Electronic Journal of Differential Equations, Vol. 2009(2009), No. 21, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

MILD SOLUTIONS FOR SEMILINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This paper concerns the existence of mild solutions for fractional semilinear differential equation with non local conditions in the α-norm. We prove existence and uniqueness, assuming that the linear part generates an analytic compact bounded semigroup, and the nonlinear part is a Lipschitz continuous function with respect to the fractional power norm of the linear part.

1. Introduction

During the past decades, fractional differential equations have attracted many authors (see for instance [16, 17, 18, 19, 21, 22, 24, 26, 27] and references therein). This, mostly because it efficiently describes many phenomena arising in Engineering, Physics, Economy, and Science.

Our aim in this paper is to discuss the existence and the uniqueness of the mild solution for fractional semilinear differential equation with nonlocal conditions :

$$\begin{gathered} D^q x(t) = -Ax(t) + f(t, x(t), Bx(t)), \quad t \in [0, T], \\ x(0) + g(x) = x_0 \end{gathered} \tag{1.1}$$

where $T > 0$, $0 < q < 1$, $-A$ generates an analytic compact semigroup $(S(t))_{t\geq 0}$ of uniformly bounded linear operators on a Banach space $\mathbb{X}$. The term $Bx(t)$ which may be interpreted as a control on the system is defined by:

$$Bx(t) := \int_0^t K(t,s)x(s)ds,$$

where $K \in C(D, \mathbb{R}^+)$, the set of all positive function continuous on $D := \{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$ and

$$B^* = \sup_{t\in[0,T]} \int_0^t K(t,s)ds < \infty, \tag{1.2}$$

2000 *Mathematics Subject Classification.* 34K05, 34A12, 34A40.
Key words and phrases. Fractional differential equation.

Submitted October 28, 2008. Published January 23, 2009.

f and g are continuous. The derivative D^q is understood here in the Riemann-Liouville sense. The non local condition

$$g(x) = \sum_{k=1}^{p} c_k x(t_k),$$

where c_k, $k = 1, 2, \dots p$, are given constants and $0 < t_1 < t_2 < \dots < t_p \leq T$. Let us recall that such local conditions were first used by Deng in [4]. In his paper, Deng indicated that using the nonlocal condition $x(0) + g(x) = x_0$ to describe for instance, the diffusion phenomenon of a small amount of gas in a transparent tube can give better result than using the usual local Cauchy Problem $x(0) = x_0$. Let's observe also that since Deng's paper, such problem has also attracted several authors including Aizicovici, Byszewski, Ezzinbi, Fan, Liu, Liang, Lin, Xiao, Hernández, Lee, etc..(see for instance [1, 2, 3, 4, 13, 11, 6, 12, 22, 19] and the references therein).

However, among the previous research on nonlocal cauchy problems, few are concerned with mild solutions of fractional semilinear differential equations. Recently, in [14], the authors prove the existence and uniqueness of a mild solution for the semilinear initial value problem of non-integer order when the linear part generates a strongly continuous semigroup. In [20], we considered the fractional semilinear differential equation with nonlocal conditions

$$\begin{gathered} D^q x(t) = Ax(t) + t^n f(t, x(t), Bx(t)), \quad t \in [0, T], \quad n \in \mathbb{Z}^+ \\ x(0) = x_0 + g(x) \end{gathered} \tag{1.3}$$

where T is a positive real, $0 < q < 1$, A is the generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on a Banach space $\mathbb{X}$, $Bx(t) := \int_0^t K(t,s)x(s)ds$, $K \in C(D, \mathbb{R}^+)$ with $D := \{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$ and $B^* = \sup_{t\in[0,T]} \int_0^t K(t,s)ds < \infty$, $f : \mathbb{R} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is a nonlinear function, $g : \mathbb{C}([0,T], \mathbb{X}) \to D(A)$ is continuous and $0 < q < 1$. The derivative D^q is understood here in the Riemann-Liouville sense.

We used the Krasnoselkii and the contraction mapping principle to show the existence and uniqueness of a mild solution for a fractional semilinear differential equation with non local conditions.

In this paper, motivated by [8, 13], we investigate the existence and the uniqueness of a mild solution for the fractional semilinear differential equation (1.1), assuming that f is defined on $[0,T] \times \mathbb{X}_\alpha \times \mathbb{X}_\alpha$ where $\mathbb{X}_\alpha = D(A^\alpha)$, for some $0 < \alpha < 1$, the domain of the fractional power of A.

The rest of this paper is organized as follows. In section 2 we give some known preliminary results on the fractional powers of the generator of an analytic compact semigroup. In Section 3, we study the existence and the uniqueness of the mild solution for the fractional semilinear differential equation (1.1).

2. Preliminaries

For the rest of this article, we set $I = [0, T]$. We denote by $\mathbb{X}$ a Banach space with norm $\|\cdot\|$ and $-A : D(A) \to \mathbb{X}$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $(S(t))_{t\geq 0}$. This means that there exists $M > 1$ such that

$$\|S(t)\| \leq M \tag{2.1}$$

We assume without loss of generality that $0 \in \rho(A)$. This allows us to define the fractional power A^α for $0<\alpha<1$, as a closed linear operator on its domain $D(A^\alpha)$ with inverse $A^{-\alpha}$(see [8]). We have the following basic properties A^α.

Theorem 2.1 ([23, pp. 69-75])**.** *(1) $\mathbb{X}_\alpha = D(A^\alpha)$ is a Banach space with the norm $\|x\|_\alpha := \|A^\alpha x\|$ for $x \in D(A^\alpha)$.*

(2) $S(t) : \mathbb{X} \to \mathbb{X}_\alpha$ for each $t > 0$.

(3) $A^\alpha S(t)x = S(t)A^\alpha x$ for each $x \in D(A^\alpha)$ and $t \geq 0$.

(4) For every $t>0$, $A^\alpha S(t)$ is bounded on $\mathbb{X}$ and there exist $M_\alpha > 0$ and $\delta > 0$ such that

$$\|A^\alpha S(t)\| \leq \frac{M_\alpha}{t^\alpha} e^{-\delta t} \tag{2.2}$$

(5) $A^{-\alpha}$ is a bounded linear operator in $\mathbb{X}$ with $D(A^\alpha) = \mathrm{Im}(A^{-\alpha})$.

(6) If $0 < \alpha \leq \beta$, then $D(A^\beta) \hookrightarrow D(A^\alpha)$.

Remark 2.2. Observe as in [13] that by Theorem 2.1 (ii) and (iii), the restriction $S_\alpha(t)$ of $S(t)$ to $\mathbb{X}_\alpha$ is exactly the part of $S(t)$ in $\mathbb{X}_\alpha$. Let $x \in \mathbb{X}_\alpha$. Since

$$\|S(t)x\|_\alpha = \|A^\alpha S(t)x\| = \|S(t)A^\alpha x\| \leq \|S(t)\|\|A^\alpha x\| = \|S(t)\|\|x\|_\alpha,$$

and as t decreases to 0

$$\|S(t)x - x\|_\alpha = \|A^\alpha S(t)x - A^\alpha x\| = \|S(t)A^\alpha x - A^\alpha x\| \to 0,$$

for all $x \in \mathbb{X}_\alpha$, it follows that $(S(t))_{t\geq 0}$ is a family of strongly continuous semigroup on $\mathbb{X}_\alpha$ and $\|S_\alpha(t)\| \leq \|S(t)\|$ for all $t \geq 0$.

We have the the following result from [13].

Lemma 2.3. *$(S_\alpha(t))_{t\geq 0}$ is an immediately compact semigroup in $\mathbb{X}_\alpha$, and hence it is immediately norm-continuous.*

Definition 2.4 ([14])**.** A continuous function $x : I \to \mathbb{X}$ satisfying the equation

$$x(t) = S(t)(x_0 - g(x)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} S(t-s)(f(s, x(s), Bx(s))\, ds \tag{2.3}$$

for $t \in [0, T]$ is called a mild solution of the equation (1.1)

In the sequel, we will also use $\|f\|_p$ to denote the L^p norm of f whenever $f \in L^p(0,T)$ for some p with $1 \leq p < \infty$. We will set $\alpha \in (0,1)$ and we will denote by $\mathcal{C}_\alpha$, the Banach space $C([0,T], \mathbb{X}_\alpha)$ endowed with the supnorm given by

$$\|x\|_\infty := \sup_{t\in I} \|x\|_\alpha, \quad \text{for } x \in \mathcal{C}.$$

3. Main Results

We assume the following conditions:

(H1) The function $f : I \times \mathbb{X}_\alpha \times \mathbb{X}_\alpha \to \mathbb{X}$ is continuous, and there exists a positive function $\mu \in L^1_{\rm loc}(I, \mathbb{R}^+)$ such that

$$\|f(t,x,y)\| \leq \mu(t), \tag{3.1}$$

(H2) $g \in C(\mathcal{C}_\alpha, \mathbb{X}_\alpha)$ is completely continuous and there exist $\lambda, \gamma > 0$ such that

$$\|g(x)\|_\alpha \leq \lambda \|x\|_\infty + \gamma.$$

Theorem 3.1. *Suppose that assumptions* (H1), (H2) *hold. If* $x_0 \in \mathbb{X}_\alpha$ *and*

$$M\lambda < \frac{1}{2} \tag{3.2}$$

then (1.1) *has a mild solution on* $[0, T]$.

Proof. We define the function $F : \mathcal{C}_\alpha \to \mathcal{C}_\alpha$ by

$$(Fx)(t) = S(t)(x_0 - g(x)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} S(t-s) f(s, x(s), Bx(s))\, ds,$$

and we choose r such that

$$r \geq 2\Big(\frac{M_\alpha T^{q-\alpha}}{(1-\alpha)\Gamma(q)} \|\mu\|_{L^1_{\rm loc}(I,\mathbb{R}_+)} + M(\|x_0\|_\alpha + \gamma)\Big).$$

Let $B_r = \{x \in \mathcal{C}_\alpha : \|x\|_\infty \leq r\}$. Then we proceed in three steps.
Step 1. We show that $FB_r \subset B_r$. Let $x \in B_r$. Then for $t \in I$, we have

$$\begin{aligned}
&\|(Fx)(t)\|_\alpha \\
&\leq \|S(t)(x_0 - g(x))\|_\alpha + \frac{1}{\Gamma(q)} \Big\| \int_0^t (t-s)^{q-1} S(t-s) f(s, x(s), Bx(s))\, ds \Big\|_\alpha \\
&\leq \|S(t)\| \|x_0 - g(x)\|_\alpha + \frac{1}{\Gamma(q)} \int_0^t \|(t-s)^{q-1} A^\alpha S(t-s) f(s, x(s), Bx(s))\| ds \\
&\leq \|S(t)\| (\|x_0\|_\alpha + \lambda \|x\|_\infty + \gamma) + \frac{T^{q-1}}{\Gamma(q)} \int_0^t \|A^\alpha S(t-s)\| \|f(s, x(s), Bx(s))\|\, ds,
\end{aligned}$$

which according to (2.1), (2.2), (3.1) and (3.2) gives

$$\begin{aligned}
&\|(Fx)(t)\|_\alpha \\
&\leq \|S(t)\| \left(\|x_0\|_\alpha + \lambda \|x\|_\infty + \gamma\right) + \frac{T^{q-1}}{\Gamma(q)} \int_0^t M_\alpha (t-s)^{-\alpha} e^{-\delta(t-s)} \mu(s)\, ds \\
&\leq \|S(t)\| \left(\|x_0\|_\alpha + \lambda \|x\|_\infty + \gamma\right) + \frac{M_\alpha T^{q-1}}{\Gamma(q)} \int_0^t (t-s)^{-\alpha} \mu(s)\, ds \\
&\leq M \left(\|x_0\|_\alpha + \lambda \|x\|_\infty + \gamma\right) + \frac{M_\alpha T^{q-\alpha}}{(1-\alpha)\Gamma(q)} \|\mu\|_{L^1_{\rm loc}(I,\mathbb{R}_+)} \leq r
\end{aligned}$$

for $t \in I$. Hence, we deduce $\|Fx\|_\infty \leq r$.
Step 2. We prove that F is continuous. Let (x_n) be a sequence of B_r such that $x_n \to x$ in B_r. Then

$$f(s, x_n(s), Bx_n(s)) \to f(s, x(s), Bx(s)), \quad n \to \infty$$

because the function f is continuous on $I \times \mathbb{X}_\alpha \times \mathbb{X}_\alpha$. Now, for $t \in I$, we have

$$\begin{aligned}
&\|Fx_n - Fx\|_\alpha \\
&\leq \|S(t)(g(x_n) - g(x))\|_\alpha \\
&\quad + \Big\| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} S(t-s) \left(f(s, x_n(s), Bx_n(s)) - f(s, x(s), Bx(s))\right) ds \Big\|_\alpha,
\end{aligned}$$

which in view of (2.1) and (2.2) gives

$$\begin{aligned}
&\|Fx_n - Fx\|_\alpha \\
&\leq \|S(t)\| \|g(x_n) - g(x)\|_\alpha
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{T^{q-1}}{\Gamma(q)} \int_0^t \|A^\alpha S(t-s)\| \|f(s, x_n(s), Bx_n(s)) - f(s, x(s), Bx(s))\| \, ds \\
&\le M\|g(x_n) - g(x)\|_\alpha \\
&\quad + \frac{M_\alpha T^{q-1}}{\Gamma(q)} \int_0^t (t-s)^{-\alpha} \|f(s, x_n(s), Bx_n(s)) - f(s, x(s), Bx(s))\| \, ds \\
&\le M\|g(x_n) - g(x)\|_\alpha \\
&\quad + \frac{M_\alpha T^{q-1}}{\Gamma(q)} \int_0^t (t-s)^{-\alpha} \|f(s, x_n(s), Bx_n(s)) - f(s, x(s), Bx(s))\| \, ds
\end{aligned}
$$

for $t \in I$. Therefore, using on the one hand the fact that

$$
\|f(s, x_n(s), Bx_n(s)) - f(s, x(s), Bx(s))\| \le 2\mu(s) \quad \text{for } s \in I,
$$

and for each $t \in I$ since f satisfies (H1) and on the other hand the fact that the function $s \mapsto 2\mu(s)(t-s)^{-\alpha}$ is integrable on I, by means of the Lebesgue Dominated Convergence Theorem one proves that

$$
\int_0^t (t-s)^{-\alpha} \|f(s, x_n(s), Bx_n(s)) - f(s, x(s), Bx(s))\| \, ds \to 0.
$$

Hence, since $g(x_n) \to g(x)$ as $n \to \infty$ because g is completely continuous on $\mathcal{C}_\alpha$, it can easily been shown that

$$
\lim_{n\to\infty} \|Fx_n - Fx\|_\infty = 0, \quad \text{as } n \to \infty.
$$

In other words F is continuous.

Step 3. We show that F is compact. To this end, we use the Ascoli-Arzela's theorem. We first prove that $\{(Fx)(t) : x \in B_r\}$ is relatively compact in $\mathbb{X}_\alpha$, for all $t \in I$. Obviously, $\{(Fx)(0) : x \in B_r\}$ is compact. Let $t \in (0, T]$. For each $h \in (0, t)$ and $x \in B_r$, we define the operator F_h by

$$
\begin{aligned}
(F_h x)(t) &= S(t)(x_0 - g(x)) + \frac{1}{\Gamma(q)} \int_0^{t-h} (t-s)^{q-1} S(t-s) f(s, x(s), Bx(s)) \, ds \\
&= S(t)(x_0 - g(x)) + \frac{S(h)}{\Gamma(q)} \int_0^{t-h} (t-s)^{q-1} S(t-h-s) f(s, x(s), Bx(s)) \, ds.
\end{aligned}
$$

Then the sets $\{(F_h x)(t) : x \in B_r\}$ are relatively compact in $\mathbb{X}_\alpha$ since by Lemma 2.3, the operators $S_\alpha(t)$, $t \ge 0$ are compact on X_α. Moreover, using (H1) and (2.2), we have

$$
\begin{aligned}
\|(Fx)(t) - (F_h x)(t)\|_\alpha &\le \frac{1}{\Gamma(q)} \int_{t-h}^t (t-s)^{q-1} \|S(t-s) f(s, x(s), Bx(s))\|_\alpha \, ds \\
&\le \frac{T^{q-1}}{\Gamma(q)} \int_{t-h}^t \|A^\alpha S(t-s)\| \|f(s, x(s), Bx(s))\| \, ds \\
&\le \frac{T^{q-1} M_\alpha}{\Gamma(q)} \|\mu\|_{L^1_{\rm loc}(I, \mathbb{R}_+)} \int_{t-h}^t (t-s)^{-\alpha} \, ds \\
&\le \frac{T^{q-1} M_\alpha \|\mu\|_{L^1_{\rm loc}(I, \mathbb{R}_+)}}{(1-\alpha)\Gamma(q)} h^{1-\alpha}
\end{aligned}
$$

Therefore, we deduce that $\{(Fx)(t) : x \in B_r\}$ is relatively compact in $\mathbb{X}_\alpha$ for all $t \in (0, T]$ and since it is compact at $t = 0$ we have the relatively compactness in $\mathbb{X}_\alpha$ for all $t \in I$. Now, let us prove that $F(B_r)$ is equicontinuous. By the compactness of the set $g(B_r)$, we can prove that the functions $Fx, x \in B_r$ are equicontinuous a $t = 0$. For $0 < t_2 < t_1 \le T$, we have

$$\begin{aligned}
&\|(Fx)(t_1) - (Fx)(t_2)\|_\alpha \\
&\le \|(S(t_1) - S(t_2))(x_0 - g(x))\|_\alpha \\
&\quad + \frac{1}{\Gamma(q)}\| \int_0^{t_2} (t_1 - s)^{q-1} \left(S(t_1 - s) - S(t_2 - s)\right) f(s, x(s), Bx(s))\, ds\|_\alpha \\
&\quad + \frac{1}{\Gamma(q)}\| \int_0^{t_2} \left((t_1 - s)^{q-1} - (t_2 - s)^{q-1}\right) S(t_2 - s) f(s, x(s), Bx(s))\, ds\|_\alpha \\
&\quad + \frac{1}{\Gamma(q)}\| \int_{t_2}^{t_1} (t_1 - s)^{q-1} S(t_1 - s) f(s, x(s), Bx(s))\, ds\|_\alpha \\
&\le I_1 + I_2 + I_3 + I_4
\end{aligned}$$

Where

$$I_1 = \|(S(t_1) - S(t_2))(x_0 - g(x))\|_\alpha$$

$$I_2 = \frac{1}{\Gamma(q)}\| \int_0^{t_2} (t_1 - s)^{q-1} \left(S(t_1 - s) - S(t_2 - s)\right) f(s, x(s), Bx(s))\, ds\|_\alpha$$

$$I_3 = \frac{1}{\Gamma(q)}\| \int_0^{t_2} \left((t_1 - s)^{q-1} - (t_2 - s)^{q-1}\right) S(t_2 - s) f(s, x(s), Bx(s))\, ds\|_\alpha$$

$$I_4 = \frac{1}{\Gamma(q)}\| \int_{t_2}^{t_1} (t_1 - s)^{q-1} S(t_1 - s) f(s, x(s), Bx(s))\, ds\|_\alpha$$

Actually, I_1, I_2, I_3 and I_4 tend to 0 independently of $x \in B_r$ when $t_2 \to t_1$. Indeed, let $x \in B_r$ and $G = \sup_{x \in \mathcal{C}_\alpha} \|g(x)\|_\alpha$. We have

$$\begin{aligned}
I_1 &= \|(S(t_1) - S(t_2))(x_0 - g(x))\|_\alpha \\
&\le \|S_\alpha(t_1) - S_\alpha(t_2)\|_\alpha \|x_0 - g(x)\|_\alpha \\
&\le \|S_\alpha(t_1) - S_\alpha(t_2)\|_\alpha \left(\|x_0\|_\alpha + G\right)
\end{aligned}$$

from which we deduce that $\lim_{t_2 \to t_1} I_1 = 0$ since by Lemma 2.3 the function $t \mapsto \|S_\alpha(t)\|_\alpha$ is continuous for $t \in (0, T]$.

$$\begin{aligned}
&I_2 \\
&\le \frac{1}{\Gamma(q)} \int_0^{t_2} \|(t_1 - s)^{q-1}\big(S(t_1 - s) - S(t_2 - s)\big) f(s, x(s), Bx(s))\|_\alpha\, ds \\
&\le \frac{T^{q-1}}{\Gamma(q)} \int_0^{t_2} \big\|\big[S\big(\frac{t_1 - t_2}{2} + \frac{t_1 - s}{2}\big) - S\big(\frac{t_2 - s}{2}\big)\big] A^\alpha S\big(\frac{t_2 - s}{2}\big) f(s, x(s), Bx(s))\big\|\, ds \\
&\le \frac{T^{q-1-\alpha}}{\Gamma(q)} \|\mu\|_{L^1_{\rm loc}(I, \mathbb{R}_+)} \int_0^{t_2} \|S\big(\frac{t_1 - t_2}{2} + \frac{t_1 - s}{2}\big) - S\big(\frac{t_2 - s}{2}\big)\|\, ds.
\end{aligned}$$

Therefore, the continuity of the function $t \mapsto \|S(t)\|$ for $t \in (0, T)$ allows us to conclude that $\lim_{t_2 \to t_1} I_2 = 0$.

$$\begin{aligned}
I_3 &\le \frac{1}{\Gamma(q)} \int_0^{t_2} \|\left((t_2 - s)^{q-1} - (t_1 - s)^{q-1}\right) S(t_2 - s) f(s, x(s), Bx(s))\|_\alpha\, ds \\
&\le \frac{1}{\Gamma(q)} \int_0^{t_2} \left|(t_2 - s)^{q-1} - (t_1 - s)^{q-1}\right| \|A^\alpha S(t_2 - s)\| \|f(s, x(s), Bx(s))\|\, ds
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(q)} \int_0^{t_2} \left|(t_2-s)^{q-1}-(t_1-s)^{q-1}\right| (t_2-s)^{-\alpha}\mu(s)\,ds \\
&\le \frac{T^{-\alpha}}{\Gamma(q)} \|\mu\|_{L^1_{\rm loc}(I,\mathbb{R}_+)} \int_0^{t_2} \left|(t_2-s)^{q-1}-(t_1-s)^{q-1}\right|\,ds \\
&\le \frac{T^{-\alpha}}{q\Gamma(q)} \|\mu\|_{L^1_{\rm loc}(I,\mathbb{R}_+)} |t_1-t_2|^q.
\end{aligned}
$$

Hence $\lim_{t_2\to t_1} I_3 = 0$.

$$
\begin{aligned}
I_4 &\le \frac{1}{\Gamma(q)} \int_{t_2}^{t_1} \|(t_1-s)^{q-1}S(t_1-s)f(s,x(s),Bx(s))\|_\alpha\,ds \\
&\le \frac{T^{q-1}}{\Gamma(q)} \int_{t_2}^{t_1} \|A^\alpha S(t_1-s)\|\|f(s,x(s),Bx(s))\|\,ds \\
&\le \frac{M_\alpha T^{q-1}}{\Gamma(q)} \int_{t_2}^{t_1} (t_1-s)^{-\alpha}\mu(s)\,ds \\
&\le \frac{M_\alpha T^{q-1}}{(1-\alpha)\Gamma(q)} \|\mu\|_{L^1_{\rm loc}(I,\mathbb{R}_+)} |t_1-t_2|^{1-\alpha}.
\end{aligned}
$$

Since $1-\alpha>0$, we deduce that $\lim_{t_2\to t_1} I_4 = 0$.

In summary, we have proven that $F(B_r)$ is relatively compact, for $t\in I$, $\{Fx : x\in B_r\}$ is a family of equicontinuous functions. Hence by the Arzela-Ascoli Theorem, F is compact. By Schauder fixed point theorem F has a fixed point $x\in B_r$. Consequently, (1.1) has a mild solution. ☐

Now we make the following assumptions.

(H1') $f : I\times\mathbb{X}_\alpha\times\mathbb{X}_\alpha\to\mathbb{X}$ is continuous and there exist functions $\mu_1,\mu_2\in L^1_{\rm loc}(I,\mathbb{R}^+)$ such that

$$\|f(t,x,u)-f(t,y,v)\| \le \mu_1(t)\|x-y\|_\alpha + \mu_2(t)\|u-v\|_\alpha,$$

for all $t\in I$, $x,y,u,v\in\mathbb{X}_\alpha$.

(H2') $g:\mathcal{C}_\alpha\to\mathbb{X}_\alpha$ is continuous and there exists a constant b such that

$$\|g(x)-g(y)\|_\alpha \le b\|x-y\|_\infty, \quad \text{for all } x,y\in\mathcal{C}_\alpha.$$

(H3) The function $\Omega_{\alpha,q} : I\to\mathbb{R}_+$, $0<\alpha,q<1$ defined by

$$\Omega_{\alpha,q} = Mb + \frac{T^{q-1}M_\alpha t^{1-\alpha}}{(1-\alpha)\Gamma(q)}\Big(\|\mu_1\|_{L^1_{\rm loc}(I,\mathbb{R}_+)} + B^*\|\mu_2\|_{L^1_{\rm loc}(I,\mathbb{R}_+)}\Big)$$

satisfies $0<\Omega_{\alpha,q}\le\tau<1$, for all $t\in I$.

Theorem 3.2. *Assume that* (H1'), (H2'), (H3) *hold. If $x_0\in\mathbb{X}_\alpha$ then* (1.1) *has a unique mild solution $x\in\mathcal{C}_\alpha$.*

Proof. Define the function $F:\mathcal{C}_\alpha\to\mathcal{C}_\alpha$ by

$$(Fx)(t) = S(t)(x_0-g(x)) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}S(t-s)f(s,x(s),Bx(s))\,ds.$$

Note that F is well defined on $\mathcal{C}_\alpha$. Now take $t\in I$ and $x,y\in\mathcal{C}_\alpha$. We have

$$
\begin{aligned}
&\|(Fx)(t)-F(y)(t)\|_\alpha \\
&\le \|S(t)\left(g(x)-g(y)\right)\|_\alpha
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|S(t-s)\left(f(s,x(s),Bx(s))-f(s,y(s),By(s))\right)\|_\alpha\,ds\\
&\le \|S(t)\|\|g(x)-g(y)\|_\alpha\\
&+\frac{T^{q-1}}{\Gamma(q)}\int_0^t \|A^\alpha S(t-s)\|\|f(s,x(s),Bx(s))-f(s,y(s),By(s))\|ds
\end{aligned}
$$

which according to (2.1), (2.2), (H1'), (H2') and (1.2) gives

$$
\begin{aligned}
&\|(Fx)(t)-F(y)(t)\|_\alpha\\
&\le Mb\|x-y\|_\infty+\frac{T^{q-1}M_\alpha}{\Gamma(q)}\int_0^t (t-s)^{-\alpha}\mu_1(s)\|x(s)-y(s)\|_\alpha ds\\
&\quad+\frac{T^{q-1}M_\alpha}{\Gamma(q)}\int_0^t (t-s)^{-\alpha}\mu_2(s)\|Bx(s)-By(s))\|_\alpha ds\\
&\le Mb\|x-y\|_\infty\\
&\quad+\frac{T^{q-1}M_\alpha}{\Gamma(q)}\|\mu_1\|_{L^1_{\rm loc}(I,\mathbb{R}_+)}\Big(\int_0^t (t-s)^{-\alpha}\,ds\Big)\|x(s)-y(s)\|_\infty\\
&\quad+\frac{T^{q-1}M_\alpha}{\Gamma(q)}\int_0^t (t-s)^{-\alpha}\mu_2(s)\Big[\int_0^s K(s,\sigma)\|A^\alpha(x(\sigma)-y(\sigma))\|d\sigma\Big]ds\\
&\le \Big(Mb+\frac{T^{q-1}M_\alpha t^{1-\alpha}}{(1-\alpha)\Gamma(q)}\|\mu_1\|_{L^1_{\rm loc}(I,\mathbb{R}_+)}\Big)\|x-y\|_\infty\\
&\quad+\frac{T^{q-1}M_\alpha B^* t^{1-\alpha}}{(1-\alpha)\Gamma(q)}\|\mu_2\|_{L^1_{\rm loc}(I,\mathbb{R}_+)}\|x-y\|_\infty\\
&\le \Big[Mb+\frac{T^{q-1}M_\alpha t^{1-\alpha}}{(1-\alpha)\Gamma(q)}\Big(\|\mu_1\|_{L^1_{\rm loc}(I,\mathbb{R}_+)}+B^*\|\mu_2\|_{L^1_{\rm loc}(I,\mathbb{R}_+)}\Big)\Big]\|x-y\|_\infty\\
&\le \Omega_{\alpha,q}(t)\|x-y\|_\infty.
\end{aligned}
$$

So we get

$$
\|(Fx)(t)-F(y)(t)\|_\infty\le \Omega_{\alpha,q}(t)\|x-y\|_\infty.
$$

Therefore, assumption (H3) allows us to conclude in view of the contraction mapping principe that, F has a unique fixed point in $\mathcal{C}_\alpha$, and

$$
x(t)=S(t)(x_0-g(x))+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}S(t-s)f(s,x(s),Bx(s))\,ds
$$

which is the mild solution of (1.1). □

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