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AN OPTIMAL EXISTENCE THEOREM FOR POSITIVE SOLUTIONS OF A FOUR-POINT BOUNDARY VALUE PROBLEM

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Abstract. We are interested in the existence of positive solutions to a four-point boundary value problem of the differential equation $y''(t)+a(t)f(y(t)) = 0$ on $[0,1]$. The value of y at 0 and 1 are each a multiple of $y(t)$ at an interior point. Many known existence criteria are based on the limiting values of $f(u)/u$ as u approaches 0 and infinity.

In this article we obtain an optimal criterion (thereby improving all existing results of kind mentioned above) by comparing these limiting values to the smallest eigenvalue of the corresponding four-point problem of the associated linear equation. In the simpler case of three-point boundary value problems, the same result has been established in an earlier paper by the first author using the shooting method.

The method of proof is based upon a variant of Krasnoselskii's fixed point theorem on cones, the classical Krein-Rutman theorem, and the Gelfand formula relating the spectral radius of a linear operator to its norm.

1. Introduction

We are interested in the existence of positive solutions of second-order nonlinear differential equations subject to four-point boundary conditions. In an earlier paper [7], we provided improvements of a result by Liu [6] on the existence of a positive solution for the equation:

$$y''(t) + a(t)f(y(t)) = 0, \quad 0 < t < 1 \tag{1.1}$$

subject to the four-point boundary conditions

$$y(0) = \alpha y(\xi), \quad y(1) = \beta y(\eta) \tag{1.2}$$

where $0 < \xi \le \eta < 1$, $a(t)$ is a continuous, and nonnegative function on $(0,1)$ and $f(y)$ is a continuous nonnegative function on $[0,\infty)$; i.e., $f \in C([0,\infty),[0,\infty))$. We proved the following result.

Theorem 1.1. *Suppose that* $a(t) \not\equiv 0$ *on* $(0,1)$,

- (H1) $0 < \alpha < \frac{1}{1-\xi}$, $0 < \beta < \frac{1}{\eta}$;
- (H2) $\Lambda = \alpha\xi(1-\beta) + (1-\alpha)(1-\beta\eta) > 0$;

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(H3) *The following two limits exist:*

$$f_0 = \lim_{u\to 0} \frac{f(u)}{u}, \quad f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}.$$

Then (1.1) (1.2) *has at least one positive solution if either*

$$f_0 < \Lambda_1, \quad f_\infty > \Lambda_2, \tag{1.3}$$

or

$$f_0 > \Lambda_2, \quad f_\infty < \Lambda_1, \tag{1.4}$$

where

$$\Lambda_1 = \Lambda\Big((1-\alpha+\alpha\xi)\int_0^1 (1-s)a(s)\,ds\Big)^{-1},$$
$$\Lambda_2 = \Lambda\Big((1-\alpha+\alpha\xi)\gamma\eta\int_\eta^1 (1-s)a(s)\,ds\Big)^{-1},$$

and $\gamma = \min\big(\eta, \beta\eta, \beta(1-\eta)/(1-\beta\eta)\big)$.

Clearly $\Lambda_2 > \Lambda_1$, so there exists a gap between the numbers f_0 and f_∞ in which existence is unknown. If $\alpha = 0$ in (1.2), then (1.1) (1.2) becomes a three-point problem which was studied in an earlier paper by the first named author [4]. Our main result is as follows.

Theorem 1.2. *Let* λ_1 *be the smallest eigenvalue of the three-point problem*

$$u''(t) + \lambda a(t)u(t) = 0, \quad 0 < t < 1 \tag{1.5}$$

subject to

$$u(0) = 0, \quad u(1) = \beta u(\eta) \tag{1.6}$$

where $0 < \beta < 1/\eta$. *If* $f(y)$ *satisfies either*

$$f_0 < \lambda_1 < f_\infty \tag{1.7}$$

or

$$f_\infty < \lambda_1 < f_0, \tag{1.8}$$

then (1.5) (1.6) *has at least one positive solution.*

Theorem 1.2 was extended in [6] to m-point problems where the boundary condition (1.6) becomes

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} k_i u(\xi_i), \tag{1.9}$$

where $0 < \xi_1 < \dots < \xi_{m-2} < 1$ and $k_i > 0$ for $i = 1, \dots, m-2$ satisfying $\sum_{i=1}^{m-2} k_i\xi_i < 1$. In both papers [4] and [6], we employed the classical shooting method which resulted in optimal criteria for f_0, f_∞ in terms of the smallest eigenvalue of the associated linear problem as given in (1.7) and (1.8). The existence of the smallest eigenvalue for the classical two-point problem (for which the corresponding eigensolution is positive on $(0, 1)$), is well-known from the Sturm-Liouville theory. The existence of λ_1 for the multi-point BVP (1.5) (1.9) was proved by the shooting method in [4].

The purpose of this paper is to prove an analogue of Theorem 1.2 for the four-point problem which includes both Theorems 1.1 and 1.2 as special cases. In fact, a result similar to Theorem 1.2 was also proved in Sun [12] using topological degree theory. However, the proof in this three-point case seems to contain an error where the definition of an infimum of a set of parameters may not exist (see [12], pp. 1058-1059). In any case, Sun's result is only concerned with symmetric solutions. In Zhang and Sun [14], optimal existence theorems given for multi-point boundary value problems were given and proved using topological degree theory. However, the conditions are more restrictive. In the three-point case, their result requires $0 \le \beta < 1$ as compared with (H1) of Theorem 1.1.

2. Main Theorem and Proof

Unlike our earlier result Theorem 1.2 where the shooting method was employed, we use the Krasnoselskii fixed point theorem on cones (see [2] and [3]) to prove our main result, similar to that used by Liu in [8] for the three-point case; i.e., BVP (1.1) (1.2) with $\alpha = 0$.

We introduce the operator $A : C[0,1] \to C^2[0,1]$ defined for $y(t) \in C[0,1]$ by

$$Ay(t) = \int_0^1 G(t,s)a(s)f(y(s))\,ds, \tag{2.1}$$

where $G(t,s)$ is given in terms of the Green's function $k(t,s)$ for the Dirichlet two-point boundary value problem:

$$x''(t) + a(t)f(x(t)) = 0, \quad x(0) = x(1) = 0.$$

Here

$$k(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1, \end{cases}$$

and

$$G(t,s) = k(t,s) + l_1(t)k(\xi,s) + l_2(t)k(\eta,s), \tag{2.2}$$

where

$$l_1(t) = \frac{\alpha}{\Lambda}[(\beta-1)t + (1-\beta\eta)], \tag{2.3}$$

$$l_2(t) = \frac{\beta}{\Lambda}[(1-\alpha)t + \alpha\xi]. \tag{2.4}$$

Define the positive number σ by

$$\sigma = \min\{l_1(t)\xi(1-\xi) + l_2(t)\eta(1-\eta) : 0 \le t \le 1\}, \tag{2.5}$$

which is positive because $l_1(t)$ and $l_2(t)$ are strictly positive on $[0,1]$. Indeed by assumption (H1), we easily obtain from (2.3) and (2.4)

$$\min\{l_1(t) : 0 \le t \le 1\} = \frac{\alpha}{\Lambda}\min(1-\beta\eta, \beta(1-\eta)) > 0$$

and

$$\min\{l_2(t) : 0 \le t \le 1\} = \frac{\beta}{\Lambda}\min(\alpha\xi, 1-\alpha+\alpha\xi) > 0.$$

Using (2.5) in (2.2), we obtain

$$G(t,s) \ge \sigma s(1-s), \quad 0 \le s,t \le 1. \tag{2.6}$$

Now let

$$\frac{M_0}{2} = \max(\alpha(1-\beta\eta), \alpha\beta(1-\eta), \alpha\beta\xi, \beta(1-\alpha+\alpha\xi)). \tag{2.7}$$

From the definition of $k(t,s)$ it is easy to see that

$$k(t,s),\ k(\xi,s),\ k(\eta,s) \le s(1-s) \tag{2.8}$$

where $0 \le t, s \le 1$ and $0 < \xi \le \eta < 1$. Upon combining (2.7) and (2.8), we have

$$G(\tau, s) \le (1 + M_0\Lambda^{-1})^{-1} s(1-s), \quad 0 \le \tau, s \le 1. \tag{2.9}$$

Using (2.9) and the definition of A given by (2.1), we find that

$$Ay(t) \ge \sigma(1 + M_0\Lambda^{-1})^{-1} Ay(\tau). \tag{2.10}$$

Define $c_0 = \sigma(1 + M_0\Lambda^{-1})^{-1}$ and the positive cone

$$K = \{y(t) \in C[0,1] : y(t) \ge c_0\|y\|\},$$

where $\|y\| = \max\{|y(t)| : 0 \le t \ge 1\}$ is the supremum norm of $C[0,1]$. By (2.10), it is clear that $A(K) \subseteq K$.

We now apply the Krasnoselskii fixed point theorem on cones given in the form by Kwong [5] as follows:

Theorem 2.1 (Krasnoselskii-Petryshyn-Benjamin). *Let $A : K \to K$ be a completely continuous operator, where K is a cone on $C[0,1]$. Suppose that*

(i) *there exists $p \in K$ such that $x - Ax \ne \mu p$, for all $\mu \ge 0$ and all $x \in K_a = \{x \in K : \|x\| = a\}$; and*
(ii) *(Leary-Schauder condition) given any $\mu \in [0,1]$, $x \ne \mu Ax$ for all $x \in K_b = \{x \in K : \|x\| = b\}$, where $b > a > 0$.*

Then A has a non-zero fixed point $x^* \in K$ satisfying $a \le \|x^*\| \le b$.

We are now ready to prove our main result:

Theorem 2.2. *Let λ_1 be the smallest eigenvalue of the linear four-point boundary value problem* (1.5) (1.2). *If $f(y)$ satisfies* (1.7) *or* (1.8)*, then* (1.1) (1.2) *has at least one positive solution.*

Consider at first the case when $f_\infty < \lambda_1 < f_0$, the so-called sublinear case. Let K be the positive cone in $C[0,1]$ defined by

$$K = \{y(t) \in C[0,1] : y(t) \ge c_0\|y\|\}$$

where $c_0 > 0$ is given in terms of (2.5) and (2.7) by $c_0 = \sigma(1 + M_0\Lambda^{-1})^{-1}$. From the definition (2.1) of the operator A and by (2.10), we know that $A : K \to K$. It is now a standard argument to prove that A is completely continuous. Next we need to verify conditions (i) and (ii) in Theorem 2.1 to establish that A has a fixed point $\hat{y} \in K$ which is clearly non-zero since $\hat{y}(t) \ge c_0\|\hat{y}\|$, $c_0 > 0$, and by the definition of K, the function $\hat{y}(t)$ is positive. It is also easy to verify that the fixed point $\hat{y}$ satisfies the four-point boundary condition (1.2).

Now consider the linear operator L defined by setting $f(y) \equiv y$ in (2.1), namely

$$Ly(t) = \int_0^1 G(t,s)a(s)y(s)\,ds.$$

Clearly L maps the cone K into itself. We quote the famous Krein-Rutman theorem.

Theorem 2.3 (Krein-Rutman [13]). *Let $L : K \to K$ be a linear, completely continuous operator which maps a cone K in a Banach space X into itself. Then the equation $Ly = \lambda y$ has a smallest positive eigenvalue $\lambda_1 > 0$ which satisfies $\lambda_1 r(L) = 1$ where $r(L)$ denotes the spectral radius of the operator L.*

Proof. Assume that $f_\infty < \lambda_1 < f_0$. We first prove that condition (i) in Theorem 2.1 holds. Let $p \in K$ be the eigensolution of the linear problem; i.e., $p = \lambda_1 Lp$, where $\lambda_1^{-1} = r(L)$ is the smallest positive eigenvalue of L, and the spectral radius $r(L)$ is given by the formula

$$r(L) = \lim_{n\to\infty} (\|L^n\|)^{1/n}.$$

The existence of such a $p \in K$ is guaranteed by Theorem 2.3 cited above. Since $f_0 > \lambda_1$, there exists $a > 0$ sufficiently small such that $f(u) \geq \lambda_1 u$ for all $u \in K_a$. Suppose that condition (i) is false, then there exists $u_0 \in K_a$ such that $\|u_0\| = a$ and

$$u_0 = Au_0 + \mu_0 p \tag{2.11}$$

for some $\mu_0 \geq 0$. From (2.11), it is clear that $\mu_0 > 0$, for otherwise u_0 is a fixed point of A. Since A is positive; i.e., $Ay(t) \geq 0$ for all $y \geq 0$, we have $u_0 \geq \mu_0 p$. Let

$$\mu^* = \sup\{\mu : u_0 \geq \mu p\}.$$

So $u_0 \geq \mu^* p \geq \mu_0 p$. Note that $Lu_0 \geq L(\mu^* p) = \mu^* Lp$ and $Au_0 \geq \lambda_1 Lu_0$. Thus

$$\begin{aligned} u_0 &= Au_0 + \mu_0 p \\ &\geq \lambda_1 Lu_0 + \mu_0 p \\ &\geq \lambda_1 \mu^* Lp + \mu_0 p \\ &= \mu^*(\lambda_1 Lp) + \mu_0 p \\ &= (\mu^* + \mu_0)p. \end{aligned} \tag{2.12}$$

Since $\mu_0 > 0$, we have $u_0 \geq (\mu^* + \mu_0)p$, contradicting the definition of μ^*. So condition (i) holds.

We now turn to condition (ii). From $f_\infty < \lambda_1$, there exist $\varepsilon > 0$ and b sufficiently large such that $f(u) \leq (\lambda_1 - \varepsilon)u$ for all $\|u\| \geq b$. Let $u_1 \in K_b$; i.e., $\|u_1\| = b$, and $u_1 = \mu Au_1$ for all $\mu \in [0,1]$. Note that

$$u_1 = \mu Au_1 \leq (\lambda_1 - \varepsilon)\mu Lu_1 = \sigma Lu_1, \quad 0 < \sigma < (\lambda_1 - \varepsilon)\mu. \tag{2.13}$$

Since L is linear, we can prove by induction from (2.13) that $u_1 \leq \sigma^n L^n u_1$ for $n = 1, 2, \dots$ from which it follows that

$$\|L^n\| \geq \ f\|L^n u_1\|\|u_1\| = \sigma^{-n}. \tag{2.14}$$

Using the Gelfand formula for the spectral radius $r(L)$ (see [13]), we obtain from (2.14)

$$r(L) = \lambda_1^{-1} = \lim_{n\to\infty} (\|L^n\|)^{1/n} \geq \sigma^{-1} \geq (\lambda_1 - \varepsilon)^{-1}. \tag{2.15}$$

Since $\varepsilon > 0$, (2.15) gives the desired contradiction. Hence, we conclude that for $x \in K_b$, $x \neq \mu Ax$ for all $\mu \in [0,1]$ and that condition (ii) holds. The existence of a non-zero fixed point of A now follows from Theorem 2.1 cited above.

In the superlinear case; i.e., $f_0 < \lambda_1 < f_\infty$, we need to apply the Krasnoselskii fixed point theorem in its expansive form with conditions (i) and (ii) in Theorem 2.1 reversed. Suppose that there exists $u_2 \in K_b$; i.e., $\|u_2\| = b$, such that $u_2 = Au_2 + \mu_2 p$ for some $\mu_2 > 0$. Since $Ay(t) \geq 0$ for all $y(t) \geq 0$, and, in particular, $Au_2 \geq 0$, so $u_2 \geq \mu_2 p$, $\mu_2 > 0$. Define $\hat{\mu} = \sup\{\mu : u_2 \geq \mu p\}$ which exists and

$\hat{\mu} \geq u_2$. Note that $Au_2 \geq \lambda_1 Lu_2$ and $Lu_2 \geq L(\hat{\mu}p) = \hat{\mu}Lp$. Using the fact that $p = \lambda_1 Lp$, we observe that

$$\begin{aligned} u_2 &= Au_2 + \mu_2 p \\ &\geq \lambda_1 Lu_2 + \mu_2 p \\ &\geq \lambda_1 \hat{\mu} Lp + \mu_2 p \\ &= (\hat{\mu} + \mu_2)p. \end{aligned} \tag{2.16}$$

Since $\mu_2 > 0$, (2.16) shows that $\hat{\mu}$ does not exist. Hence, condition (i) is satisfied.

Next we show that condition (ii) holds for $u \in K_a$ where a satisfies $0 < a < b$, if $f_0 < \lambda_1$. For $\varepsilon > 0$ sufficiently small, there exists a, $0 < a < b$ such that $f(u) \leq \lambda_1 u$ for all $0 \leq u \leq a$. If condition (ii) is not satisfied, then there exists $v_0 \in K_a$ such that $Av_0 = \bar{\mu}v_0$ for some $\bar{\mu} > 1$. Since $Ay(t) \leq \lambda_1 Ly(t)$ for all $y \in K_a$, we have $Av_0 = \bar{\mu}v_0 \leq \lambda_1 Lv_0$. By the linearity of L, we can prove by induction that $\lambda_1^n Lv_0 \geq \mu^{-n} v_0$, for $n = 2, 3, \dots$, so

$$\|L^n\| \geq \frac{\|L^n v_0\|}{\|v_0\|} = \big(\frac{\bar{\mu}}{\lambda_1}\big)^n. \tag{2.17}$$

Again by the spectral radius formula, we have from (2.17)

$$r(L) = \lambda_1^{-1} = \lim_{n\to\infty} (\|L^n\|)^{1/n} \geq \frac{\bar{\mu}}{\lambda_1} > \frac{1}{\lambda_1},$$

which gives the desired contradiction. Therefore condition (ii) of Theorem 2.1 is also true. This implies the existence of a non-zero fixed point of A and completes the proof of the theorem. □

3. Examples and Remarks

We give two examples to illustrate the usefulness of our main Theorem.

Example 3.1. Consider the three-point boundary value problem

$$y''(t) + \frac{7y^2(t) + y(t)}{1 + y(t)} = 0 \tag{3.1}$$

$$y(0) = 0, \quad y(1) = \frac{1}{2} y\big(\frac{1}{2}\big). \tag{3.2}$$

In Liu [8], it was proved that (1.1) (1.2) with $\alpha = 0$ has a positive solution if $f_0 < \overline{\Lambda}_1$ and $f_\infty > \overline{\Lambda}_2$ where

$$\overline{\Lambda}_1 = (1 - \beta\eta)\Big(\int_0^1 (1 - s)a(s)\,ds\Big)^{-1},$$

$$\overline{\Lambda}_2 = (1 - \beta\eta)\Big(\mu\eta \int_0^1 (1 - s)a(s)\,ds\Big)^{-1},$$

with $\mu = \min(\eta, \beta\eta, \beta(1 - \eta)(1 - \beta\eta))$. In case of the specific equation we have $\overline{\Lambda}_1 = 3/2$ and $\overline{\Lambda}_2 = 48$. Now consider the linear boundary value problem $y'' + \lambda y = 0$, subject to the three-point boundary condition (3.2). It is easy to determine the smallest positive eigenvalue λ_1 by equating $\sin \sqrt{\lambda}/2 = 2 \sin \sqrt{\lambda}$, yielding $\lambda_1 = 6.917$. Since $1 = f_0 < \lambda_1 < f_\infty = 7$, the BVP (3.1) (3.2) has a positive solution. Here Liu's theorem [8] does not apply since neither condition (1.3) nor (1.4) holds.

Example 3.2. Consider the four-point boundary value problem

$$y'' + \frac{aye^y}{b + e^y + \sin y} = 0, \quad a, b > 0 \tag{3.3}$$

subject to the boundary conditions

$$y(0) = \frac{1}{2}y\big(\frac{1}{3}\big), \quad y(1) = \frac{1}{3}y\big(\frac{1}{2}\big). \tag{3.4}$$

Here $f_0 = a/(1+b)$ and $f_\infty = a$. Also $\Lambda = 19/36$, $\Lambda_1 = 19/12$ and $\Lambda_2 = 76$. Since $a, b > 0$, in order to apply Theorem 1.1, we require $a > 76$ and $b > \frac{12a}{19} - 1$. This gives existence of positive solution only when $a > 76$ and $b > 47$.

We can compute the smallest positive eigenvalue of the linear BVP $y'' + \lambda y = 0$, subject to the four-point boundary condition (3.4) numerically. The answer is $\lambda_1 = 5.3163775$ and the corresponding eigenfunction is $\sin\{(2.3057271)t + 0.4971368\}$. By our main theorem, we obtain the existence of positive solution to (3.3) (3.4) if a and b satisfy

$$\frac{a}{1+b} < \lambda_1 < a. \tag{3.5}$$

Condition (3.5) gives a much greater range in a and b. If $a = 6$, we only need $b > 0.128588$. By comparison, if $a > 76$, we require $b > 13.2954$ which is better than $b > 47$ as required by the estimate (1.3) given in Theorem 1.1.

We close our discussion with a few remarks relating our work to others in the existing literature.

Remark 3.3. Our main result in this paper is closely related to our two earlier papers [6], [7], which included extensive references on the subject matter. Therefore we shall not reproduce here.

Remark 3.4. Boundary condition (1.2) is sometimes referred as separated (2;2) four point boundary conditions. It should be distinguished from the (1;3) four point boundary condition such as

$$y(0) = 0, \alpha_1 y(\xi_1) + \alpha_2 y(\xi_2) + \alpha_3 y(\xi_3) = 0,$$

where α_1, α_2, and α_3 are real constants and $0 < \xi_1 < \xi_2 < \xi_3 < 1$.

Remark 3.5. In [7], we showed that condition (H3), namely, $\Lambda = \alpha\xi(1-\beta) + (1-\alpha)(1-\beta\eta) > 0$ is a necessary condition for the existence of a positive solution. Previously, (H3) has always been assumed as a sufficient condition. When $\Lambda = 0$, the BVP (1.1) (1.2) is said to be at resonance in the sense that the associated linear homogeneous boundary value problem $x''(t) = 0$, $0 < t < 1$, $x(0) = \alpha x(\xi)$, $x(1) = \beta x(\eta)$ has non-trivial solutions. Solutions of the nonlinear BVP will not be positive. Using upper and lower solution methods coupled with Mawhin's continuation theorem, this problem was studied by Bai, Li and Ge [1] concerning the existence of non-trivial solutions, generalizing the work by Rachunkova [11].

Remark 3.6. Our method of proof is based upon a combination of Krasnoselskii's fixed point theorem and Leray-Schauder non-linear alternative which originates from the Brouwer's fixed point theorem, see [5]. This approach does not use any topological degree theory which has been used extensively in similar work on this subject.

Remark 3.7. Boundary conditions involving derivatives of solutions such as $y'(0) = \alpha y(\xi)$, $y(1) = \beta y'(\eta)$, seem more complicated. Our method of proof does not work for such Neumann type boundary conditions even in the non-resonance case. We have also not discussed multiplicity results and existence of symmetric solutions for the four point problem see e.g. Rachunkova [9, 10, 12].

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