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# THREE POSITIVE SOLUTIONS FOR A SYSTEM OF SINGULAR GENERALIZED LIDSTONE PROBLEMS

JIAFA XU, ZHILIN YANG

Abstract. In this article, we show the existence of at least three positive solutions for the system of singular generalized Lidstone boundary value problems

$$
\begin{gathered}
(-1)^m x^{(2m)} = a(t)f_1(t,x,-x'',\dots,(-1)^{m-1}x^{(2m-2)},y,-y'',\\
\dots,(-1)^{n-1}y^{(2n-2)}),\\
(-1)^n y^{(2n)} = b(t)f_2(t,x,-x'',\dots,(-1)^{m-1}x^{(2m-2)},y,-y'',\\
\dots,(-1)^{n-1}y^{(2n-2)}),\\
a_1x^{(2i)}(0)-b_1x^{(2i+1)}(0)=c_1x^{(2i)}(1)+d_1x^{(2i+1)}(1)=0,\\
a_2y^{(2j)}(0)-b_2y^{(2j+1)}(0)=c_2y^{(2j)}(1)+d_2y^{(2j+1)}(1)=0.
\end{gathered}
$$

The proofs of our main results are based on the Leggett-Williams fixed point theorem. Also, we give an example to illustrate our results.

## 1. Introduction

In this article, we study the existence positive solutions for the following system of singular generalized Lidstone boundary value problems

$$
\begin{gathered}
(-1)^m x^{(2m)} = a(t)f_1(t,x,-x'',\dots,(-1)^{m-1}x^{(2m-2)},y,-y'',\dots,(-1)^{n-1}y^{(2n-2)}),\\
(-1)^n y^{(2n)} = b(t)f_2(t,x,-x'',\dots,(-1)^{m-1}x^{(2m-2)},y,-y'',\dots,(-1)^{n-1}y^{(2n-2)}),\\
a_1x^{(2i)}(0)-b_1x^{(2i+1)}(0)=c_1x^{(2i)}(1)+d_1x^{(2i+1)}(1)=0 \quad (i=0,1,\dots,m-1),\\
a_2y^{(2j)}(0)-b_2y^{(2j+1)}(0)=c_2y^{(2j)}(1)+d_2y^{(2j+1)}(1)=0 \quad (j=0,1,\dots,n-1).
\end{gathered}
\tag{1.1}
$$

where $m,n\geq 1$, $a(t),b(t)\in C((0,1),[0,+\infty))$, $a(t)$ and $b(t)$ are allowed to be singular at $t=0$ and/or $t=1$; $f_i\in C([0,1]\times\mathbb{R}_+^{m+n},\mathbb{R}_+)(\mathbb{R}_+:=[0,+\infty))$; $a_i,b_i,c_i,d_i\in\mathbb{R}_+$ with $\rho_i:=a_ic_i+a_id_i+b_ic_i>0$, $i=1,2$.

Singular boundary value problems for ordinary differential equation describe many phenomena in applied mathematics and physical sciences, which can be found

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in the theory of nonlinear diffusion generated by nonlinear sources and in the thermal ignition of gases, see [1, 4]. Very recently, increasing attention is paid to question of positive solutions for systems of second-order or higher order singular differential equations, see for example [2, 5, 6, 7] and references therein.

In [2], by using Leggett-Williams fixed point theorem [3], Kang et al obtained at least three positive solutions to the following singular nonlocal boundary value problems for systems of nonlinear second-order ordinary differential equations.

$$\begin{gathered} u''(t) + a_1(t)f_1(t, u(t), v(t)) = 0, \quad 0 < t < 1, \\ v''(t) + a_2(t)f_2(t, u(t), v(t)) = 0, \quad 0 < t < 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \quad \gamma_1 u(1) + \delta_1 u'(1) = g_1\Big(\int_0^1 u(s)d\phi_1(s), \int_0^1 v(s)d\phi_1(s)\Big), \\ \alpha_2 v(0) - \beta_2 v'(0) = 0, \quad \gamma_2 v(1) + \delta_2 v'(1) = g_2\Big(\int_0^1 u(s)d\phi_2(s), \int_0^1 v(s)d\phi_2(s)\Big), \end{gathered} \tag{1.2}$$

where $a_i \in C((0, 1), \mathbb{R}_+)$ is allowed to be singular at $t = 0$ or $t = 1$; $f_i \in C([0, 1] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ and $g_i \in C([0, 1] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ are such that $a_1(t)f_1(t, 0, 0)$ or $a_2(t)f_2(t, 0, 0)$ does not vanish identically on any subinterval of $(0, 1)$; $\alpha_i \geq 0$, $\beta_i \geq 0$, $\gamma_i \geq 0$, $\delta_i \geq 0$, and $\rho_i = \alpha_i\gamma_i + \alpha_i\delta_i + \beta_i\gamma_i > 0$; $\int_0^1 u(s)d\phi_1(s)$ and $\int_0^1 v(s)d\phi_1(s)$ denote the Riemann-Stieltjes integrals, $i = 1, 2$.

Motivated by [2], we deal with the system of singular generalized Lidstone problems (1.1). To overcome the difficulties of (1.1) resulting from the derivatives of even orders, as in [8], we use the method of order reduction to transform (1.1) into an equivalent system of integro-integral equations, then prove the existence of positive solutions for the resulting system, thereby establishing that of positive solutions for (1.1)(see the main result in Section 3). The features of this paper mainly include the following aspects. Firstly, our study is on systems of singular generalized Lidstone problems. Secondly, $a$ and $b$ are allowed to be singular at $t = 0$ and/or $t = 1$. Finally, the system contains two equations, which can be of different orders. Thus the results presented here are different from those in [2, 5, 6, 7].

The remaining of this paper is organized as follows: Section 2 gives some preliminary results. The main result is stated and proved in Section 3, then followed by an example to illustrate the validity of our main result.

## 2. Preliminaries

Given a cone $K$ in a real Banach space $E$, a map $\alpha$ is said to be a nonnegative continuous concave functional on $K$ provided that $\alpha : K \to [0, +\infty)$ is continuous and

$$\alpha(tx + (1 - t)y) \geq t\alpha(x) + (1 - t)\alpha(y)$$

for all $x, y \in K$ and $0 \leq t \leq 1$.

Let $0 < a < b$ be given and let $\alpha$ be a nonnegative continuous concave functional on $K$. Define the convex sets $P_r$ and $P(\alpha, a, b)$ by

$$\begin{gathered} P_r := \{x \in K : \|x\| < r\}, \\ P(\alpha, a, b) := \{x \in K : a \leq \alpha(x), \|x\| \leq b\}. \end{gathered}$$

To prove our main result, we need the following Leggett-Williams fixed point theorem.

**Lemma 2.1** (see [3])**.** *Let $T : \overline{P_c} \to \overline{P_c}$ be a completely continuous operator and let $\alpha$ be a nonnegative continuous concave functional on $K$ such that $\alpha(x) \le \|x\|$ for all $x \in \overline{P_c}$. Suppose there exist $0 < a < b < d \le c$ such that*

- (C1) *$\{x \in P(\alpha, b, d) : \alpha(x) > b\} \neq \emptyset$, and $\alpha(Tx) > b$ for $x \in P(\alpha, b, d)$,*
- (C2) *$\|Tx\| < a$ for $\|x\| \le a$, and*
- (C3) *$\alpha(Tx) > b$ for $x \in P(\alpha, b, c)$ with $\|Tx\| > d$.*

*Then $T$ has at least three fixed points $x_1, x_2$, and $x_3$ such that $\|x_1\| < a$, $b < \alpha(x_2)$ and $\|x_3\| > a$ with $\alpha(x_3) < b$.*

For fixed nonnegative constants $a, b, c, d$ with $\rho := ac + ad + bc > 0$, we let

$$k(t,s) := \frac{1}{\rho}\begin{cases}(b+as)(c+d-ct), & 0 \le s \le t \le 1,\\ (b+at)(c+d-cs), & 0 \le t \le s \le 1,\end{cases} \tag{2.1}$$

By definition, $k \in C([0,1] \times [0,1], \mathbb{R}_+)$ has the following properties:

- (i) $k(t,s) \le k(s,s)$ for all $t, s \in [0,1]$.
- (ii) For any $\theta \in (0, 1/2)$, there exists $\gamma \in (0, \min\{\frac{\theta a+b}{a+b}, \frac{\theta c+d}{c+d}\}]$ such that

$$k(t,s) \ge \gamma k(s,s), \forall t \in [\theta, 1-\theta], s \in [0,1].$$

**Lemma 2.2.** *Let $f \in C[0,1], h(t) \in C(0,1)$ and $\int_0^1 k(s,s)h(s)ds < +\infty$. The boundary value problem*

$$\begin{gathered}-u'' = h(t)f(t),\\ au(0) - bu'(0) = 0,\\ cu(1) + du'(1) = 0,\end{gathered}$$

*has a unique solution*

$$u(t) = \int_0^1 k(t,s)h(s)f(s)ds,$$

*where $k(t,s)$ is given by* (2.1)*.*

Let

$$k_1(t,s) := \frac{1}{\rho_1}\begin{cases}(b_1+a_1s)(c_1+d_1-c_1t), & 0 \le s \le t \le 1,\\ (b_1+a_1t)(c_1+d_1-c_1s), & 0 \le t \le s \le 1,\end{cases} \tag{2.2}$$

$$g_1(t,s) := \frac{1}{\rho_2}\begin{cases}(b_2+a_2s)(c_2+d_2-c_2t), & 0 \le s \le t \le 1,\\ (b_2+a_2t)(c_2+d_2-c_2s), & 0 \le t \le s \le 1.\end{cases} \tag{2.3}$$

For $i, j = 2, \dots,$ define

$$k_i(t,s) := \int_0^1 k_1(t,\tau)k_{i-1}(\tau,s)d\tau, \quad g_j(t,s) := \int_0^1 g_1(t,\tau)g_{j-1}(\tau,s)d\tau \tag{2.4}$$

and the operators $A_i : C[0,1] \to C[0,1]$ and $B_j : C[0,1] \to C[0,1]$ by

$$\begin{gathered}(A_iu)(t) := \int_0^1 k_i(t,s)u(s)ds, \quad i = 1, 2, \dots, m,\\ (B_jv)(t) := \int_0^1 g_j(t,s)v(s)ds, \quad j = 1, 2, \dots, n.\end{gathered}$$

For $1 \le i \le m-1$ and $1 \le j \le n-1$, let

$$\begin{gathered}
\xi_i := \min_{\theta \le t \le 1-\theta} \int_0^1 k_i(t,s)ds, \quad \eta_i := \max_{0 \le t \le 1} \int_0^1 k_i(t,s)ds, \\
\mu_j := \min_{\theta \le t \le 1-\theta} \int_0^1 g_j(t,s)ds, \quad \nu_j := \max_{0 \le t \le 1} \int_0^1 g_j(t,s)ds, \\
\xi := \min_{1 \le i \le m-1, 1 \le j \le n-1} \{\xi_i, \mu_j\}, \quad \eta := \max_{1 \le i \le m-1, 1 \le j \le n-1} \{\eta_i, \nu_j\}.
\end{gathered}$$

## 3. Main result

Let $u(t) = (-1)^{m-1} x^{(2m-2)}$ and $v(t) = (-1)^{n-1} y^{(2n-2)}$. It is easy to see that (1.1) is equivalent to the system of integro-ordinary differential equations

$$\begin{gathered}
-u''(t) = a(t) f_1\Big(t, \int_0^1 k_{m-1}(t,s)u(s)ds, \dots, \int_0^1 k_1(t,s)u(s)ds, u(t), \\
\int_0^1 g_{n-1}(t,s)v(s)ds, \dots, \int_0^1 g_1(t,s)v(s)ds, v(t)\Big), \\
-v''(t) = b(t) f_2\Big(t, \int_0^1 k_{m-1}(t,s)u(s)ds, \dots, \int_0^1 k_1(t,s)u(s)ds, u(t), \\
\int_0^1 g_{n-1}(t,s)v(s)ds, \dots, \int_0^1 g_1(t,s)v(s)ds, v(t)\Big),
\end{gathered}$$

subject to the boundary conditions

$$a_1 u(0) - b_1 u'(0) = c_1 u(1) + d_1 u'(1) = 0, \quad a_2 v(0) - b_2 v'(0) = c_2 v(1) + d_2 v'(1) = 0.$$

Furthermore, the above system is equivalent to the system

$$\begin{gathered}
u(t) = \int_0^1 k_1(t,s) a(s) f_1\Big(s, \int_0^1 k_{m-1}(s,\tau)u(\tau)d\tau, \dots, \int_0^1 k_1(s,\tau)u(\tau)d\tau, u(s), \\
\int_0^1 g_{n-1}(s,\tau)v(\tau)d\tau, \dots, \int_0^1 g_1(s,\tau)v(\tau)d\tau, v(s)\Big)ds, \\
v(t) = \int_0^1 g_1(t,s) b(s) f_2\Big(s, \int_0^1 k_{m-1}(s,\tau)u(\tau)d\tau, \dots, \int_0^1 k_1(s,\tau)u(\tau)d\tau, u(s), \\
\int_0^1 g_{n-1}(s,\tau)v(\tau)d\tau, \dots, \int_0^1 g_1(s,\tau)v(\tau)d\tau, v(s)\Big)ds.
\end{gathered} \tag{3.1}$$

Let $E := C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ endowed with the norm $\|(u,v)\| := \|u\| + \|v\|$, where $\|u\| := \max_{0 \le t \le 1} |u(t)|$, and define the cone $K \subset E$ by

$$K := \Big\{(u,v) \in E : u(t) \ge 0, v(t) \ge 0, t \in [0,1], \min_{\theta \le t \le 1-\theta} (u(t) + v(t)) \ge \gamma \|(u,v)\|\Big\}.$$

Clearly, $(E, \|\cdot\|)$ is a real Banach space and $P$ is a cone on $E$. Define the operator $T : K \to K$ by

$$T(u,v)(t) := (T_1(u,v)(t), T_2(u,v)(t)),$$

where

$$\begin{aligned} T_1(u,v)(t) &:= \int_0^1 k_1(t,s)a(s)f_1\Big(s,(A_{m-1}u)(s),\dots,(A_1u)(s),\\ &\qquad u(s),(B_{n-1}v)(s),\dots,(B_1v)(s),v(s)\Big)ds,\\ T_2(u,v)(t) &:= \int_0^1 g_1(t,s)b(s)f_2\Big(s,(A_{m-1}u)(s),\dots,(A_1u)(s),\\ &\qquad u(s),(B_{n-1}v)(s),\dots,(B_1v)(s),v(s)\Big)ds. \end{aligned}$$

Now $f \in C([0,1]\times\mathbb{R}_+^{m+n},\mathbb{R}_+)$ and $g \in C([0,1]\times\mathbb{R}_+^{m+n},\mathbb{R}_+)$ imply that $T: K \to K$ is a completely continuous operator. In our setting, the existence of positive solutions for (3.1) is equivalent to that of positive fixed points of $T$.

**Lemma 3.1.** *The operator $T$ maps $K$ into $K$.*

*Proof.* If $(u,v)\in K$, then

$$\begin{aligned} &(T_1(u,v)(t))''\\ &= -a(t)f_1\Big(t,(A_{m-1}u)(t),\dots,(A_1u)(t),u(t),(B_{n-1}v)(t),\dots,(B_1v)(t),v(t)\Big)\\ &\le 0,\\ &(T_2(u,v)(t))''\\ &= -b(t)f_2\Big(t,(A_{m-1}u)(t),\dots,(A_1u)(t),u(t),(B_{n-1}v)(t),\dots,(B_1v)(t),v(t)\Big)\\ &\le 0. \end{aligned}$$

So $T_1(u,v)(t)$ and $T_2(u,v)(t)$ are concave on [0,1]. If $(u,v)\in K$, then from the properties of $k_1(t,s)$ and $g_1(t,s)$, we have

$$\begin{aligned} \|T_1(u,v)(t)\| \le \int_0^1 & k_1(s,s)a(s)f_1\Big(s,(A_{m-1}u)(s),\dots,(A_1u)(s),\\ & u(s),(B_{n-1}v)(s),\dots,(B_1v)(s),v(s)\Big)ds, \end{aligned}$$

and

$$\begin{aligned} \|T_2(u,v)(t)\| \le \int_0^1 & g_1(s,s)b(s)f_2\Big(s,(A_{m-1}u)(s),\dots,(A_1u)(s),\\ & u(s),(B_{n-1}v)(s),\dots,(B_1v)(s),v(s)\Big)ds. \end{aligned}$$

On the other hand,

$$\begin{aligned} \min_{\theta\le t\le 1-\theta} T_1(u,v)(t) &\ge \gamma\int_0^1 k_1(s,s)a(s)f_1\Big(s,(A_{m-1}u)(s),\dots,(A_1u)(s),u(s),\\ &\qquad (B_{n-1}v)(s),\dots,(B_1v)(s),v(s)\Big)ds\\ &\ge \gamma\|T_1(u,v)(t)\|, \end{aligned}$$

and

$$\begin{aligned}\min_{\theta\le t\le 1-\theta} T_2(u,v)(t) &\ge \gamma \int_0^1 g_1(s,s)b(s)f_2\Big(s,(A_{m-1}u)(s),\dots,(A_1u)(s),u(s),\\ &\qquad (B_{n-1}v)(s),\dots,(B_1v)(s),v(s)\Big)ds\\ &\ge \gamma\|T_2(u,v)(t)\|.\end{aligned}$$

Combining the preceding inequalities, we arrive at

$$\begin{aligned}\min_{\theta\le t\le 1-\theta}(T_1(u,v)(t)+T_2(u,v)(t)) &\ge \min_{\theta\le t\le 1-\theta}(T_1(u,v)(t))+\min_{\theta\le t\le 1-\theta}(T_2(u,v)(t))\\ &\ge \gamma(\|T_1(u,v)(t)\|+\|T_2(u,v)(t)\|).\end{aligned}$$

This completes the proof. $\square$

In this paper, we use the following assumptions:

((H1) $a(t)$ and $b(t)$ do not vanish identically on any subinterval of $(0,1)$, and there exists $t_0\in(0,1)$ such that $a(t_0)>0$, $b(t_0)>0$ and $0<\int_0^1 k_1(s,s)a(s)ds<+\infty$, $0<\int_0^1 g_1(s,s)b(s)ds<+\infty$.

Finally, we define the nonnegative continuous concave functional

$$\alpha(u,v):=\min_{\theta\le t\le 1-\theta}(u(t)+v(t)).$$

We observe here that, for each $(u,v)\in K$, $\alpha(u,v)\le\|(u,v)\|$. Let

$$\begin{gathered}\widetilde{\xi}_1:=\min_{\theta\le t\le 1-\theta}\int_0^1 k_1(t,s)a(s)ds,\quad \widetilde{\eta}_1:=\max_{0\le t\le 1}\int_0^1 k_1(t,s)a(s)ds,\\ \widetilde{\mu}_1:=\min_{\theta\le t\le 1-\theta}\int_0^1 g_1(t,s)b(s)ds,\quad \widetilde{\nu}_1:=\max_{0\le t\le 1}\int_0^1 g_1(t,s)b(s)ds\,.\end{gathered}$$

**Theorem 3.2.** *Let $1\le i\le m-1$, and $1\le j\le n-1$.Assume there exist nonnegative numbers $a,b,c$ such that $0<a<b\le\min\{\gamma,\widetilde{\xi}_1/\widetilde{\eta}_1,\xi_i/\eta_i,\widetilde{\mu}_1/\widetilde{\nu}_1,\mu_j/\nu_j\}c$, and $f(t,x_1,\dots,x_{m-1},x_m,y_1,\dots,y_{n-1},y_n)$, $g(t,x_1,\dots,x_{m-1},x_m,y_1,\dots,y_{n-1},y_n)$ satisfy the following growth conditions:*

- (H2) $f_1(t,x_1,\dots,x_{m-1},x_m,y_1,\dots,y_{n-1},y_n)\le c/2\widetilde{\eta}_1$,
  $f_2(t,x_1,\dots,x_{m-1},x_m,y_1,\dots,y_{n-1},y_n)\le c/2\widetilde{\nu}_1$, *for all* $t\in[0,1]$, $x_i+y_j\in[0,\eta c]$, $x_m+y_n\in[0,c]$.
- (H3) $f_1(t,x_1,\dots,x_{m-1},x_m,y_1,\dots,y_{n-1},y_n)<a/2\widetilde{\eta}_1$,
  $f_2(t,x_1,\dots,x_{m-1},x_m,y_1,\dots,y_{n-1},y_n)<a/2\widetilde{\nu}_1$, *for all* $t\in[0,1]$, $x_i+y_j\in[0,\eta a]$, $x_m+y_n\in[0,a]$.
- (H4) $f_1(t,x_1,\dots,x_{m-1},x_m,y_1,\dots,y_{n-1},y_n)\ge b/2\widetilde{\xi}_1$,
  $f_2(t,x_1,\dots,x_{m-1},x_m,y_1,\dots,y_{n-1},y_n)\ge b/2\widetilde{\mu}_1$, *for all* $t\in[\theta,1-\theta],x_i+y_j\in[\xi b,\eta b/\gamma]$, $x_m+y_n\in[b,b/\gamma]$.

*Then* (1.1) *has at least three positive fixed points $(u_1,v_1)$, $(u_2,v_2)$, $(u_3,v_3)$ such that $\|(u_1,v_1)\|<a$, $b<\min_{\theta\le t\le 1-\theta}(u_2(t)+v_2(t))$, and $\|(u_3,v_3)\|>a$, with $\min_{\theta\le t\le 1-\theta}(u_3(t)+v_3(t))<b$.*

*Proof.* We note first that $T:\overline{P_c}\to\overline{P_c}$ is a completely continuous operator. If $(u,v)\in\overline{P_c}$, then $\|(u,v)\|\le c$. For $1\le i\le m-1$ and $1\le j\le n-1$,

$$(A_iu)(t)+(B_jv)(t)=\int_0^1 k_i(t,s)u(s)ds+\int_0^1 g_j(t,s)v(s)ds\le\eta_i\|u\|+\nu_j\|v\|\le\eta c.$$

From (H2), we have

$$\begin{aligned}
\|T(u,v)(t)\| &= \max_{0\le t\le 1}|T_1(u,v)(t)| + \max_{0\le t\le 1}|T_2(u,v)(t)| \\
&= \max_{0\le t\le 1}\int_0^1 k_1(t,s)a(s)f_1\Big(s,(A_{m-1}u)(s),\dots,(A_1u)(s), \\
&\quad u(s),(B_{n-1}v)(s),\dots,(B_1v)(s),v(s)\Big)ds \\
&\quad + \max_{0\le t\le 1}\int_0^1 g_1(t,s)b(s)f_2\Big(s,(A_{m-1}u)(s),\dots,(A_1u)(s), \\
&\quad u(s),(B_{n-1}v)(s),\dots,(B_1v)(s),v(s)\Big)ds \\
&\le c/2\widetilde{\eta}_1 \max_{0\le t\le 1}\int_0^1 k_1(t,s)a(s)ds + c/2\widetilde{\nu}_1\max_{0\le t\le 1}\int_0^1 g_1(t,s)b(s)ds \le c.
\end{aligned}$$

Therefore, $T:\overline{P_c}\to\overline{P_c}$. In an analogous argument, one can verify that (H3) implies condition (C2) of Lemma 2.1. Clearly, $\{(u,v)\in P(\alpha,b,b/\gamma):\alpha(u,v)>b\}\neq\emptyset$. If $(u,v)\in P(\alpha,b,b/\gamma)$, then $b\le u(t)+v(t)\le b/\gamma$, $t\in[\theta,1-\theta]$.

For $1\le i\le m-1$ and $1\le j\le n-1$, we have

$$(A_iu)(t)+(B_jv)(t)=\int_0^1 k_i(t,s)u(s)ds+\int_0^1 g_j(t,s)v(s)ds\ge \xi b,$$
$$(A_iu)(t)+(B_jv)(t)=\int_0^1 k_i(t,s)u(s)ds+\int_0^1 g_j(t,s)v(s)ds\le \eta b/\gamma.$$

By (H4),

$$\begin{aligned}
\alpha(T(u,v)(t)) &= \min_{\theta\le t\le 1-\theta}(T_1(u,v)(t)+T_2(u,v)(t)) \\
&\ge \min_{\theta\le t\le 1-\theta}T_1(u,v)(t) + \min_{\theta\le t\le 1-\theta}T_2(u,v)(t) \\
&\ge \min_{\theta\le t\le 1-\theta}\int_0^1 k_1(t,s)a(s)f_1\Big(s,(A_{m-1}u)(s),\dots,(A_1u)(s), \\
&\quad u(s),(B_{n-1}v)(s),\dots,(A_1v)(s),v(s)\Big)ds \\
&\quad + \min_{\theta\le t\le 1-\theta}\int_0^1 g_1(t,s)b(s)f_2\Big(s,(A_{m-1}u)(s),\dots,(A_1u)(s), \\
&\quad u(s),(B_{n-1}v)(s),\dots,(B_1v)(s),v(s)\Big)ds \\
&\ge b/2\widetilde{\xi}_1\min_{\theta\le t\le 1-\theta}\int_0^1 k_1(t,s)a(s)ds + b/2\widetilde{\mu}_1\min_{\theta\le t\le 1-\theta}\int_0^1 g_1(t,s)b(s)ds \\
&= b.
\end{aligned}$$

Therefore, condition (C1) of Lemma 2.1 is satisfied.

Finally, we show that (C3) is also satisfied. If $(u,v)\in P(\alpha,b,c)$ and $\|T(u,v)\|>b/\gamma$, then

$$\alpha(T(u,v)(t))=\min_{\theta\le t\le 1-\theta}(T_1(u,v)(t)+T_2(u,v)(t))\ge\gamma\|T(u,v)\|>b.$$

Therefore, (C3) of Lemma 2.1 is also satisfied. By Lemma 2.1, the system (1.1) has at least three positive fixed points $(u_1,v_1),(u_2,v_2),(u_3,v_3)$ such that $\|(u_1,v_1)\|<a$,

$b < \min_{\theta \le t \le 1-\theta}(u_2(t)+v_2(t))$, and $\|(u_3, v_3)\| > a$, with $\min_{\theta \le t \le 1-\theta}(u_3(t)+v_3(t)) < b$. $\square$

**An example.** Consider a system of nonlinear second-order and fourth-order ordinary differential equations (with $m = 1, n = 2$):

$$\begin{gathered}
-x'' = a(t)f_1(t, x, y, -y'') = 0,\\
y^{(4)} = b(t)f_2(t, x, y, -y'') = 0,\\
x(0) - x'(0) = x(1) + x'(1) = 0,\\
y(0) - y'(0) = y(1) + y'(1) = 0,\\
y''(0) - y'''(0) = y''(1) + y'''(1) = 0.
\end{gathered} \tag{3.2}$$

Let $u := x$ and $v := -y''$, then the problem (3.2) is equivalent to the following system of nonlinear integral equations

$$\begin{aligned}
u(t) &= \int_0^1 k_1(t, s)a(s)f_1(s, u(s), \int_0^1 g_1(s, \tau)v(\tau)d\tau, v(s)),\\
v(t) &= \int_0^1 g_1(t, s)b(s)f_2(s, u(s), \int_0^1 g_1(s, \tau)v(\tau)d\tau, v(s)),
\end{aligned} \tag{3.3}$$

where

$$k_1(t, s) = g_1(t, s) := \frac{1}{3}\begin{cases} (1 + s)(2 - t), 0 \le s \le t \le 1,\\ (1 + t)(2 - s), 0 \le t \le s \le 1.\end{cases}$$

We choose $a(t) := (1 - t)^{-1/2}$, $b(t) := t^{-1/2}$, $\theta := 1/4$, $f_1(t, x_1, x_2, x_3)$ equals

$$\begin{cases}
\frac{t}{100} + \frac{1}{10}(x_1 + x_3)^2, & t \in [0, 1],\ 0 \le x_1 + x_3 \le 2,\ x_2 \ge 0,\\
\frac{t}{100} + 6[(x_1 + x_3)^2 - 2(x_1 + x_3)] + \frac{2}{5}, & t \in [0, 1],\ 2 < x_1 + x_3 < 4,\ x_2 \ge 0,\\
\frac{t}{100} + 20\log_2(x_1 + x_3) + 2(x_1 + x_3) + \frac{2}{5}, & t \in [0, 1],\ 4 \le x_1 + x_3 \le 16,\ x_2 \ge 0,\\
\frac{t}{100} + \frac{1}{4}(x_1 + x_3) + \frac{542}{5}, & t \in [0, 1],\ x_1 + x_3 > 16, x_2 \ge 0.
\end{cases}$$

and $f_2(t, x_1, x_2, x_3)$ equals

$$\begin{cases}
\frac{t}{100} + \frac{1}{10}(x_1 + x_3)^2, & t \in [0, 1],\ 0 \le x_1 + x_3 \le 2,\ x_2 \ge 0,\\
\frac{t}{100} + \frac{13}{8}[(x_1 + x_3)^2 - 2(x_1 + x_3)] + \frac{2}{5}, & t \in [0, 1],\ 2 < x_1 + x_3 < 4,\ x_2 \ge 0,\\
\frac{t}{100} + \frac{5}{2}\log_2(x_1 + x_3) + 2(x_1 + x_3) + \frac{2}{5}, & t \in [0, 1],\ 4 \le x_1 + x_3 \le 16,\ x_2 \ge 0,\\
\frac{t}{100} + \frac{1}{4}(x_1 + x_3) + \frac{192}{5}, & t \in [0, 1],\ x_1 + x_3 > 16, x_2 \ge 0.
\end{cases}$$

Then by direct calculation, we obtain

$$\xi \approx 0.4120, \quad \eta = 0.5, \quad \widetilde{\xi}_1 \approx 0.0417, \quad \widetilde{\mu}_1 \approx 0.9273, \quad \widetilde{\eta}_1 \approx 0.8889, \quad \widetilde{\nu}_1 \approx 1.1111.$$

It is easy to check that (H1) holds. Choose $\gamma = \frac{1}{4}$, $a = 1$, $b = 4$, $c = 800$. Also, it is easy to verify that $f_1$ and $f_2$ satisfy conditions (H2)-(H4). So the system (3.2) has at least three positive solutions $(u_1, v_1), (u_2, v_2), (u_3, v_3)$ such that $\|(u_1, v_1)\| < 1$, $4 < \min_{1/4 \le t \le 3/4}(u_2(t) + v_2(t))$, and $\|(u_3, v_3)\| > 1$, with $\min_{1/4 \le t \le 3/4}(u_3(t) + v_3(t)) < 4$.

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