R}$, then (2.9) is automatically satisfied, as $\mu(t)\equiv 0$.

Lemma 2.5. *Assume that* (2.6) *holds, that P is given by* (2.8)*, and that* (2.9) *holds. If for some real number $\lambda<1$ we have*

$$\int_{t_1}^\infty q(r)P^{\lambda}(r)\Delta r = \infty \quad \textit{for} \quad t_1 \ge \sigma(t_0), \tag{2.15}$$

then nonlinear system (1.3) *is oscillatory.*

Proof. By Lemma 2.3, we can focus on $\lambda \in (0,1)$. Assume that (x,y) is a nonoscillatory solution of nonlinear system (1.3), and assume that $x>0$ on $[t_0,\infty)_{\mathbb{T}}$; the case where $x<0$ on $[t_0,\infty)_{\mathbb{T}}$ is similar and consequently omitted. As in the proof of Lemma 2.2, $y>0$ with $y^\Delta \le 0$ and $x^\Delta > 0$ on $[t_0,\infty)_{\mathbb{T}}$. Let $w := y/x$. Then $w>0$, and suppressing the argument, we have by the delta quotient rule and (1.3) that on $[t_0,\infty)_{\mathbb{T}}$,

$$w^\Delta = \frac{x^\sigma y^\Delta - y^\sigma x^\Delta}{xx^\sigma} = -q\frac{g(x)}{x} - pww^\sigma\frac{f(y)}{y} \le -qG - pww^\sigma F < 0. \tag{2.16}$$

In fact this gives us

$$w^\Delta \le -qG - p(w^\sigma)^2F, \tag{2.17}$$

and from the previous line we obtain on $[t_0,\infty)_{\mathbb{T}}$ that

$$\Big(\frac{1}{w}\Big)^\Delta = \frac{-w^\Delta}{ww^\sigma} \ge \frac{qG+pww^\sigma F}{ww^\sigma} \ge pF;$$

delta integrating from t_0 to t we see that

$$1 > 1 - \frac{w(t)}{w(t_0)} \ge Fw(t)\int_{t_0}^t p(r)\Delta r = Fw(t)P(t) \ge 0, \quad t\in[t_0,\infty)_{\mathbb{T}}. \tag{2.18}$$

Again by the mean value theorem, $\big(P^\lambda\big)^\Delta \le \lambda p P^{\lambda-1}$ for $\lambda \in (0,1)$. Multiplying (2.17) by P^λ and delta integrating from $t_1 \ge \sigma(t_0)$ to t we obtain

$$\begin{aligned}
G\int_{t_1}^t q(r)P^\lambda(r)\Delta r &\le -\int_{t_1}^t P^\lambda(r)w^\Delta(r)\Delta r - F\int_{t_1}^t p(r)P^\lambda(r)(w^\sigma)^2(r)\Delta r\\
&\overset{\text{parts}}{=} -P^\lambda(t)w(t) + P^\lambda(t_1)w(t_1) + \int_{t_1}^t \big(P^\lambda\big)^\Delta(r)w^\sigma(r)\Delta r\\
&\quad - F\int_{t_1}^t p(r)P^\lambda(r)(w^\sigma)^2(r)\Delta r\\
&\le -P^\lambda(t)w(t) + P^\lambda(t_1)w(t_1) + \int_{t_1}^t \lambda p(r)P^{\lambda-1}(r)w^\sigma(r)\Delta r\\
&\quad - F\int_{t_1}^t p(r)P^\lambda(r)(w^\sigma)^2(r)\Delta r\\
&= -P^\lambda(t)w(t) + P^\lambda(t_1)w(t_1)\\
&\quad + \int_{t_1}^t p(r)P^{\lambda-2}(r)\big[P(r)w^\sigma(r)\Big(\lambda - FP(r)w^\sigma(r)\Big)\big]\Delta r.
\end{aligned} \tag{2.19}$$

Since by (2.18) we have

$$0 \le FP(t)w^\sigma(t) \le FP(t)w(t) < 1, \quad t \in [t_0,\infty)_{\mathbb{T}}, \tag{2.20}$$

there exists a positive real number k such that

$$\big|P(r)w^\sigma(r)\big(\lambda - FP(r)w^\sigma(r)\big)\big| < k.$$

As a result we have $\lim_{t\to\infty} -P^\lambda(t)w(t) = 0$ by (2.18) for $0<\lambda<1$, and

$$\begin{aligned}
\Big|\int_{t_1}^t p(r)P^{\lambda-2}(r)\left[P(r)w^\sigma(r)\big(\lambda - FP(r)w^\sigma(r)\big)\right]\Delta r\Big| &< k\int_{t_1}^\infty p(r)P^{\lambda-2}(r)\Delta r\\
&\overset{(2.11)}{\le} k\frac{(1+\epsilon)^{2-\lambda}}{1-\lambda}P^{\lambda-1}(t_1)
\end{aligned}$$

for all $t \in [t_1,\infty)_{\mathbb{T}}$. Therefore,

$$\int_{t_1}^\infty q(r)P^\lambda(r)\Delta r < \infty,$$

a contradiction of (2.15). $\square$

Due to (2.6) and the establishment of Lemma 2.5, we will henceforth restrict our analysis to the case

$$\int_{t_0}^\infty p(r)\Delta r = \infty, \quad \text{and} \quad \int_{t_1}^\infty q(r)P^\lambda(r)\Delta r < \infty \quad \text{for} \quad \lambda < 1, \quad t_1 \ge \sigma(t_0). \tag{2.21}$$

We also adopt the following notation. Set

$$g(t,\lambda) := G\begin{cases} P^{1-\lambda}(t)\int_t^\infty q(r)P^\lambda(r)\Delta r & : \lambda < 1,\\ P^{1-\lambda}(t)\int_{t_0}^t q(r)P^\lambda(r)\Delta r & : \lambda > 1.\end{cases}$$

In either case, take

$$g_*(\lambda) := \liminf_{t\to\infty} g(t,\lambda) \quad \text{and} \quad g^*(\lambda) := \limsup_{t\to\infty} g(t,\lambda).$$

Lemma 2.6. *Assume that* (2.21) *holds, that P is given by* (2.8)*, and that* (2.9) *holds. If (x,y) is a nonoscillatory solution of nonlinear system* (1.3)*, then*

$$\liminf_{t\to\infty} w(t)P(t) \ge \frac{1}{2F}\Big(1-\sqrt{1-4Fg_*(0)}\Big), \tag{2.22}$$

$$\limsup_{t\to\infty} w(t)P(t) \le \frac{1}{2F}\Big(1+\sqrt{1-4Fg_*(2)}\Big), \tag{2.23}$$

where again $w := y/x$.

Proof. By (2.18), we can introduce the constants

$$r := \liminf_{t\to\infty} w(t)P(t), \quad R := \limsup_{t\to\infty} w(t)P(t), \tag{2.24}$$

and by (2.21), we must have

$$\lim_{t\to\infty} w(t) = 0. \tag{2.25}$$

From (2.16) we have $w^\Delta \le -qG - pww^\sigma F$; delta integrate this from t to ∞, use (2.25), and multiply by P to see that

$$w(t)P(t) \ge GP(t)\int_t^\infty q(\tau)\Delta\tau + FP(t)\int_t^\infty p(\tau)w(\tau)w^\sigma(\tau)\Delta\tau \tag{2.26}$$

holds for $t\in[t_1,\infty)_{\mathbb{T}}$. From (2.24) this yields

$$r \ge g_*(0). \tag{2.27}$$

This time multiply (2.17) by P^2 and delta integrate from t_1 to t to get

$$\begin{aligned} G\int_{t_1}^t q(\tau)P^2(\tau)\Delta\tau &\le -\int_{t_1}^t P^2(\tau)w^\Delta(\tau)\Delta\tau - F\int_{t_1}^t p(\tau)P^2(\tau)\big(w^\sigma\big)^2(\tau)\Delta\tau \\ &= -P^2(t)w(t) + P^2(t_1)w(t_1) + \int_{t_1}^t (P^2)^\Delta(\tau)w^\sigma(\tau)\Delta\tau \\ &\quad - F\int_{t_1}^t p(\tau)P^2(\tau)\big(w^\sigma\big)^2(\tau)\Delta\tau \\ &= -P^2(t)w(t) + P^2(t_1)w(t_1) + \int_{t_1}^t \mu(\tau)p^2(\tau)w^\sigma(\tau)\Delta\tau \\ &\quad + \int_{t_1}^t p(\tau)P(\tau)w^\sigma(\tau)[2-FP(\tau)w^\sigma(\tau)]\Delta\tau \end{aligned}$$

for $t\in[t_1,\infty)_{\mathbb{T}}$, which leads to

$$\begin{aligned} &w(t)P(t) \\ &\le -GP^{-1}(t)\int_{t_1}^t q(\tau)P^2(\tau)\Delta\tau + P^{-1}(t)\int_{t_1}^t \mu(\tau)p^2(\tau)w^\sigma(\tau)\Delta\tau \\ &\quad + P^{-1}(t)P^2(t_1)w(t_1) + P^{-1}(t)\int_{t_1}^t p(\tau)P(\tau)w^\sigma(\tau)[2-FP(\tau)w^\sigma(\tau)]\Delta\tau. \end{aligned} \tag{2.28}$$

Using (2.20), $0<(1-FPw^\sigma)^2$, leading to $FPw^\sigma[2-FPw^\sigma]<1$. Thus for large $t\in\mathbb{T}$,

$$P^{-1}(t)\int_{t_1}^t p(\tau)P(\tau)w^\sigma(\tau)\left[2-P(\tau)w^\sigma(\tau)\right]\Delta\tau \le 1/F.$$

Applying L'Hôpital's rule [4, Theorem 1.120], (2.20) again, and (2.9) we have

$$0 \le \lim_{t\to\infty} \frac{\int_{t_1}^t \mu(\tau)p^2(\tau)w^\sigma(\tau)\Delta\tau}{P(t)} = \lim_{t\to\infty} \mu(t)p(t)w^\sigma(t) \le \lim_{t\to\infty} \frac{\mu(t)p(t)}{P(t)} = 0.$$

Altogether then, inequality (2.28) implies that

$$R \le 1/F - g_*(2). \tag{2.29}$$

If $g_*(0) = 0 = g_*(2)$, then estimates (2.22) and (2.23) follow directly from (2.27) and (2.29), respectively. Thus we pick a real number $\epsilon \in \big(0, \min\{g_*(0), g_*(2)\}\big)$ and $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that for $t \in [t_2,\infty)_{\mathbb{T}}$,

$$r - \epsilon < w(t)P(t) < R + \epsilon, \quad w(t)P(t) \ge GP(t)\int_t^\infty q(\tau)\Delta\tau > g_*(0) - \epsilon,$$
$$GP^{-1}(t)\int_{t_0}^t q(\tau)P^2(\tau)\Delta\tau > g_*(2) - \epsilon.$$

From (2.26) and L'Hôpital's rule we have for $t \in [t_2,\infty)_{\mathbb{T}}$ that

$$w(t)P(t) \ge g_*(0) - \epsilon + F(r-\epsilon)^2.$$

Multiply (2.16) by P^2 and delta integrate from t_1 to t to see that this leads to

$$\begin{aligned} w(t)P(t) \le{}& -GP^{-1}(t)\int_{t_1}^t q(\tau)P^2(\tau)\Delta\tau + P^{-1}(t)\int_{t_1}^t \mu(\tau)p^2(\tau)w^\sigma(\tau)\Delta\tau \\ &+ P^{-1}(t)P^2(t_1)w(t_1) + P^{-1}(t)\int_{t_1}^t p(\tau)P(\tau)w^\sigma(\tau)[2 - Fw(\tau)P(\tau)]\Delta\tau. \end{aligned} \tag{2.30}$$

From (2.30) we have for $t \in [t_2,\infty)_{\mathbb{T}}$ that

$$w(t)P(t) \le \frac{P^2(t_1)w(t_1) + \int_{t_1}^t \mu(\tau)p^2(\tau)w^\sigma(\tau)\Delta\tau}{P(t)} - g_*(2) + \epsilon + (R+\epsilon)(2 - F(R+\epsilon)),$$

since $Fw^\sigma P \le FwP < 1$. These two inequalities lead to

$$r \ge g_*(0) + Fr^2, \quad R \le R(2 - FR) - g_*(2). \tag{2.31}$$

Consequently,

$$r \ge \frac{1}{2F}\left(1 - \sqrt{1 - 4Fg_*(0)}\right), \quad R \le \frac{1}{2F}\left(1 + \sqrt{1 - 4Fg_*(2)}\right),$$

and the lemma is proven. $\square$

3. Main oscillation results

We use the lemmas obtained previously to prove our main results.

Theorem 3.1. *Assume that* (2.21) *holds, that* P *is given by* (2.8), *and that* (2.9) *holds. If*

$$g_*(0) = \liminf_{t\to\infty} P(t)\int_t^\infty q(\tau)\Delta\tau > \frac{1}{4F}, \quad \text{or} \tag{3.1}$$

$$g_*(2) = \liminf_{t\to\infty} \frac{1}{P(t)}\int_{t_0}^t q(\tau)P^2(\tau)\Delta\tau > \frac{1}{4F}, \tag{3.2}$$

then every solution of nonlinear system (1.3) *is oscillatory.*

Proof. Suppose to the contrary that (x,y) is a nonoscillatory solution of (1.3) with $x(t)>0$ for $t\in[t_0,\infty)_{\mathbb{T}}$. Let

$$r:=\liminf_{t\to\infty} w(t)P(t),\quad R:=\limsup_{t\to\infty} w(t)P(t),$$

where $w=y/x$. By Lemma 2.6 and its proof (in particular (2.31)) and simple calculus, we have

$$g_*(0)\le r-Fr^2\le\frac{1}{4F}\quad\text{and}\quad g_*(2)\le R-FR^2\le\frac{1}{4F},$$

a contradiction of both (3.1) and (3.2). The case with $x(t)<0$ for $t\in[t_0,\infty)_{\mathbb{T}}$ is similar. □

Theorem 3.2. *Assume that* (2.21) *holds, that P is given by* (2.8)*, and that* (2.9) *holds. Let $g_*(2)\le 1/(4F)$, and assume there exists a real number $\lambda\in[0,1)$ such that*

$$g^*(\lambda)>\frac{\lambda^2}{4F(1-\lambda)}+\frac{1}{2F}\Big(1+\sqrt{1-4Fg_*(2)}\Big). \tag{3.3}$$

Then every solution of nonlinear system (1.3) *is oscillatory.*

Proof. Suppose to the contrary that (x,y) is a nonoscillatory solution of (1.3) with $x(t)>0$ for $t\in[t_0,\infty)_{\mathbb{T}}$. By (2.17) we have

$$Gq(t)\le -w^{\Delta}(t)-Fp(t)(w^{\sigma})^2(t),\quad t\in[t_0,\infty)_{\mathbb{T}},$$

where $w=y/x$; multiply this by P^{λ} and delta integrate from t to infinity to get

$$\begin{aligned}
G\int_t^{\infty} q(\tau)P^{\lambda}(\tau)\Delta\tau &\le -\int_t^{\infty} w^{\Delta}(\tau)P^{\lambda}(\tau)\Delta\tau - F\int_t^{\infty} p(\tau)(w^{\sigma})^2(\tau)P^{\lambda}(\tau)\Delta\tau\\
&= P^{\lambda}(t)w(t)+\int_t^{\infty}\big(P^{\lambda}\big)^{\Delta}(\tau)w^{\sigma}(\tau)\Delta\tau\\
&\quad - F\int_t^{\infty} p(\tau)P^{\lambda}(\tau)(w^{\sigma})^2(\tau)\Delta\tau\\
&= P^{\lambda}(t)w(t)+\frac{1}{4F}\int_t^{\infty}\frac{\big((P^{\lambda})^{\Delta}\big)^2(\tau)}{p(\tau)P^{\lambda}(\tau)}\Delta\tau\\
&\quad -\int_t^{\infty}\Big(\sqrt{Fp(\tau)}P^{\lambda/2}(\tau)w^{\sigma}(\tau)-\frac{\big(P^{\lambda}\big)^{\Delta}(\tau)}{2\sqrt{Fp(\tau)}P^{\lambda/2}(\tau)}\Big)^2\Delta\tau\\
&\le P^{\lambda}(t)w(t)+\frac{1}{4F}\int_t^{\infty}\frac{\big((P^{\lambda})^{\Delta}\big)^2(\tau)}{p(\tau)P^{\lambda}(\tau)}\Delta\tau.
\end{aligned}$$

It follows that

$$P^{1-\lambda}(t)G\int_t^{\infty} q(\tau)P^{\lambda}(\tau)\Delta\tau < P(t)w(t)+\frac{P^{1-\lambda}(t)}{4F}\int_t^{\infty}\frac{\big((P^{\lambda})^{\Delta}\big)^2(\tau)}{p(\tau)P^{\lambda}(\tau)}\Delta\tau. \tag{3.4}$$

By (2.10), (2.23), and (3.4),

$$g^*(\lambda)\le\frac{1}{2F}\Big(1+\sqrt{1-4Fg_*(2)}\Big)+\frac{\lambda^2}{4F(1-\lambda)},$$

a contradiction of (3.3). Similarly if $x(t)<0$ for $t\in[t_0,\infty)_{\mathbb{T}}$. □

Corollary 3.3. *Assume that* (2.21) *holds, that* P *is given by* (2.8), *and that* (2.9) *holds. If* $g_*(2) \le 1/(4F)$ *and* $g^*(0) > \frac{1}{2F}\left(1 + \sqrt{1 - 4Fg_*(2)}\right)$, *then every solution of nonlinear system* (1.3) *is oscillatory.*

Theorem 3.4. *Assume that* (2.21) *holds, that* P *is given by* (2.8), *and that* (2.9) *holds. Let* $g_*(0), g_*(2) \le 1/(4F)$, *and assume there exists a real number* $\lambda \in [0, 1)$ *such that*

$$g_*(0) > \frac{\lambda(2-\lambda)}{4F}, \quad \textit{and} \tag{3.5}$$

$$g^*(\lambda) > \frac{g_*(0)}{1-\lambda} + \frac{1}{2F}\left(\sqrt{1-4Fg_*(0)} + \sqrt{1-4Fg_*(2)}\right). \tag{3.6}$$

Then every solution of nonlinear system (1.3) *is oscillatory.*

Proof. Suppose to the contrary that (x, y) is a nonoscillatory solution of (1.3) with $x(t) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$; the case with $x(t) < 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$ is omitted. Let $r = \liminf_{t\to\infty} w(t)P(t)$ and $R = \limsup_{t\to\infty} w(t)P(t)$, where $w = y/x$. By (2.22) and (2.23),

$$r \ge m := \frac{1}{2F}\left(1 - \sqrt{1-4Fg_*(0)}\right), \quad R \le M := \frac{1}{2F}\left(1 + \sqrt{1-4Fg_*(2)}\right). \tag{3.7}$$

Using (3.5) and (3.7) we find that $m > \lambda/(2F)$, whence given $\epsilon \in \left(0, m - \frac{\lambda}{2F}\right)$, there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$m - \epsilon < w(t)P(t) < M + \epsilon, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.8}$$

Similar to what we did in (2.19), multiply (2.17) by P^λ and delta integrate from t to infinity to get

$$\begin{aligned} &G\int_t^\infty q(\tau)P^\lambda(\tau)\Delta\tau \\ &\le w(t)P^\lambda(t) + \int_t^\infty p(\tau)P^{\lambda-2}(\tau)\left[\lambda w^\sigma(\tau)P(\tau) - F\big(P(\tau)w^\sigma(\tau)\big)^2\right]\Delta\tau; \end{aligned}$$

this leads to

$$\begin{aligned} P^{1-\lambda}(t)G\int_t^\infty q(\tau)P^\lambda(\tau)\Delta\tau \le w(t)P(t) + P^{1-\lambda}(t)\int_t^\infty p(\tau)P^{\lambda-2}(\tau) \\ \times \left[\lambda w^\sigma(\tau)P(\tau) - F\big(P(\tau)w^\sigma(\tau)\big)^2\right]\Delta\tau. \end{aligned} \tag{3.9}$$

Since the function $\gamma(z) := \lambda z - Fz^2$ is decreasing over the real interval $[\frac{\lambda}{2F}, \infty)$, it follows from (3.8), (3.9), and Lemma 2.4 that

$$\begin{aligned} &P^{1-\lambda}(t)G\int_t^\infty q(\tau)P^\lambda(\tau)\Delta\tau \\ &< M + \epsilon + (m-\epsilon)(\lambda - F(m-\epsilon))P^{1-\lambda}(t)\int_t^\infty p(\tau)P^{\lambda-2}(\tau)\Delta\tau \\ &< M + \epsilon + \frac{(m-\epsilon)(\lambda - F(m-\epsilon))(1+\epsilon)^{2-\lambda}}{1-\lambda}. \end{aligned}$$

This in tandem with (3.7) yields

$$g^*(\lambda) \le M + \frac{m(\lambda - Fm)}{1-\lambda} = \frac{g_*(0)}{1-\lambda} + \frac{1}{2F}\left(\sqrt{1-4Fg_*(0)} + \sqrt{1-4Fg_*(2)}\right),$$

a contradiction of (3.6). □

Corollary 3.5. *Assume that* (2.21) *holds, that* P *is given by* (2.8), *and that* (2.9) *holds. Let* $0 < g^*(0) \le 1/(4F)$ *and* $g_*(2) \le 1/(4F)$. *If*

$$g^*(0) > g_*(0) + \frac{1}{2F}\left(\sqrt{1-4Fg_*(0)} + \sqrt{1-4Fg_*(2)}\right),$$

then every solution of nonlinear system (1.3) *is oscillatory.*

4. Example

We illustrate Theorem 3.1 with the following example.

Example 4.1. Let $\mathbb{T}$ be an arbitrary time scale unbounded above, and let p and F be positive constants. Then the linear system

$$x^\Delta(t) = pFy(t), \quad y^\Delta(t) = \frac{-1}{t\sigma(t)}x(t), \quad t \in [t_0,\infty)_\mathbb{T} \tag{4.1}$$

for $t_0 > 0$, is nonoscillatory for $0 < p \le 1/(4F)$ and oscillatory for $p > 1/(4F)$. In other words, the inequality in (3.1) is sharp on all time scales.

Proof. Note that $p(t) \equiv p$, $f(y) = Fy$, $q(t) = \frac{1}{t\sigma(t)}$, and $g(x) = x$. Thus we have $P(t) = p(t-t_0)$, $f(y)/y = F$, and $G \equiv 1$, so that

$$g_*(0) = \liminf_{t\to\infty} GP(t)\int_t^\infty q(r)\Delta r = \liminf_{t\to\infty} \frac{p(t-t_0)}{t} = p.$$

By Theorem 3.1 and (3.1), any solution (x,y) of (4.1) oscillates if $p > 1/(4F)$. Converting (4.1) to a second-order dynamic equation for x, we arrive at a Cauchy-Euler equation [5, Section 2.3] of the form

$$t\sigma(t)x^{\Delta\Delta}(t) + pFx(t) = 0,$$

with general solution

$$x(t) = Ae_{\frac{1+\sqrt{1-4Fp}}{2t}}(t,t_0) + Be_{\frac{1-\sqrt{1-4Fp}}{2t}}(t,t_0), \tag{4.2}$$

where we have used a linear combination involving the time-scale exponential function [4, Section 2.2]. From elementary analysis and Euler's formula we know that x is nonoscillatory for $p \le 1/(4F)$ and oscillatory for $p > 1/(4F)$, showing in particular that the $1/(4F)$ in (3.1) is sharp for all time scales $\mathbb{T}$. $\square$

Remark 4.2. In Example 4.1 we can identify the exponential functions that occur in (4.2) for specific time scales [5, Example 2.19]. Letting $\lambda = \frac{1+\sqrt{1-4Fp}}{2}$, we get that

$$\mathbb{T} = \mathbb{R} : e_{\frac{1+\sqrt{1-4Fp}}{2t}}(t,t_0) = \left(\frac{t}{t_0}\right)^\lambda,$$

$$\mathbb{T} = q^\mathbb{Z} : e_{\frac{1+\sqrt{1-4Fp}}{2t}}(t,t_0) = \left(\frac{t}{t_0}\right)^{\log_q[1+(q-1)\lambda]},$$

$$\mathbb{T} = \mathbb{Z} : e_{\frac{1+\sqrt{1-4Fp}}{2t}}(t,t_0) = \frac{\Gamma(t+\lambda)\Gamma(t_0)}{\Gamma(t)\Gamma(t_0+\lambda)},$$

where Γ is the gamma function.

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